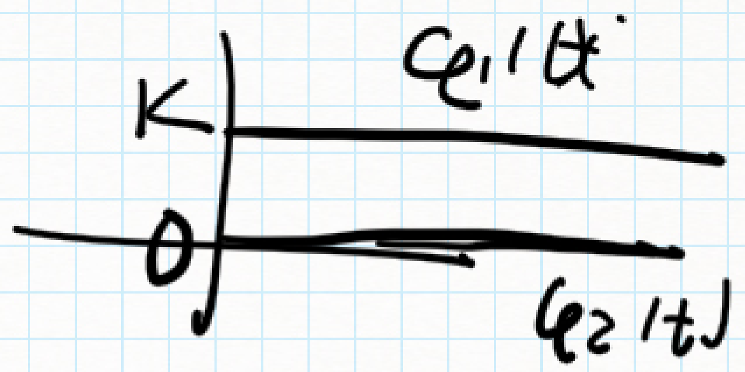
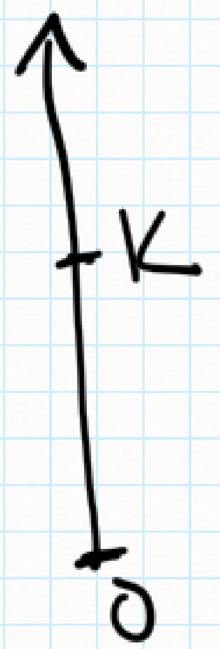
$$y' = r\left(1 - \frac{y}{K}\right)y \qquad \underline{r>0}$$

1) Find equilibrium slns $\underline{y' = 0}$

$$\Rightarrow \quad y(t) = K \quad \Rightarrow \quad \varphi_1(t) \equiv K$$
$$y = 0 \qquad \varphi_2(t) \equiv 0$$



2) 

investigate behav of y' in

a) $(K, +\infty)$

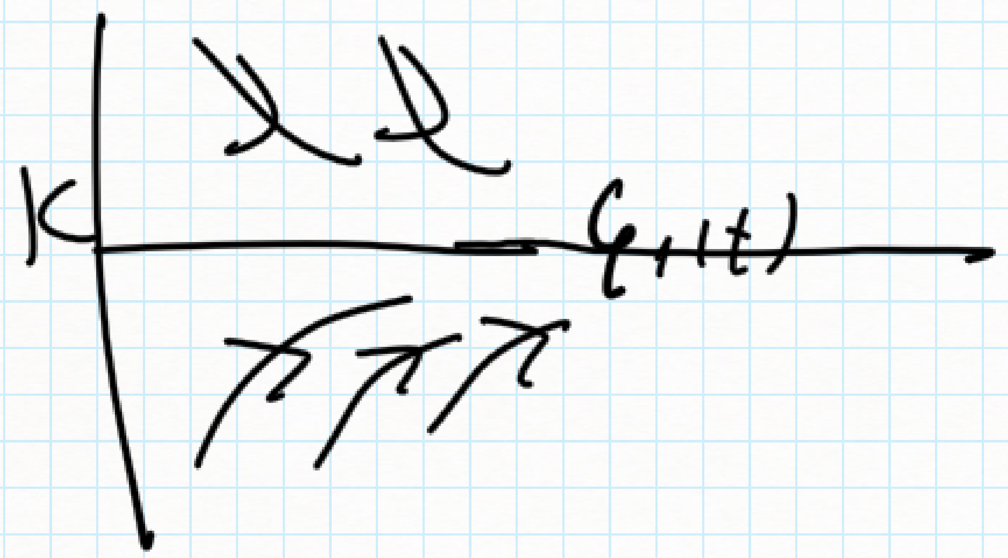
b) $(0, K)$

c) $(-\infty, 0)$

3) $y \in [K, +\infty)$ $\quad y' = r\left(1 - \frac{y}{K}\right) y < 0 \quad y > K \Rightarrow 1 - \frac{y}{K} < 0$

