Outline

1. Repeated roots
2. Stability
   - Stability criterion
   - Price adjustment mechanism
3. Method of undetermined coeffic

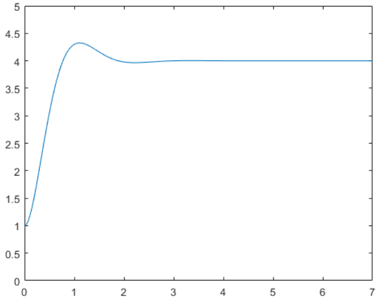

Figure: stabilization around market price

Repeated roots

In some cases the roots are equal when $b^2 - 4ac = 0\big)$. For example, suppose that $\gamma^2 \approx 4km$ $\big($called **critically damped**$\big)$, then the roots will be

$$r_1 = r_2 = -\frac{\gamma}{2m} =: r.$$

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$$r_1 = r_2 = -\frac{\gamma}{2m} =: r.$$

If the ode $ay'' + by' + cy = 0$ has a characteristic equation with repeated root $r := \frac{-b}{2a}$, then its fundamental solution is of the form:

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

Proof.

For $y_2 := g(t)y_1$ we will first find which ode g(t) must satisfy in order that y_2 is a solution of our ode

$a(g(t)y_1)'' + b(g(t)y_1)' + c = 0 \Rightarrow$

$a(g''(t)e^{rt} + 2g're^{rt}) + bg'e^{rt} = 0 \Rightarrow$

where we used that y_1 satisfies the ode

$0 = a(g''(t) + g'(a2r + b) = ag'' + g'(2a\frac{-b}{2a} + b) = ag'' \Rightarrow \quad ag'' = 0 \Rightarrow$

$g = c_1 + c_2 t.$

This is indeed the general solution because:

$$W(y_1, y_2, t) = e^{rt}(e^{rt} + te^{rt}) - re^{rt}te^{rt} = e^{rt} \neq 0.$$

□

In class example

Consider the IVP

$$y'' - 2y' + y = 0, y(0) = 1, y'(0) = 2$$

In class example

Consider the IVP

$$y'' - 6y' + 9y = 0, y(0) = 0, y'(0) = 2$$

Consider nonhomogeneous equation of the form

$$y'' + ay' + by = c,$$

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$$y = y_h + \frac{c}{b}.$$

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The solution $y_s := \frac{c}{b}$ is called **globally stable** when $y \to y_s$, which is equivalent to saying $y_h \to 0$ as $t \to +\infty$.

Stability criterion

Consider non-homogeneous equation

$$y'' + ay' + by = f(t).$$

Then the solution is of the form $y = y_h + y_s$, where y_h solves the homogeneous problem and y_s is any solution of the nonhomogeneous problem.

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$$\lim_{t\to\infty} y = y_s \text{ iff } a > 0, b > 0.$$

In other words, y_s is globally stable iff $a > 0, b > 0$ iff the real parts of the roots of the characteristic equation are both negative.

As with constant f(t), we again obtain that the generalized solution is of the form

$$y = c_1 y_1 + c_2 y_2 + y_s =: y_h + y_s.$$

So by studying when $y_h \to 0$, we can identify when y_s is the globally stable solution.

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So by studying when $y_h \to 0$, we can identify when y_s is the globally stable solution.

(1) If the characteristic equation has two real distinct roots r_1, r_2 then

$$y_h = c_1 e^{r_1} + c_2 e^{r_2 t}$$

and so y_s is stable iff $r_1, r_2 < 0$. The roots are

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

We have $r_1 < 0 \Leftrightarrow b > 0$ and $r_2 < 0 \Leftrightarrow a > 0$.

(2) If the characteristic equation has two complex roots r_1, r_2 then

$$y_h = e^{-\frac{a}{2}}(c_1 cos(\beta t) + c_2 sin(\beta t))$$

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(3) If the characteristic equation has a double root $r = r_1 = r_2$ then

$$y_h = e^{rt}(c_1 + c_2 t)$$

and so y_s is stable iff $r = \frac{-a}{2} < 0 \Rightarrow a > 0$. The condition $b > 0$ follows from $a^2 = 4b$.

