Outline

1. Method of undetermined coeffic
2. Variation of parameters

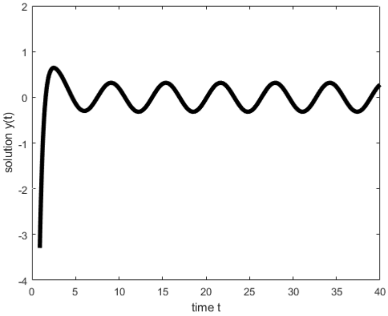

Figure: Nonhomogeneous periodic forcing

$f = ct^m e^{r_* t}$

If $f = ct^m e^{r_* t}$, then the ansatz is

$$y_0 = (a_0 + a_1 t + \ldots + a_m t^m) e^{r_* t} \cdot \begin{cases} 1 & \text{if } r_* \text{ is not a root} \\ t & \text{if } r_* \text{ is a simple root} \\ t^2 & \text{if } r_* \text{ is a double root} \end{cases}$$

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$$y_0 = e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m) cos(\beta t) + (b_0 + b_1 t + ... + b_m t^m) sin(\beta t)]$$

$$\cdot \begin{cases} 1 & if \;\; \alpha + i\beta \;\; is \;\; not \;\; a \;\; root \\ t & if \;\; \alpha + i\beta \;\; is \;\; a \;\; simple \;\; root \end{cases}$$

Presenting method example

Consider the spring system governed by

$$y'' + 2y' - 3y = 3te^{t}.$$

Find the solution and its asymptotic behaviour.

In class example

Consider the spring system governed by

$$y'' + 2y' - 3y = 2te^t sin(t).$$

Determine what form the solution will take.

In class example

Solve the IVP and determine long term behaviour

$$y'' + 5y' + 6y = 2te^{t}sin(t), y(0) = 2, y'(0) = 1$$

Variation of parameters

We will now consider non-homogeneous equations with coefficients of the form

$$ay'' + by' + cy = f\left(t\right),$$

where $\mathrm{f}\left(\mathrm{t}\right)$ is any continuous function and a,b,c are also functions with $a\left(t\right) \neq 0$.

Example-presenting the method

Returning to the spring example suppose that it is damping free $\gamma = 0$ and the exernal force is $f(t) = tan(t)$:

$$y'' + y = tan(t).$$

(1) First we find independent solutions for the homogeneous problem:

$$y'' + y = 0.$$

(2) One can easily check that $cos(t), sin(t)$ are solutions for it and computing their Wronskian gives:

$$W(cos(t), sin(t), t) = cos^2 t + sin^2(t) = 1 \neq 0.$$

(3) Therefore, we make a guess

$$y_{nh} = v_1 cos(t) + v_2 sin(t).$$

Example presenting the method

(4) Using our system of equations we obtain

$$
\begin{aligned}
& v_1' y_1 + v_2' y_2 = 0 \\
& y_1' v_1' + y_2' v_2' = \frac{f}{a} \Rightarrow \\
& v_1' cos(t) + v_2' sin(t) = 0 \\
& -cos(t) v_1' + cos(t) v_2' = tan(t) \Rightarrow \\
& v_1' = -tan(t) sint \\
& v_2' = tan(t) cos(t) = sin(t).
\end{aligned}
$$

Therefore, by integrating we obtain

$$v_1 = -\int tan(t) sin(t)\, dt = -\int \frac{sin^2(t)}{cos(t)} dt$$

$$= \int cos(t) - \frac{1}{cos(t)} dt = sin(t) - ln|\frac{1 + sin(t)}{cos(t)}| + c_1 \text{ and}$$

$$v_2 = \int sin(t)\, dt = -cos(t) + c_2.$$

For simplicity we take $c_1 = c_2 = 0$ and we get:

$$y_{nh} = (sin(t) - ln|\frac{1 + sin(t)}{cos(t)}|) cos(t) - cos(t) sin(t)$$

$$= ln|\frac{cos(t)}{1 + sin(t)}| cos(t).$$

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$$y_{nh} = v_1(t)y_1 + v_2(t)y_2$$

and plug it into our ODE.This will gives one equation for v_1, v_2.

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This equation is helpful because it simplifies the first equation (proved in detail below)

$$y_{nh}' = y_1'v_1 + y_2'v_2 + 0 \Rightarrow ay_g'' + by_{nh}' + cy_{nh} = a(y_1'v_1' + y_2'v_2').$$

Method formal steps

(4) Therefore, we can obtain v_1, v_2 from the system

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ y_1' v_1' + y_2' v_2' = \frac{f}{a} \end{cases}$$

In class example

Consider the equation

$$ty'' - \big(1 + t\big)y' + y = t^2 e^{2t}$$

with given fundamental solutions $y_1 = 1 + t, y_2 = e^t$ for the homogeneous problem $ty'' - \big(1 + t\big)y' + y = 0.$

In class example

Consider the equation

$$x^2y'' - 3xy' + 4y = x^2 lnx$$

with given fundamental solutions $y_1 = x^2, y_2 = x^2 ln(x)$ for the homogeneous problem $x^2y'' - 3xy' + 4y = 0$.

In class example

Consider the equation

$$ty'' - (1+t)y' + y = t^2 e^{2t}$$

with given fundamental solutions $y_1 = 1 + t, y_2 = e^t$ for the homogeneous problem $ty'' - (1+t)y' + y = 0$.

The End