Outline

1. Autonomous equations
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   - In class example

2. 2nd order equations
   - Real roots
   - Presenting the method: Sprin

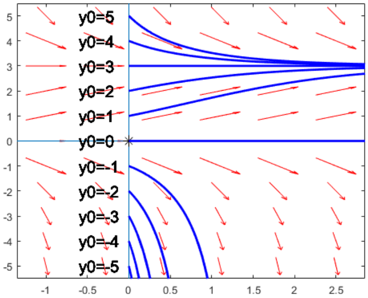

Figure: Direction field for population logistic equation.

Autonomous equations

The equations of the form

$$\frac{dy}{dt} = f(y)$$

are called **autonomous**. Such equations might not have explicit solutions, but it is possible to draw qualitative solutions for them.

Example-Presenting the method: Population logistics

En route to studying the competing species we will need the population logistic equation. Let $y(t)$ be the population of a given species at time t then

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y,$$

where $r > 0$ is called the intrinsic growth rate and K the saturation level. Since y is a physical quantity, the $y < 0$ is ignored.

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Population logistics

(1) First we find the equilibrium solutions:

$$r(1-\frac{y}{K})y = 0 \Rightarrow y = K \text{ or } y = 0.$$

So the equilibrium solutions are $\phi_1(t) \equiv 0, \phi_2(t) \equiv K$.

(2) We have $y' = (1-\frac{y}{K})y > 0$ when $K > y$ and $y > 0$ ($y < 0$ is ignored). Therefore, the solutions started from below K will be growing upwards to $y = K$.

(3) On the other hand, $y' = (1-\frac{y}{K})y < 0$ when $K < y$ and $y > 0$.Therefore, the solutions started from above K will be decaying downwards to $y = K$.

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Matlab simulation for $K = 3$

Method formal steps

1. First we draw the curves $\phi_i(t) = (t, y(t))$ where $f(y) = 0$ (called the **equilibrium solutions** or **critical points**).
2. These will separate the regions into $y' = f(y) > 0$ and $y' = f(y) < 0$.
3. We classify each ϕ_i as **asymptotically stable** if for $y(t)$ starting close to ϕ_i (i.e. $|y_0 - \phi_i(0)| < \varepsilon$)

$$\lim_{t \to \infty} y(t) = \phi_i$$

irrespective of whether $y_0 < \phi_i(0), y_0 > \phi_i(0)$ and **asymptotically unstable** if solutions that start close to the $\phi_i(t)$ curve, move away from it.

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In class example

Consider the autonomous equation:

$$\frac{dy}{dt} = y(2 - y).$$

Identify the equilibrium solutions and classify them as stable or unstable.

In class example

1. First we identify the equilibrium solutions:

$$\frac{dy}{dt} = 0 \Rightarrow \phi_1(t) \equiv 0, \phi_2(t) \equiv 2.$$

2. Second we check the stability close to each solution. For $y > 2$ we have, $\frac{dy}{dt} < 0$ and so the solution will decay towards ϕ_2. For $y \in [0, 2]$ we have $\frac{dy}{dt} > 0$ and so the solution will increase towards ϕ_2. For $y < 0$ we have $\frac{dy}{dt} < 0$ and so the solution will decay to minus infinity.

3. Matlab simulation

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The general form of 2nd order equations is

$$y'' = f(t, y, y').$$

We call them **linear non-homogeneous** if

$$y'' + p(t)y' + q(t)y = g(t)$$

and **linear homogeneous** if $g(t) = 0$

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Spring oscillation

Consider a mass m hanging at the rest on the end of a vertical spring of length L, spring constant k and damping constant γ.

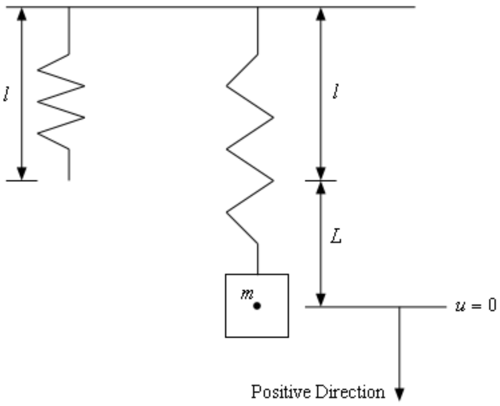

Figure: Spring mass

Spring oscillation

Let $u(t)$ denote the displacement from the equilibrium position. Then by Newton's law one can obtain the equation

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where $F(t)$ is any external force, which for simplicity we will assume to be zero.

