Week 1

Exercises marked (*) are harder and will not appear on quizzes. But variations of them may appear on the midterm. Exercises marked (**) are above the required level and will not appear on midterms or final exam. Most of the exercises are in the main textbook.

- **Linear integrating factor (2.1)**
  - Find the general solution and use it to determine how solutions behave as $t \to +\infty$
    1. $y' + 3y = t + e^{-2t}$
    2. $y' + y = te^{-t} + 1,$
    3. (*) $y' + \frac{1}{t}y = 3cos(2t), t > 0$
  - Find the general solution and use it to determine the asymptotic behavior for different values of a
    1. $y' - \frac{1}{2}y = 2cos(t), y(0) = a$ as $t \to +\infty,$
    2. (*) $ty' + (t+1)y = 2te^{-t}, y(1) = a, 0 < t$ as $t \to 0.$
    3. (*)A rock contains two radioactive isotopes R_1, R_2 with R_1 decaying into R_2 with rate $5e^{-20t}kg/sec$. So if $y(t)$ is the total mass of R_2, we obtain:

       $$\begin{aligned}\frac{dy}{dt} &= \text{rate of creation of } R_2 \text{ - rate of decay of } R_2\\ &= 5e^{-20t} - ky(t),\end{aligned}$$

       where $k > 0$ is the decay constant for R_2. Also assume that $y(0) = 40kg$.

    4. (*) For demand $D(P) = 2 - 2P(t)$ and supply $S(P) = 1 + P(t)$ let

       $$\frac{dP}{dt} = (D(P) - S(P)).$$

       Find $P(t)$ and its limit for $t \to +\infty$.

- **Separable (2.2):**
  - a)Find the solution of the given initial value problem and b)determine the interval in which the solution is defined:
    1. $y' = (1-2x)y^2, y(0) = -1/6,$
    2. $y' = (1-2x)/y, y(1) = -2,$
    3. (*) $\frac{dr}{d\theta} = r^2/\theta, r(1) = 2,$
    4. (*) $y' = 2x/(1+2y), y(2) = 0,$
    5. (*) $y' = y^2 + 1, y(0) = 0$ (only the interval containing 0).
  - a)Find the solution of the given initial value problem and b)determine the behaviour as $t \to +\infty$:
    1. $y' = cos^2(y), y(0) = 2,$
    2. (*) $y' = t\frac{y(4-y)}{3}, y(0) = 1.$
  - (*) Homogeneous equations problem 2.2-(25): Consider the equation

    $$\frac{dy}{dx} = \frac{y-4x}{x-y}.$$

1. Rewrite it as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}$$

   and set $\nu(x) := y/x$ to get

$$\nu + x\frac{d\nu}{dx} = \frac{\nu - 4}{1 - \nu} \Rightarrow x\frac{d\nu}{dx} = \frac{\nu^2 - 4}{1 - \nu}.$$

2. This is a separable equation. Solve to find an implicit solution for ν and then plug in y to get:

$$|y + 2x|^3 |y - 2x| = c.$$

– (*)In the capital-labor ratio example assume instead that the labour grows polynomially in time ie. $L = b(t + a)^p$ for positive constants b, a, p. The differential equation we get is:

$$K' = sX = c(t + a)^{p\alpha} K^{1-\alpha},$$

where where $c := Asb^\alpha$. Solve the equation and obtain the limit of the capital-labor ratio $\frac{K}{L}$ as $t \to +\infty$. What does that mean for the output per worker ratio X/L?