Outline

1. Wronskian

2. Complex roots

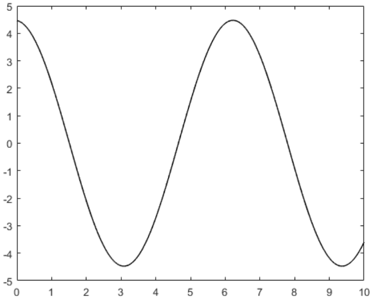

Figure: oscillating system

Generalized solution

Suppose that y_1, y_2 are solutions of

$$y'' + p(t)y' + y = 0.$$

Then the family of solutions

$$y = c_1 y_1 + c_2 y_2$$

for arbitrary c_1, c_2, includes all possible solutions if and only if there is a t_* where the Wronskian of $y_1(t_*), y_2(t_*)$ is not zero.

Spring example

Going back to the spring example, the characteristic equation is

$$mr^2 + \gamma r + k = 0.$$

Assume that it has two distinct real roots r_1, r_2 and so we can easily check that $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W(e^{r_1 t}, e^{r_2 t}, t) = e^{(r_1 + r_2)t}(r_2 - r_1) \neq 0.$$

Therefore, all solutions will be of the form: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

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In class examples

Consider the equation $y'' - 2y' + y = 0$ and functions $y_1 := e^t, y_2 := te^t$.
Consider the ode $y'' - y' - 2y = 0$ and functions $y_1 := e^{2t}, y_2 := -2e^{2t}$.

Complex roots

In some cases the roots are complex $\left(\text{when } b^2 - 4ac < 0\right)$. For example, suppose that there is no damping in the above example $\left(\gamma = 0\right)$, then the equation will be:

$$mu'' + ku = 0.$$

Therefore, the roots will be $r = \pm\sqrt{-k/m} = \pm i\sqrt{k/m} =: \pm i\omega$, where we define $i := \sqrt{-1}$ called the imaginary i. The main result we will need is **Euler's formula**

$$e^{i\omega t} = cos\left(\omega t\right) + isin\left(\omega t\right).$$

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$$e^{i\omega t} = cos\left(\omega t\right) + isin\left(\omega t\right).$$

Here we can easily check that $y_1(t) = cos(\omega t)$ and $y_2(t) = sin(\omega t)$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W(cos(\omega t), sin(\omega t), t) = \omega cos^2(\omega t) + \omega sin^2(\omega t) = \omega \neq 0.$$

Therefore, all solutions will be of the form: $y = c_1 cos(\omega t) + c_2 sin(\omega t)$, where c_i could be complex constants. Physically this periodicity is expected because there is no external force or damping to remove energy from the spring and so it can keep oscillating forever.

Oscillation

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In-class example

Consider the equation $y'' + y = 0, y\left(\pi/3\right) = 2, y'\left(\pi/3\right) = -4$

In-class example

1. The roots are $r^2 + 1 = 0 \Rightarrow r = \pm i$ and so the general solution is
$$y(t) = c_1 e^{it} + c_2 e^{-it} = a_1 cos(t) + a_2 sin(t).$$
2. Using the initial conditions we obtain:
$$2 = a_1 \frac{1}{2} + a_2 \frac{\sqrt{3}}{2} \text{ and } -4 = -a_1 \frac{\sqrt{3}}{2} + a_2 \frac{1}{2}.$$
3. Solving these two equations gives: $a_1 = (1 + 2\sqrt{3}), a_2 = -(2 - \sqrt{3})$ and so the solution for our IVP is:
$$y(t) = (1 + 2\sqrt{3}) cos(t) - (2 - \sqrt{3}) sin(t).$$

In-class example

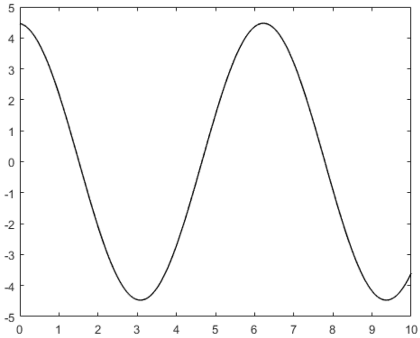

Figure: Spring mass

So as $t \to \infty$ the system simply keeps oscillating steadily. Physically this is because it is damping free $\gamma = 0$.

In-class example

Consider the equation $y'' - 2y' + 5y = 0, y\left(\pi/2\right) = 0, y'\left(\pi/2\right) = 2$

In-class example

1. The roots are $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm i2$ and so the general solution is

$$y(t) = c_1 e^{t(1+2i)} + c_2 e^{t(1-2i)} = e^t\left(a_1 cos(2t) + a_2 sin(2t)\right).$$

2. Using the initial conditions we obtain:

$$0 = e^{\frac{\pi}{2}}\left(a_1 \cdot 0 + a_2 \cdot 1\right) \text{ and } 2 = a_2 e^{\frac{\pi}{2}} + e^{\frac{\pi}{2}}\left(-a_1 2\right).$$

3. Solving these two equations gives: $a_1 = 0, a_2 = -e^{-\pi/2}$ and so the solution for our IVP is:

$$y(t) = -e^{t-\pi/2} sin(2t).$$

In-class example

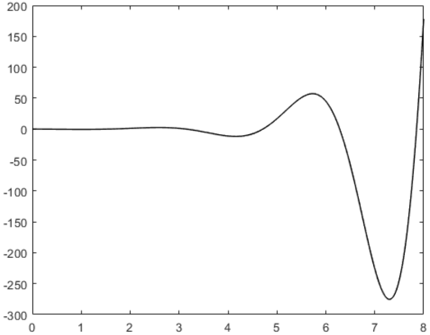

Figure: Spring mass

So as $t \to \infty$ the system simply keeps oscillating with increasing amplitude. Physically this is because the damping is negeative $\gamma = -2 < 0$ and so instead of removing energy, it adds.

The End