Contents

1 Inverse of a matrix

Given a 2×2 matrix $\mathbf{A}$ we will sometimes have to compute its inverse. To do this efficiently we provide an algorithm which gives the formula for the inverse of $\mathbf{A}$. We recall that if $\mathbf{v} = \binom{v_1}{v_2}$ then $\mathbf{v}^\perp = \binom{v_2}{-v_1}$ is its perpendicular vector.

1.1 Formal steps

1. We are given a 2×2 matrix $\mathbf{A}$ which takes the form

$$\mathbf{A} = [\boldsymbol{\xi} \quad \boldsymbol{\eta}]$$

   where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are vectors in $\mathbb{R}^2$. We compute $\xi^\perp$ and $\eta^\perp$.

2. Next we compute $\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp$.

3. Finally, we form the inverse matrix

$$\mathbf{A}^{-1} = \frac{1}{\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp} \begin{bmatrix} (\boldsymbol{\eta}^\perp)^T \\ -(\boldsymbol{\xi}^\perp)^T \end{bmatrix}.$$

1.2 Example of the method

1. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 2 & -7 \end{pmatrix}$$

   for which $\boldsymbol{\xi} = \binom{3}{2}$ and $\boldsymbol{\eta} = \binom{5}{-7}$. We compute that $\boldsymbol{\xi}^\perp = \binom{2}{-3}$ and $\boldsymbol{\eta}^\perp = \binom{-7}{-5}$.

2. $\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp = -31$.

3. Finally we obtain that

$$\mathbf{A}^{-1} = \frac{-1}{31} \begin{pmatrix} -7 & -5 \\ -2 & 3 \end{pmatrix}.$$

We can check that this is correct by multipling $\mathbf{A}^{-1}$ and $\mathbf{A}$ to obtain $\mathbf{I}_2$.

1.3 Reasoning behind the method

This algorithm comes from the following reasoning. If we have a matrix $\mathbf{A}$ and we want to find its inverse $\mathbf{A}^{-1}$ then we require that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{pmatrix} (\boldsymbol{\xi} \quad \boldsymbol{\eta}) = \begin{pmatrix} \boldsymbol{\alpha}\cdot\boldsymbol{\xi} & \boldsymbol{\alpha}\cdot\boldsymbol{\eta} \\ \boldsymbol{\beta}\cdot\boldsymbol{\xi} & \boldsymbol{\beta}\cdot\boldsymbol{\eta} \end{pmatrix}.$$

Comparing both sides we see that we must have $\boldsymbol{\alpha}\cdot\boldsymbol{\eta} = 0$ as well as $\boldsymbol{\beta}\cdot\boldsymbol{\xi} = 0$. The easiest way to achieve this is to choose $\boldsymbol{\alpha} = \boldsymbol{\eta}^T$ and $\boldsymbol{\beta} = \boldsymbol{\xi}^T$. This, however, ignores that we need $\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = 1$ and $\boldsymbol{\beta}\cdot\boldsymbol{\eta} = 1$. Fortunately, with the choices $\boldsymbol{\alpha} = \boldsymbol{\eta}^\perp$ and $\boldsymbol{\beta} = \boldsymbol{\xi}^\perp$ we have

$$\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = \boldsymbol{\eta}^\perp\cdot\boldsymbol{\xi} = \xi_1\eta_2 - \xi_2\eta_1$$

as well as

$$\boldsymbol{\beta}\cdot\boldsymbol{\eta} = \boldsymbol{\xi}^\perp\cdot\boldsymbol{\eta} = \xi_2\eta_1 - \xi_1\eta_2$$

which differ by a factor of -1. If we now choose $\boldsymbol{\beta} = -\boldsymbol{\xi}^\perp$ (essentially multiplying both sides of the last computation by -1) then we have $\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = \boldsymbol{\beta}\cdot\boldsymbol{\eta}$. Thus, the matrix $\mathbf{B}$ defined by

$$\mathbf{B} = \begin{pmatrix} (\boldsymbol{\eta}^\perp)^T \\ -(\boldsymbol{\xi})^T \end{pmatrix}$$

satisfies

$$\mathbf{BA} = \begin{pmatrix} \boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp & 0 \\ 0 & \boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp \end{pmatrix} = \boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Dividing both sides by $\boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp$ we get that

$$\left(\frac{1}{\boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp}\mathbf{B}\right)\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives us the inverse matrix.