4) $y \in [0, K]$

$$y' = r\left(1 - \frac{y}{K}\right) y \geq 0$$



$$1 - \frac{y}{K} > 0$$

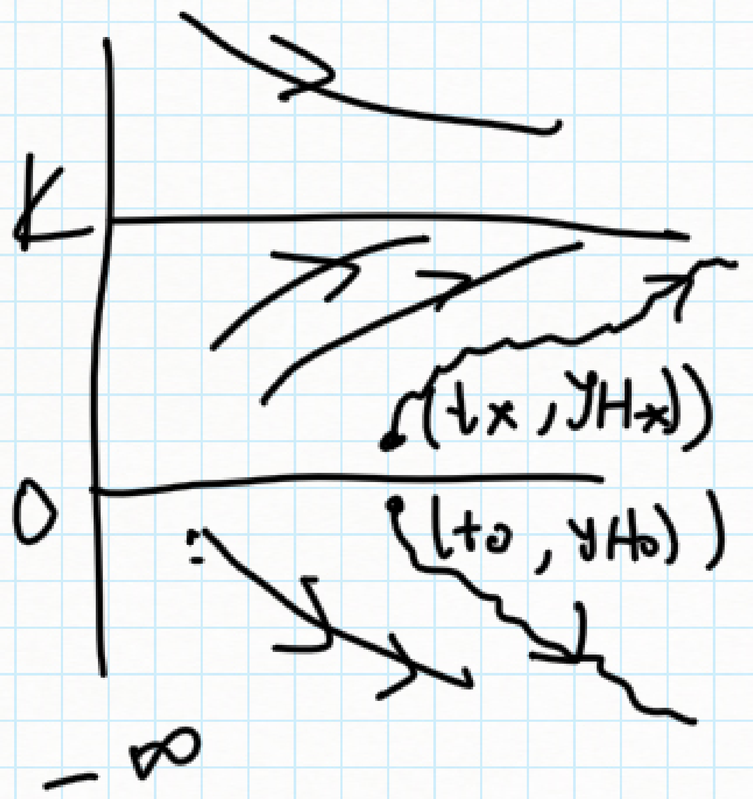
$$\Leftrightarrow 1 > \frac{y}{K}$$

$$\Leftrightarrow K > y$$

Asymptotically stable b/c for $y(0)$ close to $\varphi_1(t)$ we have $y \to \varphi_1(t)$

5) $y \in [-\infty, 0]$ $\quad y' = r\left(1-\frac{y}{K}\right)y < 0$

$y < 0$



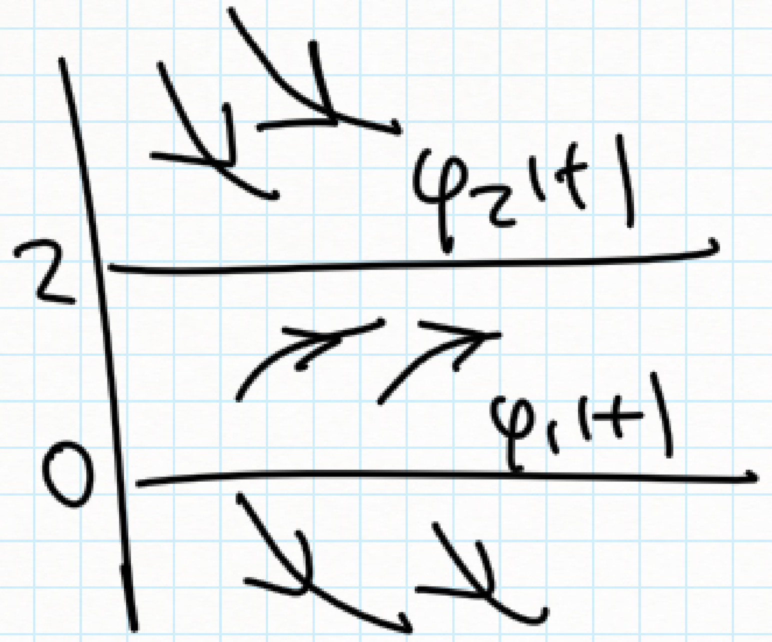
$\Rightarrow$ $\varphi_1(t)$ as. stable

$\varphi_2(t)$ as. unstable

$y' = y(2-y)$ $\qquad$ $y' = (y-1)(y-2)(y-3)$

1) equilibrium sltns $\quad y' = 0 \Rightarrow \varphi_1(t) = 0$

$\varphi_2(t) = 2$

2) 

