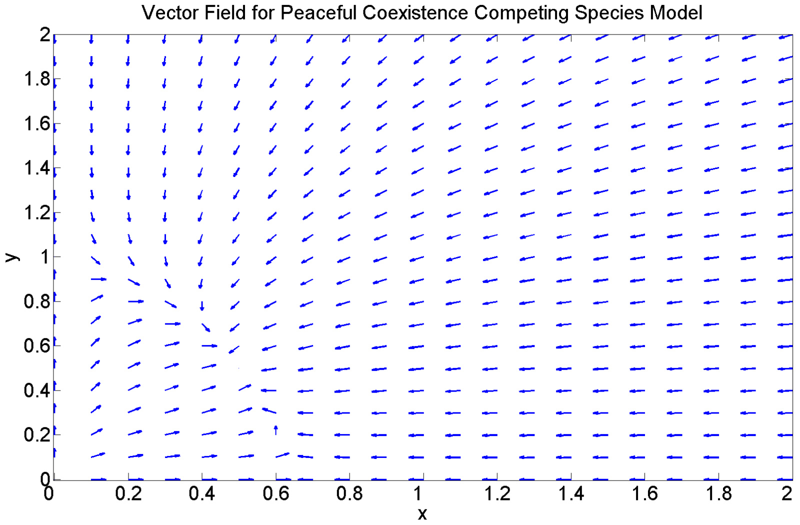
MAT244 Ordinary Differential Equations



Overview

Term work

- Syllabus posted on Quercus.
- There will be 6 biweekly quizzes, two midterms, one assignment and a final exam.
- On portal you can now find all the weekly exercises. Quizzes will be drawn from them and midterms will be variations of them.
- First Quiz will be next week and it will be from the current week's exercises.

Outline

What is a linear ODE? A linear ordinary differential equation of order n is a relation of the form

$$y^{(n)} = F(t, x, y', ..., y^{(n-1)}),$$

where F is some nice function, $t \in \mathbb{R}$, y=y(t) is a function of t and

$$y' := \frac{dy}{dt}, ..., y^{(n)} := \frac{d^n y}{dt^n}.$$

Examples include:

- y'=y(t) whose solution is the exponential $y(t) = e^t + c$,
- Solow-Swan model of economic growth (elasticity α) and constants c_i

  $$y' = c_1 y^\alpha e^{c_2 t} - c_3 y.$$

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Outline

- May: Techniques in solving first order equations $\frac{dy}{dt} = F(y, t)$. Midterm 1 will be on them.
- June: Techniques in solving second order equations $\frac{d^2y}{d^2t} = F(y, t)$ and intro to systems of equations eg. $y(t)' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} y(t)$. Midterm 2 will be on them.
- July: Continuation in studying systems of equations and then studying stability results for locally linear systems (eg. Competing species). The assignment will be on them.

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Compound interest

Let $P(t)$ be the number of dollars in a savings account at time t and suppose that the interest is compounded continuously at an annual interest rate r(t), that varies in time. Then

$$P(t+\Delta t) = P(t) + r(t)\cdot P(t)\cdot \Delta t \Rightarrow \frac{dP}{dt} = r(t)P(t).$$

(1) We separate

$$\frac{\mathrm{d}P}{\mathrm{d}t} = r(t)P \Rightarrow \frac{1}{P}\mathrm{d}P = r(t)\mathrm{d}t$$

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Compound interest

(2) We integrate

$$\int \frac{1}{P}\mathrm{d}P = \int r(t)\mathrm{d}t \Rightarrow$$

$$\ln|P| = \int r(t)\mathrm{d}t + c$$

for some constant $c = P(0)$. Since $P \geq 0$ we obtain

$$P = \$P(0)exp\{\int r(t)\mathrm{d}t\}.$$

(3) For example if $r(t) = t^2$ and $P(0) = \$10^3$ we get

$$P(t) = \$10^3 \cdot exp\{\frac{t^3}{3}\}$$

(4) In other words, we got the formula for the future value of $P(0)$.

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Separable equations form

There are equations of the form

$$y' = f(t, y) = f_1(t) \cdot f_2(y)$$

or in more standard form:

$$M(t)dt + N(y)dy = 0.$$

Here are the formal steps for solving such equations:

1. Separate variables to either side

$$M(t) + N(y)\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow N(y)\mathrm{d}y = -M(t)\mathrm{d}t$$

2. Integrate both sides

$$\int N(y)\mathrm{d}y = -\int M(t)\mathrm{d}t.$$

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Nonlinear

It can even tackle non-linear equations:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t-5}{y^2}, \quad y(0) = 1.$$

(1) We first separate and integrate

$$\int y^2 \mathrm{d}y = \int (t-5)\mathrm{d}t.$$

(2) This gives

$$\frac{y^3}{3} = \frac{t^2}{2} - 5t + c.$$

(3) and using $y(0) = 1$ we obtain

$$y(t) = \left(\frac{3t^2}{2} - 15t + 1\right)^{1/3}.$$

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Restricted solution

Consider IVP (initial value problem)

$$y' = \frac{2x-3}{2y}, \quad y(0) = 2.$$

(1) We separate and integrate

$$\int y \mathrm{d}y = \int (2x-3)\,\mathrm{d}x \Rightarrow \frac{y^2}{2} = x^2 - 3x + c.$$