2 Exponential of matrix

2.1 Identities and formulas

Proposition 2.1. *We let $e^{t\mathbf{A}}$ denote the unique*[1] *matrix which solves*

$$\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{I}_n.$$

I claim that, in this case, $e^{t\mathbf{A}}$ satisfies:

1. $e^{\mathbf{0}} = I_n$
2. *The unique solution to the problem* $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{X}_0$ *is* $e^{t\mathbf{A}}\mathbf{X}_0$.
3. $\mathbf{A}e^{t\mathbf{A}} = e^{t\mathbf{A}}\mathbf{A}$
4. *The exponential of matrix is invertible and we have that* $(e^{t\mathbf{A}})^{-1} = e^{-t\mathbf{A}}$
5. *If* $\mathbf{AB} = \mathbf{BA}$ *then* $e^{t(\mathbf{A}+\mathbf{B})} = e^{t\mathbf{A}}e^{t\mathbf{B}}$
6. $(e^{t\mathbf{A}})^{\mathrm{T}} = e^{t\mathbf{A}^{\mathrm{T}}}$

[1] This is due to the uniqueness theorem for linear matrix ODEs.

7. *If T is an invertible matrix then* $e^{t\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T}$.
8. *If* $\mathbf{A}^2 = \mathbf{A}$ *then* $e^{t\mathbf{A}} = \mathbf{I}_n + (e^t - 1)\mathbf{A}$.
9. *Formulas of* $e^{t\mathbf{A}}$ *for* $n = 2$*:*
   (a) *If the eigenvalues are distinct then*
   $$exp\{t\mathbf{A}\} := e^{\lambda_1 t}\frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}_2) - e^{\lambda_2 t}\frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_1\mathbf{I}_2).$$
   (b) *If* $\lambda = \lambda_1 = \lambda_2$ *then*
   $$exp\{t\mathbf{A}\} := e^{\lambda t}\mathbf{I}_2 + e^{\lambda t}t(\mathbf{A} - \lambda\mathbf{I}_2).$$
   (c) *If* $\lambda_1 = a + ib, \lambda_2 = a - ib$ *then*
   $$exp\{t\mathbf{A}\} := \frac{e^{at}}{b}\{b\cos(bt)\mathbf{I}_2 + \sin(bt)(\mathbf{A} - a\mathbf{I}_2)\}.$$

Proof.

1. By definition we have
   $$e^{\mathbf{0}} = e^{0\mathbf{A}} = \mathbf{X}(0) = \mathbf{I}_n.$$
2. Let $\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{X}_0$. Then,
   $$\mathbf{Y}'(t) = \left(e^{t\mathbf{A}}\mathbf{X}_0\right)' = \mathbf{A}e^{t\mathbf{A}}\mathbf{X}_0 = \mathbf{A}\mathbf{Y}(t)$$
   and
   $$\mathbf{Y}(0) = e^{\mathbf{0}}\mathbf{X}_0 = \mathbf{I}_n\mathbf{X}_0 = \mathbf{X}_0.$$
   By uniqueness of solutions of linear matrix IVPs we have that $\mathbf{Y}$ is the only solution to this matrix IVP.
3. Let $\mathbf{Y}(t) = \mathbf{A}e^{t\mathbf{A}}$. Observe that
   $$\mathbf{Y}'(t) = \mathbf{A}\left(e^{t\mathbf{A}}\right)' = \mathbf{A}\left(\mathbf{A}e^{t\mathbf{A}}\right) = \mathbf{A}\mathbf{Y}(t)$$
   and
   $$\mathbf{Y}(0) = \mathbf{A}e^{\mathbf{0}} = \mathbf{A}\mathbf{I}_n = \mathbf{A}.$$
   By 2 we must have
   $$\mathbf{A}e^{t\mathbf{A}} = \mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{A}$$
   for all t.
4. Let $\mathcal{F}(t) = e^{t\mathbf{A}}e^{-t\mathbf{A}}$ for $t \in \mathbb{R}$. Observe that
   $$\mathcal{F}'(t) = (e^{t\mathbf{A}})'e^{-t\mathbf{A}} + e^{t\mathbf{A}}(e^{-t\mathbf{A}})' = \mathbf{A}e^{t\mathbf{A}}e^{-t\mathbf{A}} - e^{t\mathbf{A}}\mathbf{A}e^{-t\mathbf{A}} = \mathbf{A}e^{t\mathbf{A}}e^{-t\mathbf{A}} - \mathbf{A}e^{t\mathbf{A}}e^{-t\mathbf{A}} = \mathbf{0}.$$
   where I have used 3. Thus, $\mathcal{F}$ is constant. In particular, we have, by evaluating at $t = 0$
   $$\mathcal{F}(0) = e^{\mathbf{0}}e^{\mathbf{0}} = \mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n.$$
   By reversing the roles of $e^{t\mathbf{A}}$ and $e^{-t\mathbf{A}}$ we obtain the desired conclusion.
5. For this proof we let $\mathbf{X}(t) = e^{t\mathbf{A}}$, $\mathbf{Y}(t) = e^{t\mathbf{B}}$, and $\mathbf{Z}(t) = e^{t(\mathbf{A}+\mathbf{B})}$. Define $\mathcal{G}(t) = \mathbf{Z}(t) - \mathbf{X}(t)\mathbf{Y}(t)$. Differentiating we obtain
   $$\mathcal{G}'(t) = \mathbf{Z}'(t) - \mathbf{X}'(t)\mathbf{Y}(t) - \mathbf{X}(t)\mathbf{Y}'(t)$$