3) $y \in [2, +\infty]$

$\Rightarrow y' < 0$

$$m u'' + \gamma u' + k u = 0 \qquad m=1,\ \gamma=5,\ k=6$$

$$u_1 = e^{-2t} \quad u_2 = e^{-3t}$$

1) Make the ansatz that $\boxed{u(t) = e^{r \cdot t}}$ (unknown r)

$$m r^2 e^{rt} + \gamma r e^{rt} + k e^{rt} = 0 \Longrightarrow$$

$$\Rightarrow \underset{\neq 0}{e^{rt}} (m r^2 + \gamma r + k) = 0 \Rightarrow$$ 2nd order eqn

$$m r^2 + \gamma r + k = 0$$

$$r^2 + 5r + 6 = 0 \Rightarrow r_1 = -2, \; r_2 = -3$$

$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$= C_1 e^{-2t} + C_2 e^{-3t}$$

$y(0) = 0$

$y'(0) = 1$

$$\Rightarrow \left.\begin{aligned} C_1 + C_2 &= 0 \\ C_1(-2) + C_2(-3) &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 1 \\ C_2 &= -1 \end{aligned}$$

$$\Rightarrow u(t) = e^{-2t} - e^{-3t} \longrightarrow 0 \text{ as } t \to +\infty$$

If y_1, y_2 are s/ns to 2nd order thm

$ay_1 + by_2$ is also. Let $L = a_1 \partial^2 + a_2 \partial + a_3$

$$a_1(ay_1 + by_2)'' + a_2(ay_1 + by_2)' + a_3(ay_1 + by_2) = 0$$

$$Ly_1 + Ly_2 = 0$$

$$\shortparallel$$

$$L(y_1 + y_2)$$

$$4y'' - y = 0 \qquad y(-2) = 1 \,,\; y'(-2) = -1$$

1) $$4r^2 - 1 = 0 \Rightarrow r = \pm \frac{1}{2}$$

2) $$y(t) = c_1 e^{-\frac{1}{2}\cdot t} + c_2 e^{\frac{1}{2}t}$$

3) $$\left.\begin{aligned} c_1 \cdot e + c_2 e^{-1} &= 1 \\ -\frac{c_1}{2} e + \frac{c_2}{2} e^{-1} &= -1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= \\ c_2 &= \end{aligned}$$

$$y'' + 5y' + 6y = 0 \qquad y(0) = 1,\ y'(0) = 2$$

1) $y(t) = e^{rt}$ $\qquad (e^{rt})'' + 5(e^{rt})' + 6e^{rt} = 0$

$$\Rightarrow r^2 e^{rt} + 5r e^{rt} + 6 e^{rt} = 0$$

$e \neq 0$

$$\Rightarrow \boxed{r^2 + 5r + 6 = 0} \Rightarrow \begin{array}{l} r_1 = -3 \\ r_2 = -2 \end{array}$$

2) $y = C_1 e^{-3t} + C_2 e^{-2t}$

$y(0) = 1$

3) $\Rightarrow C_1 + C_2 = 1$

$y'(0) = 2$

$$\left.\begin{aligned} -3C_1 - 2C_2 &= 2 \\ 2C_1 + 2C_2 &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= -4 \\ C_2 &= 5 \end{aligned}$$

$$y(0)=2\ ,\ y'(0)=1$$

$$\left.\begin{aligned} c_1+c_2&=2\\ -3c_1+(-2)c_2&=1\end{aligned}\right\} \Rightarrow \begin{aligned} c_1&=7\\ c_2&=-5\end{aligned}$$

$$y(t)=7e^{-3t}+(-5)\,e^{-2t}$$

We'll show $\overbrace{\boxed{z(t) := a y_1 + b y_2}}^{\text{Linear}} \Rightarrow \boxed{z = \varphi}$

1) $\begin{bmatrix} y_1(t_*) & y_2(t_*) \\ y_1'(t_*) & y_2'(t_*) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varphi(t_*) \\ \varphi'(t_*) \end{pmatrix}$