(2) Using the initial condition we get

$$y = \sqrt{2(x^2-3x+2)} = \sqrt{2(x-1)(x-2)}.$$

(3) and since the square root is only defined for positive numbers, we require

$$x > 2 \text{ or } x < 1.$$

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Asymptotic solution

Consider IVP

$$\frac{dy}{dt} = t(1 + b \cdot y), \quad y(0) = 0$$

for $b \neq 0$

(1) We separate and integrate

$$\frac{1}{b}\ln(1 + b \cdot y) = \frac{t^2}{2} + c.$$

(2) Using the initial condition we obtain

$$y = \frac{1}{b}\left(exp\{b\left(\frac{t^2}{2}\right)\} - 1\right).$$

(3) So the asymptotic behaviour, as $t \to \pm\infty$, depends on b:

$$y \to +\infty \text{ if } b > 0$$

$$y \to \frac{-1}{b} \text{ if } b < 0 .$$

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Implicit solution

Consider IVP

$$\frac{dy}{dt} = \frac{t^2+1}{cos(y)+e^y}, y(0) = \pi.$$

(1) We separate and integrate

$$\int (cos(y)+e^y)dy = \int (t^2+1)dt \Rightarrow sin(y)+e^y = \frac{t^3}{3}+t+c.$$

(2) Using $y(0) = \pi$ we get

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(3) This equation cannot be directly solved in terms of y. One only has an implicit solution that is obtained numerically.

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Cobb-Douglas production model

Let $X(t)$ denote the national product, K(t) the capital stock and L(t) the number of workers in a country at time t. We have the following relations:

$$X = AK^{1-\alpha}L^{\alpha}, K' = sX \text{ and } L = L_0 e^{\lambda t},$$

where A, s, L_0 are constants and $0 < \alpha < 1$ is called elasticity. The first is the Cobb-Douglas production model. The second says that aggregate investment is proportional to output.The third says that the labour forces grows exponentially.Using these three we obtain the equation

$$K' = sX = ce^{\alpha\lambda t}K^{1-\alpha},$$

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Cobb-Douglas production model: Solution

(1) We first separate and integrate

$$\int K^{\alpha-1} dK = \int c e^{\alpha\lambda t} dt.$$

(2) This gives

$$\frac{K^{\alpha}}{\alpha} = c\frac{e^{\alpha\lambda t}}{\alpha\lambda} + C.$$

(3) We find the constant C by plugging in the initial condition, so we get $C := K_0^{\alpha} - \frac{sAL_0^{\alpha}}{\lambda}$ and in turn

$$K = [K_0^{\alpha} + \frac{sAL_0^{\alpha}}{\lambda}(\frac{e^{\alpha\lambda t}}{\alpha\lambda} - 1)]^{1/\alpha}.$$

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Cobb-Douglas production model: Asymptotic behaviour

(4) Next we study the asymptotic behaviour of the ratio $\frac{K}{L}$ (also called the capital-labor ratio) as $t \to +\infty$:

$$\frac{K}{L} = \frac{1}{L_0 e^{\lambda t}}[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(\frac{e^{\alpha\lambda t}}{\alpha\lambda} - 1)]^{1/\alpha}$$

(5) For simplicity we will first compute the asymptotic of

$$(\frac{K}{L})^\alpha = \frac{1}{L_0^\alpha e^{\lambda\alpha t}}[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(\frac{e^{\alpha\lambda t}}{\alpha\lambda} - 1)].$$

(6) We note that the only surving term is the following:

$$\approx \frac{1}{L_0^\alpha e^{\lambda\alpha t}} \frac{sAL_0^\alpha}{\lambda} \frac{e^{\alpha\lambda t}}{\alpha\lambda} = \frac{sA}{\lambda}.$$

(7) This means that in the long term the national product per worker will be approximately constant

$$\frac{X}{L} = A\frac{K}{L}(\frac{L}{K})^\alpha \approx A(\frac{sA}{\lambda})^{1/\alpha}(\frac{sA}{\lambda})^{-1} = A(\frac{sA}{\lambda})^{1/\alpha - 1}.$$

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Radioactive isotopes

A rock contains two radioactive isotopes R_1, R_2 with R_1 decaying into R_2 with rate $5e^{-10t} kg/sec$. So if $y(t)$ is the total mass of R_2, we obtain:

$$\frac{dy}{dt} = \text{rate of creation of } R_2 \text{ - rate of decay of } R_2$$

$$= 5e^{-10t} - ky(t),$$

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Radioactive isotopes

(1) We first multiply by yet unknown $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)ky(t) = \mu(t)5e^{-10t}.$$

(2) We want to find a μ s.t. $\mu'(t) = \mu(t)k$ because then

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$$y(t) = \mu(t)^{-1}\int \mu(s)5e^{-10s}ds.$$

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(3) Then by product rule we have

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(1) Starting from $y' + p(t)y = g(t)$, we multiply both sides by a function $\mu(t)$ that we will determine later:

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Consider the equation

$y' - 2y = t^2 e^{2t}.$

(1) We multiply by $\mu(t)$

$$\mu y' - 2\mu y = \mu t^2 e^{2t}.$$

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example 2

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Price adjument mechanism

When the price of a good is p, the total demand is D(p) = a - bp and the total supply is $S(p) = \alpha + \beta p$, where a, b, α, and β are positive constants. When demand exceeds supply, price rises, and when supply exceeds demand it falls. The speed at which the price changes is proportional to the difference between supply and demand. Specifically

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$$\mu p' + \mu\lambda(b+\beta)p = \lambda(a-\alpha).$$

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$$\mu' = \lambda(b+\beta)\mu \Rightarrow \mu = exp\{\lambda(b+\beta)t\}.$$

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$$\frac{\partial}{\partial t}(\mu p) = exp\{\lambda(b+\beta)t\}\lambda(a-\alpha) \Rightarrow p(t) = cexp\{-\lambda(b+\beta)t\} + \frac{(a-\alpha)}{(b+\beta)}.$$

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The End