$$= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - \mathbf{A}\mathbf{X}(t)\mathbf{Y}(t) - \mathbf{X}(t)\mathbf{B}\mathbf{Y}(t).$$

Note that if we can show that $\mathbf{X}(t)\mathbf{B} = \mathbf{B}\mathbf{X}(t)$ then we get

$$\begin{aligned}
&= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - \mathbf{A}\mathbf{X}(t)\mathbf{Y}(t) - \mathbf{X}(t)\mathbf{B}\mathbf{Y}(t)\\
&= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - \mathbf{A}\mathbf{X}(t)\mathbf{Y}(t) - \mathbf{B}\mathbf{X}(t)\mathbf{Y}(t)\\
&= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - (\mathbf{A}+\mathbf{B})\mathbf{X}(t)\mathbf{Y}(t)\\
&= (\mathbf{A}+\mathbf{B})(\mathbf{Z}(t) - \mathbf{X}(t)\mathbf{Y}(t))\\
&= (\mathbf{A}+\mathbf{B})\mathcal{G}(t).
\end{aligned}$$

We also have

$$\mathcal{G}(0) = \mathbf{Z}(0) - \mathbf{X}(0)\mathbf{Y}(0) = \mathbf{I}_n - \mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n - \mathbf{I}_n = \mathbf{0}.$$

By 2 we have

$$\mathcal{G}(t) = e^{t(\mathbf{A}+\mathbf{B})}\mathbf{0} = \mathbf{0}.$$

Thus,

$$e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}e^{t\mathbf{B}} = \mathbf{Z}(t) - \mathbf{X}(t)\mathbf{Y}(t) = \mathbf{0}$$

as we wanted. Now we show that $\mathbf{X}(t)\mathbf{B} = \mathbf{B}\mathbf{X}(t)$. Observe that

$$(\mathbf{B}\mathbf{X}(t))' = \mathbf{B}\mathbf{X}'(t) = \mathbf{B}\mathbf{A}\mathbf{X}(t) = \mathbf{A}\mathbf{B}\mathbf{X}(t) = \mathbf{A}(\mathbf{B}\mathbf{X}(t))$$

and

$$\mathbf{B}\mathbf{X}(0) = \mathbf{B}\mathbf{I}_n = \mathbf{B}.$$

By 2 we must have

$$\mathbf{B}\mathbf{X}(t) = \mathbf{X}(t)\mathbf{B}$$

for all $t \in \mathbb{R}$.

6. Observe that by 3 we have

$$(e^{t\mathbf{A}})' = \mathbf{A}e^{t\mathbf{A}} = e^{t\mathbf{A}}\mathbf{A}.$$

By transposing the previous equation we have

$$((e^{t\mathbf{A}})^T)' = \mathbf{A}^T(e^{t\mathbf{A}})^T.$$

Observe also that $(e^{0\mathbf{A}})^T = \mathbf{I}_n^T = \mathbf{I}_n$. Hence, by definition we have

$$e^{t\mathbf{A}^T} = (e^{t\mathbf{A}})^T.$$

7. Observe that

$$(\mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T})' = \mathbf{T}^{-1}(e^{t\mathbf{A}})'\mathbf{T} = \mathbf{T}^{-1}\mathbf{A}e^{t\mathbf{A}}\mathbf{T} = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})(\mathbf{T}^{-1}e^{t\mathbf{A}}T)$$

and $\mathbf{T}^{-1}e^{o\mathbf{A}}\mathbf{T} = \mathbf{T}^{-1}\mathbf{I}_n\mathbf{T} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}_n$. Thus, by definition we have

$$e^{t\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T}.$$

8. We have that

$$(e^{t\mathbf{A}})' = \mathbf{A}e^{t\mathbf{A}}$$

and so

$$(\mathbf{A}e^{t\mathbf{A}})' = \mathbf{A}(e^{t\mathbf{A}})' = \mathbf{A}^2e^{t\mathbf{A}} = \mathbf{A}e^{t\mathbf{A}}.$$

