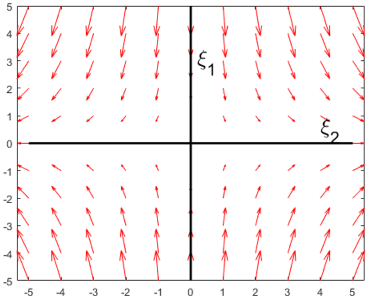
Outline

1. Real roots stability possibilities

2. Richardson's arms race model



Both eigenvalues positive: unstable source

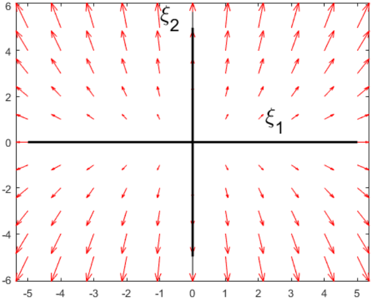
The system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}$$

has solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}.$$

The origin is a source and the y-direction $\binom{0}{1}$ dominates.



Both eigenvalues negative: stable sink

The system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{x}$$

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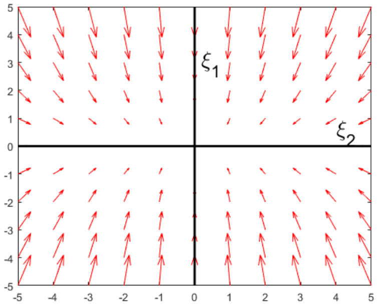
$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

The origin is a sink and the x-direction $\binom{1}{0}$ dominates.



Opposite signs eigenvalues: unstable saddle point

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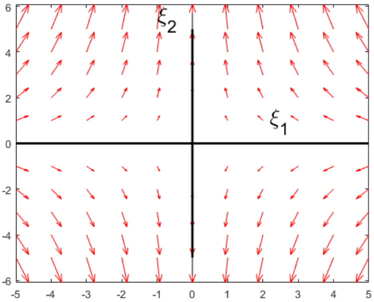

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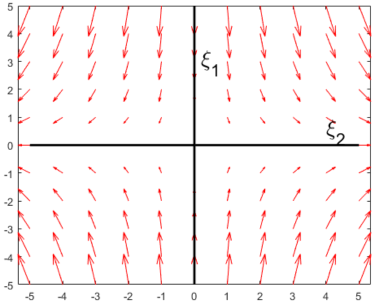

One zero eigenvalue and one negative: stable sink line

The system

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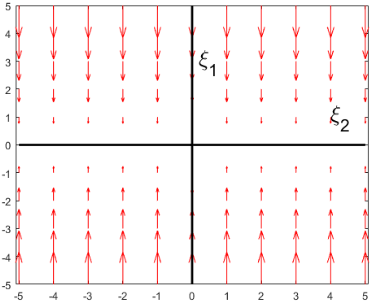

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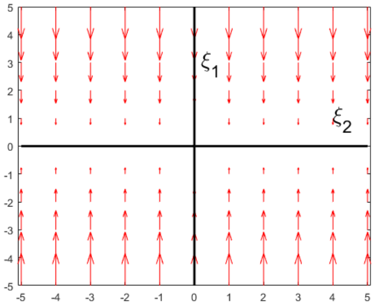

In class example

Consider the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Find general solution and sketch the phase portrait:

1. Which term dominates?
2. If $\mathbf{x}(0) = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$ what is the limit of the solution?

For matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ here are the general formulas for its eigenvalues and eigenvectors:

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4det(A)}$$

and if $c \neq 0$ (cf. other cases see notes) then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}.$$

Richardson's arms race model

Consider countries A, B with $x(t), y(t)$ amount of weaponry respectively. The model for the rate of change of weaponry is:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -a \cdot x + b \cdot y + e_1$$
$$\frac{\mathrm{d}y}{\mathrm{dt}} = c \cdot x - d \cdot y + e_2.$$

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3. The constant e_1 represents the distrust country A has for country B and the reverse for e_2.

For concreteness consider the system:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

1. A constant solution for this nonhomogeneous problem is $v(t) = \binom{1}{1}$, which we obtained by setting $\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \binom{0}{0}$ and solving for $\mathbf{x}$. Therefore, as explained above the general solution will be:

$$\mathbf{x} = c_1\boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2\boldsymbol{\xi}_2 e^{\lambda_2 t} + \begin{pmatrix}1\\1\end{pmatrix}.$$

2. The eigenvalues for this matrix are the solutions to

$$0 = \lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 4\lambda - 5$$
$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -5.$$

3. The corresponding eigenvectors are $\boldsymbol{\xi}_1 = \binom{1}{1}$ and $\boldsymbol{\xi}_2 = \binom{-1}{2}$. So the general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}-1\\2\end{pmatrix}e^{-5t} + \begin{pmatrix}1\\1\end{pmatrix}.$$

1. Therefore, $\binom{1}{1}$ becomes a saddle point. That is, if a solution starts from $\binom{1}{1}$ in a direction parallel to $\binom{-1}{2}$ (i.e. choose $c_1 = 0$), then the solution will converge to the constant $\binom{1}{1}$ at an exponential rate (like e^{-5t}). For example, this happens if $\mathbf{x}(0) = \binom{\frac{1}{2}}{2}$. However for $c_1 \neq 0$ the solution will diverge to infinity like e^{t} in the direction $\binom{1}{1}$ away from the starting point $\binom{1}{1}$.

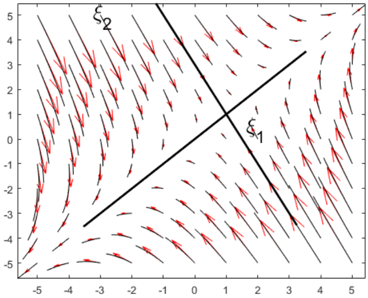

Figure: The $\binom{1}{1}$ is a saddle point

1. This is reasonable because if the amount of distrust is negative $e_1 = -1, e_2 = -1 < 0$ (i.e. positive trust), then for appropriate initial conditions the solutions will converge to peaceful coexistence $(x(t), y(t)) \to (1, 1)$.

The End