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Spring oscillation

(1) Assume the solution is of the form $y(t) = e^{rt}$ then inserting into our ode we obtain:

$$mr^2 + \gamma r + k = 0,$$

which is called the characteristic equation for our ode.

(2) Suppose that $m = 1lb, \gamma = 5lb/ft/s$ and $k = 6lb/ft$then we obtain the roots $r_1 = -2, r_2 = -3$.

(3) Therefore, the general solution will be

$$u(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

(4) Further if $y(0) = 0, y'(0) = 1$ we obtain $c_1 = 1, c_2 = -1$:

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Method formal steps

1. We assume that the solution is of the form $y(t) = e^{rt}$ (this is called making an ansatz). This gives

$$(ar^2 + br + c)e^{rt} = 0 \Rightarrow ar^2 + br + c = 0,$$

which equation is called the **characteristic equation.**

2. So to solve the above ode, it suffices to find the two roots r_1, r_2.
3. Then the general solution is of the form:

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In class example

Consider the IVP

$$4y'' - y = 0, y(-2) = 1, y'(-2) = -1.$$

Solve and determine long term behaviour.

In class example

1. We obtain the characteristic equation $4r^2 - 1 = 0 \Rightarrow r = \pm\frac{1}{2}$ and so the general solution will be
$$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}}.$$
2. Using the initial conditions we obtain:
$$1 = c_1 e^{-1} + c_2 e \text{ and } 1 = \frac{1}{2}(c_1 e^{-1} - c_2 e).$$
3. Solving these two equations gives: $c_1 = -\frac{1}{2}e, c_2 = \frac{3}{2}e^{-1}$ and so the solution for our IVP is:
$$y(t) = -\frac{1}{2}e^{1+\frac{t}{2}} + \frac{3}{2}e^{-\frac{t}{2}-1}.$$
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$$y'' + 5y' + 6y = 0, y(0) = 2, y'(0) = 1.$$

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$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

   Using the initial conditions we obtain:

$$2 = c_1 + c_2 \text{ and } 1 = -2c_1 - 3c_2.$$

2. Solving these two equations gives: $c_1 = 7, c_2 = -5$ and so the solution for our IVP is:

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2. Solving these two equations gives: $c_1 = 7, c_2 = -5$ and so the solution for our IVP is:

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where y_i are solutions for it that satisfy a linear independence condition that is called the **Wronskian**. Then $\{y_1, y_2\}$ will be called the **fundamental solution** because it can generate all others.

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Generalized solution

Suppose that y_1, y_2 are solutions of

$$y'' + p(t)y' + y = 0.$$

Then the family of solutions

$$y = c_1 y_1 + c_2 y_2$$

for arbitrary c_1, c_2, includes all possible solutions if and only if there is a t_* where the Wronskian of $y_1(t_*), y_2(t_*)$ is not zero.

Consider general solution $\phi(t)$ of the above ODE. We will show that there are constants a,b s.t. $\phi(t) = ay_1 + by_2$. Let t_* be the time for which $W(y_1, y_2, t_*) \neq 0$ and let $K_0 = \phi(t_*), K_1 = \phi'(t_*)$. Then

$$\begin{bmatrix} y_1(t_*) & y_2(t_*) \\ y_1'(t_*) & y_2'(t_*) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$$

has a solution $\binom{a}{b}$ because the matrix is invertible. So if $\zeta(t) := ay_1(t) + by_2(t)$ we have $\zeta(t_*) = K_0, \zeta'(t_*) = K_1$. Therefore, the existence and uniqueness theorem for 2nd order odes gives us $\phi(t) = \zeta(t) = ay_1(t) + by_2(t)$ for all t.

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Going back to the spring example, the characteristic equation is

$$mr^2 + \gamma r + k = 0.$$

Assume that it has two distinct real roots r_1, r_2 and so we can easily check that $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W(e^{r_1 t}, e^{r_2 t}, t) = e^{(r_1+r_2)t}(r_2 - r_1) \neq 0.$$

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Method formal steps

Consider arbitrary initial condition be $y(t_0) = y_0$.

1. Assuming that $y = c_1 y_1 + c_2 y_2$ we obtain the following system

$$W_{matrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

2. Then we compute the determinant of this system

$$W = det(W_{matrix}) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0).$$

3. If it is not zero, then the general solution will be of the form $y = c_1 y_1 + c_2 y_2$.
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