Also $\mathbf{A}e^{0\mathbf{A}} = \mathbf{A}$ so by 2 we have

$$\mathbf{A}e^{t\mathbf{A}} = e^t\mathbf{A}.$$

Observe that

$$(\mathbf{I}_n - \mathbf{A})\mathbf{A} = \mathbf{A} - \mathbf{A}^2 = \mathbf{0}.$$

This means that

$$((\mathbf{I}_n - \mathbf{A})e^{t\mathbf{A}})' = (\mathbf{I}_n - \mathbf{A})(e^{t\mathbf{A}})' = (\mathbf{I}_n - \mathbf{A})\mathbf{A}e^{t\mathbf{A}} = \mathbf{0}.$$

We conclude that $(I_n - \mathbf{A})e^{t\mathbf{A}}$ is constant and equal to $\mathbf{I}_n - \mathbf{A}$ at $t = 0$. Thus,

$$\mathbf{e^{tA}} = \mathbf{A}e^{t\mathbf{A}} + (\mathbf{I}_n - \mathbf{A})e^{t\mathbf{A}} = e^t\mathbf{A} + (\mathbf{I}_n - \mathbf{A}) = \mathbf{I}_n + (e^t - 1)\mathbf{A}.$$

9. (a)
   (b)
   (c)

□

We observe that identity 7 allows for easier computation of the matrix exponential when the matrix $\mathbf{A}$ is diagonalizable. To see this, observe that in this case we can find an invertible matrix $\mathbf{T}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. Identity 7 gives

$$e^{t\mathbf{D}} = e^{t\cdot\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T}$$

which means

$$e^{t\mathbf{A}} = \mathbf{T}e^{t\mathbf{D}}\mathbf{T}^{-1}.$$

Note that solving the vector ODE $\mathbf{x}' = \mathbf{D}\mathbf{x}$ is much simpler since $\mathbf{D}$ is a diagonal matrix.

2.2 Liouville's Formula

Proposition 2.2. *Suppose $\mathbf{A} : \mathbb{R} \to M_{n\times n}(\mathbb{R})$ is a matrix-valued function. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A}(t))) = \sum_{i=1}^{n} \det(\mathbf{A}^i(t))$$

where

$$\mathbf{A}^i(t) = \begin{pmatrix} A_{1,1}(t) & A_{1,2}(t) & \cdots & A_{1,n}(t) \\ A_{2,1}(t) & A_{2,2}(t) & \cdots & A_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A'_{i,1}(t) & A'_{i,2}(t) & \cdots & A'_{i,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}(t) & A_{n,2}(t) & \cdots & A_{n,n}(t) \end{pmatrix}$$

for $i = 1, \ldots, n$.

Proof. We proceed by induction. For $n = 1$, since $\mathbf{A}(t)$ will be a 1×1 matrix then $\det(\mathbf{A}(t)) = A_{1,1}(t)$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A}(t))) = A'_{1,1}(t) = \sum_{i=1}^{1} \det(A^i(t))$$

where the last equality follows from the fact that there is only one row to take the derivative of and

$$\det(A^1(t)) = \det(A'_{1,1}(t)) = A'_{1,1}(t).$$

Now we presume this formula holds for $n-1$, $n \geq 2$ and we show it holds for n. Observe that

$$\det(\mathbf{A}(t)) = \sum_{i=1}^{n} (-1)^{1+i} A_{1,i}(t) \det(\tilde{\mathbf{A}}_{1,i}(t))$$

where $\tilde{\mathbf{A}}_{1,i}(t)$ denotes the matrix, of size $(n-1)\times(n-1)$ obtained from $\mathbf{A}(t)$ which has row 1 and column i removed. Differentiating and using the induction hypothesis we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A}(t))) &= \sum_{i=1}^{n}(-1)^{1+i}A'_{1,i}(t)\det(\tilde{\mathbf{A}_{1,i}}(t)) + \sum_{i=1}^{n}(-1)^{1+i}A_{1,i}(t)\frac{\mathrm{d}}{\mathrm{d}t}(\det(\tilde{\mathbf{A}}_{1,i}(t)))\\
&= \sum_{i=1}^{n}(-1)^{1+i}A'_{1,i}(t)\det(\tilde{\mathbf{A}_{1,i}}(t)) + \sum_{i=1}^{n}(-1)^{1+i}A_{1,i}(t)\sum_{j=1}^{n-1}\det(\tilde{\mathbf{A}}^{j}_{1,i}(t))\\
&= \det(A^1(t)) + \sum_{i=1}^{n}(-1)^{1+i}A_{1,i}(t)\sum_{j=1}^{n-1}\det(\tilde{\mathbf{A}}^{j}_{1,i}(t))\\
&= \det(A^1(t)) + \sum_{j=1}^{n-1}\sum_{i=1}^{n}(-1)^{1+i}A_{1,i}(t)\det(\tilde{\mathbf{A}}^{j}_{1,i}(t))\\
&= \det(A^1(t)) + \sum_{j=1}^{n-1}\det(\mathbf{A}^{j+1}(t))\\
&= \det(A^1(t)) + \sum_{j=2}^{n}\det(\mathbf{A}^{j}(t))\\
&= \sum_{j=1}^{n}\det(\mathbf{A}^{j}(t))
\end{aligned}$$