Price adjustment mechanism

Buyers may also base their behavior on whether the price is increasing or decreasing.

$$D(P) = a - bP + mP' + nP'' \text{ and } S(P) = \alpha + \beta P + uP' + wP'',$$

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where $a, b, \alpha, \beta > 0$ and m, n, u, w can be any sign. For now we will study it from the buyers perspective and set $u = w = 0$. To obtain an ODE for it, we assume that the market is cleared and thus $D(P) = S(P)$.

$$a - bP + mP' + nP'' = \alpha + \beta P \Rightarrow P'' + \frac{m}{n}P' - \frac{b+\beta}{n}P = \frac{\alpha - a}{n}.$$

For example, suppose $a = 40, b = 2, m = -2, \alpha = -5, \beta = 3, n = -1$ then our ode will be:

$$P'' + 2P' + 5P = 45$$

and for $m = 2$

$$P'' - 2P' + 5P = 45.$$

The corresponding solutions will be

$$P(t) = e^{-t}[a_1 cos(2t) + a_2 sin(2t)] + 9$$

and

$$P(t) = e^{t}[a_1 cos(2t) + a_2 sin(2t)] + 9.$$

Matlab simulation

In class example

Solve the IVP and determine long term behaviour

$$y'' + 5y' + 6y = 3, y(0) = 2, y'(0) = 1.$$

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Solve the IVP and determine long term behaviour

$$y'' + y = 9, y(\pi/3) = 2, y'(\pi/3) = -4$$

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By managing to find a particular solution y_{nh}, then we can generate every other one. Let v be any another solution, then

$$a(v - y_{nh})'' + b(v - y_{nh})' + c(v - y_{nh}) = f(t) - f(t) = 0.$$

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Therefore, by finding the fundamental set of solutions y_1, y_2 for the homogeneous problem we have

$$v - y_{nh} = c_1 y_1 + c_2 y_2 \Rightarrow v = y_{nh} + c_1 y_1 + c_2 y_2.$$

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So we managed to generate any solution starting from y_{nh}, y_1, y_2. Here we will find y_{nh} for $f(t)$ of the following possible forms:

$$f_1(t) := Ct^m e^{r_* t}, f_2(t) := Ct^m e^{\alpha} cos(\beta t), f_3(t) := Ct^m e^{\alpha t} sin(\beta t).$$

forcing term $f = ct^m e^{r_* t}$

If $f = ct^m e^{r_* t}$ then we make the ansatz $\big($assume the solution to be of the form$\big)$

$$y_{nh}(t) = t^s\big(a_0 + a_1 t + \ldots + a_m t^m\big)e^{r_* t}.$$

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Now the way we pick the exponent s, depends on whether or not r_* is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.

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1. If r_* is not a root, then we set $s := 0$.
2. If r_* is a simple root, then we set $s := 1$.
3. If r_* is a double root, then we set $s := 2$.

First we will work with

$ay'' + by' + c = Ct^m e^{r_* t}$.

We make the guess

$y_{nh}(t) = (a_0 + a_1 t + ... + a_n t^n) e^{rt}$.

for some yet undetermined n. Then plugging it into our ode we obtain

$$\begin{aligned} &ay''_{nh} + by'_{nh} + cy_{nh} \\ &= a_n(ar^2 + br + c)t^n e^{rt} + (a_n n(2ar + b) + a_{n-1}(ar^2 + br + c))t^{n-1}e^{rt} \\ &+ [a_n(n-1)a + a_{n-1}(n-1)(2ar + b) + a_{n-2}(ar^2 + br + c)]t^{n-2}e^{rt} \\ &+ lower\ order\ terms. \end{aligned}$$

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Case 2: If r is a simple root, then $ar^2 + br + c = 0$ and we are left with t^{n-1} being the leading term and so $n - 1 := m$.

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Case 3: If r is a double root, then $ar^2 + br + c = 0, 2ar + b = 0$ and we are left with t^{n-2} being the leading term and so $n - 2 := m$.

