Outline

1. Systems of odes
   - General solution
   - Example presenting the method

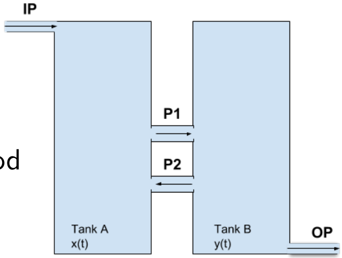

Figure: Tanks A and B containing salt

Intro

Consider system of equations:

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\begin{aligned}
x_1' &= p_{11}(t)x_1 + \ldots + p_{1n}x_n + g_1(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + \ldots + p_{nn}x_n + g_n(t)
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where $p_{ij}(t), g_i(t)$ are continuous functions.We can rewrite this system as

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t).$$

Linear independence

For the homogeneous problem

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where $\mathbf{x}_i = i^{\text{th}}$ $row = (x_{1i}, ..., x_{ni})$, then

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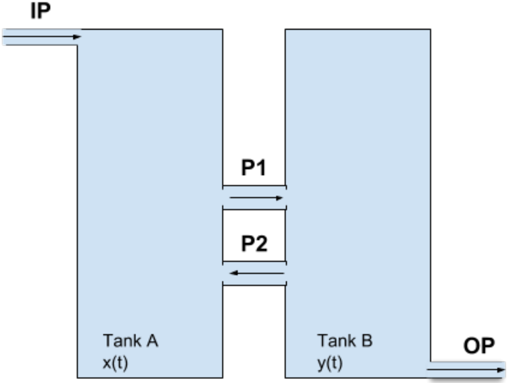

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$$\frac{\mathrm{d}y}{\mathrm{dt}} = \text{Input rate} - \text{Output rate} = 1 \cdot \frac{x(t)}{1000} - 2 \cdot \frac{y(t)}{1000} - 1 \cdot \frac{y(t)}{1000}.$$

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In matrix form our system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{1000} \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

1. First we compute the eigenvalues

$$\det \begin{bmatrix} -1-\lambda & 2 \\ 1 & -3-\lambda \end{bmatrix} = 0 \;\; \Rightarrow (-1-\lambda)(-3-\lambda) - 2\cdot 1 = 0$$

$$\Rightarrow \lambda_1 = -2+\sqrt{3},\; \lambda_2 = -2-\sqrt{3}.$$

2. Second we find the corresponding eigenvectors. To find $\boldsymbol{\xi}_1 := \binom{\xi_{1,1}}{\xi_{2,1}}$ we solve the system (up to multiples):

$$\begin{bmatrix} -1-\lambda_1 & 2 \\ 1 & -3-\lambda_1 \end{bmatrix} \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Connected tanks

By solving the system directly we obtain the solution (up to multiples). For example, we rewrite the above to get:

$$\begin{cases} (1-\sqrt{3})\xi_{1,1} + 2\xi_{2,1} = 0 \\ \xi_{1,1} + (-1-\sqrt{3})\xi_{2,1} = 0 \end{cases} \implies \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}.$$

Similarly to obtain $\boldsymbol{\xi}_2$ we have to solve

$$\begin{bmatrix} -1-(-2-\sqrt{3}) & 2 \\ 1 & -3-(-2-\sqrt{3}) \end{bmatrix} \begin{pmatrix} \xi_{1,2} \\ \xi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we get

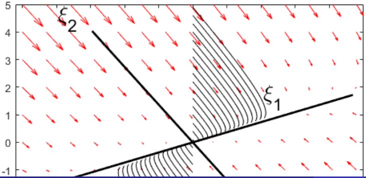
$$\boldsymbol{\xi}_2 = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}.$$

Connected tanks

1. Therefore, by the discussion above the general solution will be

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{1000} c_1 \cdot \boldsymbol{\xi}_1 e^{\lambda_1 t} + \frac{1}{1000} c_2 \cdot \boldsymbol{\xi}_2 e^{\lambda_2 t} = c_1 \cdot \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix} \frac{e^{(-2+}}{100}$$

2. Since $2 > \sqrt{3}$, both the eigenvalues are negative and in turn the salt concentrations $x(t), y(t)$ will go to zero as $t \to +\infty$. This is reasonable because through pipe IP we are injecting salt-free water that over time transports the tanks' salt out through pipe OP.
3. Next we study the stability. Since $-2+\sqrt{3} > -2-\sqrt{3}$, we get $e^{(-2+\sqrt{3})t} > e^{(-2-\sqrt{3})t}$ and so as $t \to +\infty$ the first eigenvector $\binom{1+\sqrt{3}}{1}$ will dominate.



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