Outline

1. Real roots
2. Complex
3. Repeated eigenvalues

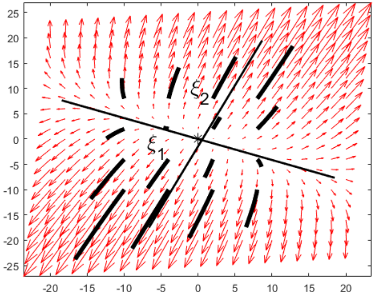


Figure:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The corresponding eigenvectors are $\boldsymbol{\xi}_1 = \binom{1}{1}$ and $\boldsymbol{\xi}_2 = \binom{-1}{2}$. So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-5t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solve and sketch:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

For matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ here are the general formulas for its eigenvalues and eigenvectors:

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4det(A)}$$

and if $c \neq 0$ (cf. for other cases see notes) then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}.$$

The corresponding eigenvectors are $\boldsymbol{\xi}_1 = \binom{1}{3}$ and $\boldsymbol{\xi}_2 = \binom{1}{1}$. So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Solve and sketch:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \mathbf{x}.$$

For matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ here are the general formulas for its eigenvalues and eigenvectors:

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4det(A)}$$

and if $c \neq 0$ (cf. for other cases see notes) then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}.$$

The corresponding eigenvectors are $\boldsymbol{\xi}_1 = \binom{1}{2}$ and $\boldsymbol{\xi}_2 = \binom{3}{4}$. So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Consider the system

$$\mathbf{x}' = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

$$\begin{aligned} x_1(t) &= \xi_1 e^{(\lambda+i\mu)t} \\ &= (a+ib)e^{\lambda t}(cos(\mu t) + isin(\mu t)) \\ &= e^{\lambda t}(acos(\mu t) - bsin(\mu t)) + ie^{\lambda t}(bcos(\mu t) + asin(\mu t)). \end{aligned}$$

similarly for x_2.

$$x = c_1 e^{\lambda t}(acos(\mu t) - bsin(\mu t)) + c_2 e^{\lambda t}(bcos(\mu t) + asin(\mu t))$$

We first compute its eigenvalues: $\lambda^2 - a\lambda + 1 = 0 \Rightarrow \lambda = \frac{a \pm \sqrt{a^2-4}}{2}$. So to explore the complex case we assume that $a^2 < 4$.
Let ξ_1, ξ_2 be the corresponding eigenvectors: $\xi_1 = \binom{\lambda_1}{-1}, \xi_2 = \binom{\lambda_2}{-1}$

Solve and sketch:

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We first compute its eigenvalues: $\lambda_1 = -1 + 2i, \lambda_1 = -1 - 2i$ Let ξ_1, ξ_2 be the corresponding eigenvectors: $\xi_1 = \binom{\lambda_1}{-1} = \binom{2i}{1}, \xi_2 = \binom{\lambda_2}{-1} = \binom{-2i}{1}$

Solve and sketch for the

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

different values of a that give different behaviour:

$$\mathbf{x}' = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} \mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We first compute its eigenvalues: $\lambda = a - i, a + i$.
Let ξ_1, ξ_2 be the corresponding eigenvectors:
$\xi_1 = \binom{\lambda_1}{-1} = \binom{i}{1}, \xi_2 = \binom{\lambda_2}{-1} = \binom{-i}{1}$.

Repeated eigenvalues

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

Repeated eigenvalues

1. We first find the eigenvalues:

$$\lambda = \frac{\mathsf{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathsf{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})} = 2$$

   and so we have a repeated eigenvalue.

2. Second, we find the corresponding eigenvector:

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

1. Assuming the solution is of the form $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ and plugging into our ODE we obtain:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi} \Longrightarrow \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solving this system gives us:

$$\eta_1 + \eta_2 = -1 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} k \\ -k-1 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

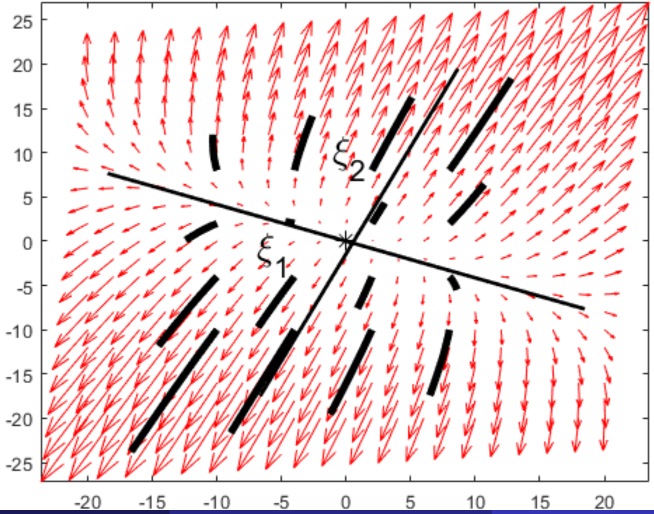
where k is any real number. We can rewrite $\boldsymbol{\eta}$ as:

$$\boldsymbol{\eta} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Therefore, the general solution is:

$$
\begin{aligned}
\mathbf{x} &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \\
&= c_1 e^{2t}\boldsymbol{\xi} + c_2(\boldsymbol{\xi} e^{2t}\cdot t + \boldsymbol{\eta} e^{2t}) \\
&= c_1 e^{2t}\boldsymbol{\xi} + c_2\left[\boldsymbol{\xi} e^{2t}\cdot t + \left\{k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} e^{2t}\right] \\
&= e^{2t}\left[(c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}\right].
\end{aligned}
$$

The vector $\boldsymbol{\xi}_1 = \binom{1}{-1}$ dominates the long term behaviour due to the extra term $\binom{1}{-1}t$ (provided we do not choose $c_2 = 0$). So we see that, essentially, all solutions are diverging away from the linear span of $\binom{1}{-1}$.



1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.

2. For the second term we make the ansatz

$$\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}.$$

1. Plugging this into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ we obtain the stystem:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

2. By determining $\boldsymbol{\eta}$ we obtain:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\boldsymbol{\xi}e^{\lambda t} + c_2(\boldsymbol{\xi}e^{\lambda t} \cdot t + \boldsymbol{\eta}e^{\lambda t}).$$

The End