$z(t_*) = a\, y_1(t_*) + y_2(t_*)\, b = \varphi(t_*)$

$z'(t_*) = a\, y_1'(t_*) + b\, y_2'(t_*) = \varphi'(t_*)$

$\xrightarrow{\text{E\&V of 2nd ord}} z(t) = \varphi(t) \quad \forall t$

$y_1 = e^{r_1 t}$ $\quad y_2 = e^{r_2 t}$

$$W = \left| \begin{bmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{bmatrix} \right| = e^{r_1 t} \cdot e^{r_2 t} \cdot r_2 - e^{r_2 t} \cdot r_1 e^{r_1 t}$$

$$\forall t \quad = \underbrace{e^{(r_1+r_2)t}}_{\neq 0} \underbrace{(r_2 - r_1)}_{\neq 0}$$

$\Rightarrow W \neq 0$

$\Rightarrow$ So any sln of our ode is of the form

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$m y'' + \gamma y' + k y = 0$$

$\Rightarrow$ 1) $m r^2 + \gamma \cdot r + k = 0$

2) $r_1, r_2 \Rightarrow y_1 = e^{r_1 t}$

$y_2 = e^{r_2 t}$

3) If φ solves $m \varphi'' + \gamma \cdot \varphi' + k\varphi = 0$

do we have some a, b s.t $\varphi = a e^{r_1 t} + b e^{r_2 t}$?

$$W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

$$= e^{r_1 t} e^{r_2 t} r_2 - e^{r_2 t} r_1 e^{r_1 t}$$

$$= e^{(r_1 + r_2)t} (r_2 - r_1) \neq 0$$

So indeed $\exists$ a, b st $\varphi = a y_1 + b y_2$.

<u>notes</u>

$$y_1 = e^t,\ y_2 = te^t$$

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ (e^t)' & e^t + e^t \cdot t \end{vmatrix}$$

$$= e^t(e^t(1+t)) - e^{t} \cdot 2\,t$$

$$= e^{2t} \neq 0 \text{ for any } t.$$

$$y_1 = e^{2t}, \quad y_2 = -2e^{2t}$$

$$W(y_1, y_2) = \det \begin{bmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & -4e^{2t} \end{bmatrix}$$

$$= e^{4t}(-4) - (-4)e^{4t}$$

$$= e^{4t}(-4+4) = 0 \quad \underline{\forall\, t}$$

$$m u'' + k u = 0$$

$$\Rightarrow \quad m r^2 + k = 0 \quad \Rightarrow r^2 = \frac{-k}{m}$$

$$\Rightarrow r = \pm \sqrt{\frac{-k}{m}}$$

$$= \pm i \sqrt{\frac{k}{m}} = \pm i w$$

$$\Rightarrow \quad y_1 = e^{iwt}$$

$$y_2 = e^{-iwt}$$

$$c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

$$= c_1 \left(\cos(\omega t) + i \sin(\omega t) \right) + c_2 \left(\cos(\omega t) + i \left(-\sin(\omega t) \right) \right)$$

$$= a_1 \cos(\omega t) + a_2 \sin(\omega t)$$

a_1, a_2 complex

$$W(y_1, y_2) = \det \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ (\cos(\omega t))' & (\sin(\omega t))' \end{bmatrix}$$

$$= \omega \cos(\omega t) \cdot \cos(\omega t) - \omega \sin(\omega t)(-\sin(\omega t))$$

$$= \omega(\cos^2(\omega t) + \sin^2(\omega t)) = \omega \cdot 1 \neq 0.$$

So any sln $y = a\cos(\omega t) + b\sin(\omega t)$.

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0$$

$$\Rightarrow r = \pm i$$

$$y = c_1 e^{it} + c_2 e^{-it}$$

Euler's formula

$$= a_1 \cos(t) + a_2 \sin(t)$$

$$y'' - 2y' + 5y = 0$$

$$\Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm i4}{2}$$

$$= 1 \pm 2i$$

$$\Rightarrow y_1 = e^{(1+2i)t}$$

$$y_2 = e^{(1-2i)t} \Rightarrow y = e^{t}(c\cos\omega t + \sin\omega t)$$