□

Proposition 2.3. *Let $\mathbf{X}(t)$ denote the matrix of fundamental solutions to the problem*

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

where $\mathbf{x}(t)$ is an n-vector. Then

$$\det(\mathbf{X}(t)) = \det(\mathbf{X}(0))e^{t\cdot\mathrm{tr}(\mathbf{A})}.$$

As a consequence we have

$$\det(e^{t\mathbf{A}}) = e^{t\cdot\mathrm{tr}(\mathbf{A})}.$$

Proof. We first notice that by proposition 2.2 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{X}(t)) = \sum_{i=1}^{n}\det(\mathbf{X}^{i}(t)).$$

Observe that

$$\begin{aligned}
\mathbf{X}^{i}(t) &= \begin{pmatrix} X_{1,1}(t) & X_{1,2}(t) & \cdots & X_{1,n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ X'_{i,1}(t) & X'_{i,2}(t) & \cdots & X'_{i,n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ X_{n,1}(t) & X_{n,2}(t) & \cdots & X_{n,n}(t)\end{pmatrix}\\
&= \begin{pmatrix} X_{1,1}(t) & X_{1,2}(t) & \cdots & X_{1,n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{j=1}^{n}A_{i,j}X_{j,1}(t) & \sum_{j=1}^{n}A_{i,j}X_{j,2}(t) & \cdots & \sum_{j=1}^{n}A_{i,j}X_{j,n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ X_{n,1}(t) & X_{n,2}(t) & \cdots & X_{n,n}(t)\end{pmatrix}.
\end{aligned}$$

Recall that subtracting multiples of one row from another does not change the value of the determinant. So subtracting $A_{i,1}$ times row 1 of $\mathbf{X}(t)$ from row i and then subtracting $A_{2,i}$ times row 2 of $\mathbf{X}(t)$ from row i and so on does not change the value of the determinant but leads us to

$$\det(\mathbf{X}^i(t)) = \det \begin{pmatrix} X_{1,1}(t) & X_{1,2}(t) & \cdots & X_{1,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{i,i}X_{i,1}(t) & A_{i,i}X_{i,2}(t) & \cdots & A_{i,i}X_{i,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ X_{n,1}(t) & X_{n,2}(t) & \cdots & X_{n,n}(t) \end{pmatrix} = A_{i,i} \det(\mathbf{X}(t)).$$

The above conclusions are true for each $i = 1, \ldots, n$. We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{X}(t))) = \sum_{i=1}^{n} A_{i,i} \det(\mathbf{X}(t)) = \mathrm{tr}(\mathbf{A}) \det(\mathbf{X}(\mathrm{t})).$$

We conclude that

$$\det(\mathbf{X}(t)) = Ce^{t \cdot \mathrm{tr}(\mathbf{A})}.$$

Evaluating at $t = 0$ gives

$$C = \det(\mathbf{X}(0)).$$

To obtain the second identity notice that the first identity can be written as

$$\det(\mathbf{X}(t)(\mathbf{X}(0))^{-1}) = e^{t \cdot \mathrm{tr}(\mathbf{A})}$$

and notice that $\mathbf{X}(t)(\mathbf{X}(0))^{-1}$ solves

$$\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{I}_n.$$

□

2.3 Remarks

The construction of the matrix exponential given in section 2 is not the standard development. Generally, one defines this matrix through the use of infinite series of matrices. It is then a theorem that the matrix exponential solves the matrix ODE $\mathbf{X}' = \mathbf{A}\mathbf{X}$ with $\mathbf{X}(0) = \mathbf{I}_n$. The construction given in section 2 is probably not new though the authors of these notes have no citations for this technique. It is, perhaps, worth noting that many of the standard identities involving the exponential matrix can be obtained from 2.1 and 2.2 by simply setting $t = 1$.

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