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$$y_{nh} = \left(a_0 + a_1 t + ... + a_n t^{m+2}\right) e^{rt}.$$

Moreover, since r is a repeated root, then e^{rt}, te^{rt} are both solutions of the homogeneous equation $ay'' + by' + c = 0$ and so we can ignore them. Thus,

$$y_{nh} = \left(a_2 t^2 + ... + a_n t^{m+2}\right) e^{rt} = t^2 \left(a_2 + ... + a_n t^m\right) e^{rt}.$$

Next we will work with

$$ay'' + by' + c = Ct^m e^{\alpha t} sin\left(\beta t\right) = \frac{1}{2i} Ct^m e^{\alpha t + i\beta t} - \frac{1}{2i} Ct^m e^{\alpha t - i\beta t}.$$

Therefore, from the previous we make the guess

$$y_{nh} = \left(a_0 + a_1 t + ... + a_n t^n\right) e^{\left(\alpha + i\beta\right) t} + \left(b_0 + b_1 t + ... + b_n t^n\right) e^{\left(\alpha - i\beta\right) t}.$$
$$= e^{\alpha t} \left(c_0 + c_1 t + ... + c_n t^n\right) cos\left(\beta t\right) + \left(d_0 + d_1 t + ... + d_n t^n\right) e^{\alpha t} sin\left(\beta t\right).$$

So as above we check whether $r_* = \alpha + i\beta$ is a root and the same analysis shows the result.

Example presenting the method

Resuming the spring example, let u(t) denote the displacement from the equilibrium position. Then by Newton's law one can obtain the equation

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where $F(t)$ is any external force. Above we assumed that $F(t) = 0$, and now we will take it to be any of the above mentioned functions. For example,consider the equation

$$y'' + 3y' + 2y = sin(t).$$

Here we are shaking the spring system periodically in time.

1. First we are in the second case and so we make the ansatz

$$y_{nh}(t) = t^s e^{\alpha t}[(a_0+a_1t+...+a_mt^m)cos(\beta t)+(b_0+a_1t+...+b_mt^m)sin(\beta$$

which simplifies because $m = 0, \alpha = 0$ and $\beta = 1$:

$$y_{nh}(t) = t^s(a_0cos(t) + b_0sin(t)).$$

2. Next we pick s, depending on whether $a + i\beta = i$ is a root of our ODE's characteristic equation:

$$r^2 + 3r + 2 = 0.$$

3. Its roots are $r_1 = -2, r_2 = -1$ and so we set $s = 0$ and have

$$y_{nh}(t) = a_0cos(t) + b_0sin(t).$$

1. Plugging into our ode we obtain

$$
\begin{aligned}
&y'' + 3y' + 2y \\
&= -(a_0 cos(t) + b_0 sin(t)) + 3(-a_0 sin(t) + b_0 cos(t)) \\
&+ 2(a_0 cos(t) + b_0 sin(t)) \\
&= (a_0 + 3b_0) cos(t) + (-3a_0 + b_0) sin(t)
\end{aligned}
$$

and so to have this be equal to sin(t) we require

$$
\begin{cases} a_0 + 3b_0 = 0 \\ -3a_0 + b_0 = 0 \end{cases} \Rightarrow a_0 = -0.3, b_0 = 0.1.
$$

2. So the solution will be

$$
y_{nh}(t) = -0.3cos(t) + 0.1sin(t).
$$

3. Therefore, the general solution will be:

$$
y = y_{nh} + c_1 e^{-2t} + c_2 e^{-t}.
$$

4. But why is it periodic given that damping is involved $(\gamma \neq 0)$? The sinusoidal external force keeps pumping energy into the system.
5. Matlab simulation

In class example

Consider the spring system governed by

$$y'' + 2y' - 3y = 3te^t.$$

Find the solution and its asymptotic behaviour.

In class example

Consider the spring system governed by

$$y'' + 2y' - 3y = 2te^t sin(t).$$

Determine what form the solution will take.

The End