Outline

1. Repeated eigenvalues
2. nonhomogeneous linear first ord

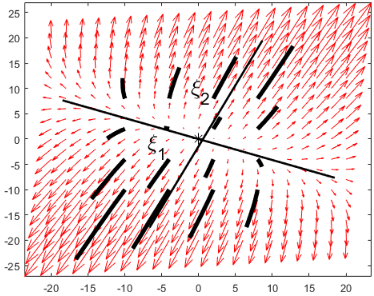


Figure: repeated eigenvalue

Repeated eigenvalues

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

1 We first find the eigenvalues:

$$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})} = 2$$

and so we have a repeated eigenvalue.

2 Second, we find the corresponding eigenvector:

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

1. Assuming that $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t$ fails because it implies $\xi = 0$.

3 Assuming the solution is of the form $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ and plugging into our ODE we obtain:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi} \Longrightarrow \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solving this system gives us:

$$\eta_1 + \eta_2 = -1 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} k \\ -k-1 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

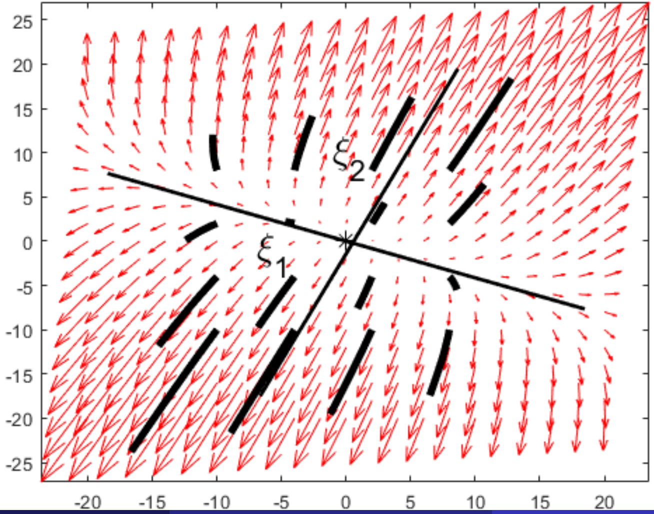
where k is any real number. We can rewrite $\boldsymbol{\eta}$ as:

$$\boldsymbol{\eta} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Therefore, the general solution is:

$$
\begin{aligned}
\mathbf{x} &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \\
&= c_1 e^{2t}\boldsymbol{\xi} + c_2(\boldsymbol{\xi} e^{2t} \cdot t + \boldsymbol{\eta} e^{2t}) \\
&= c_1 e^{2t}\boldsymbol{\xi} + c_2\left[\boldsymbol{\xi} e^{2t} \cdot t + \left\{k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} e^{2t}\right] \\
&= e^{2t}\left[(c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}\right].
\end{aligned}
$$

The vector $\boldsymbol{\xi}_1 = \binom{1}{-1}$ dominates the long term behaviour due to the extra term $\binom{1}{-1}t$ (provided we do not choose $c_2 = 0$). So we see that, essentially, all solutions are diverging away from the linear span of $\binom{1}{-1}$.



1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.

1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.
2. For the second term we make the ansatz

$$\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}.$$

1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.
2. For the second term we make the ansatz

$$\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}.$$

3. Plugging this into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ we obtain the stystem:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.
2. For the second term we make the ansatz

$$\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}.$$

3. Plugging this into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ we obtain the stystem:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

4. By determining $\boldsymbol{\eta}$ we obtain:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}).$$

In class example

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Find η from

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Then

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}).$$

In class example

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Find η from

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}).$$

Consider nonhomogeneous linear first order systems:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a vector of continuous functions and $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with eigenvalues $\{\lambda_i\}_{i=1,\ldots,n}$. The latter assumption means that if $\mathbf{T}$ has the eigenvectors of $\mathbf{A}$ as columns, then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$ is a diagonal matrix.

Using diagonalization

Plugging in $\mathbf{x} = \mathbf{T}\mathbf{y}$ for some yet unknown $\mathbf{y}$ we obtain

$$\begin{aligned} \mathbf{T}\mathbf{y}' &= \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t) \\ \implies \mathbf{y}' &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t). \end{aligned}$$

As a result, we decoupled the system. From this decoupled system we obtain the first order equations:

$$y_i' = \lambda_i y_i(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_i \quad \text{for } i = 1, ..., n.$$

For $h_i(t) := (\mathbf{T}^{-1}\mathbf{g}(t))_i$ we have (by the method of integrating factors)

$$y_i(t) = e^{\lambda_i t}\left[\int_0^t e^{-\lambda_i s} h_i(s) ds + c_i\right].$$

Therefore, we found the solution $\mathbf{x} = \mathbf{T}\mathbf{y}$.

The End