$$\dot{x} = \begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

1) Find eigenvalue :

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{TrA^2 - 4\det A}$$

$$= \frac{-4}{2} \pm \frac{1}{2}\sqrt{16 - 4\cdot(-4-8)}$$

$$= -\frac{4}{2} \pm \frac{1}{2} 2\sqrt{4+12} = = -5, 1$$

$$\lambda_1 = -5, \quad \lambda_2 = 1$$

2) $V_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix}, V_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad c \neq 0$

$$V_1 = \begin{pmatrix} -5-(-3) \\ 4 \end{pmatrix} = \underset{\sim}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 1-(-3) \\ 4 \end{pmatrix} = \underset{\sim}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3) General sln

$$X = c_1 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X' = AX + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow X' = 0$$

$$\Rightarrow AX + V = 0$$

$$X' = AX + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow X' = 0$$

$$\Rightarrow AX + V = 0$$

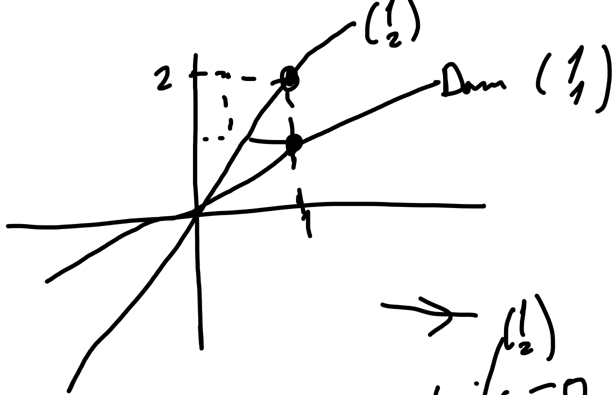
$$\Rightarrow X = -\underline{A^{-1}V} = -A^{-1}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $Av = \lambda v \Rightarrow A^{-1}v = \frac{1}{\lambda} v$ $\quad \overset{\lambda=1}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$X = c_1 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

4) Dominating $X \approx a(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

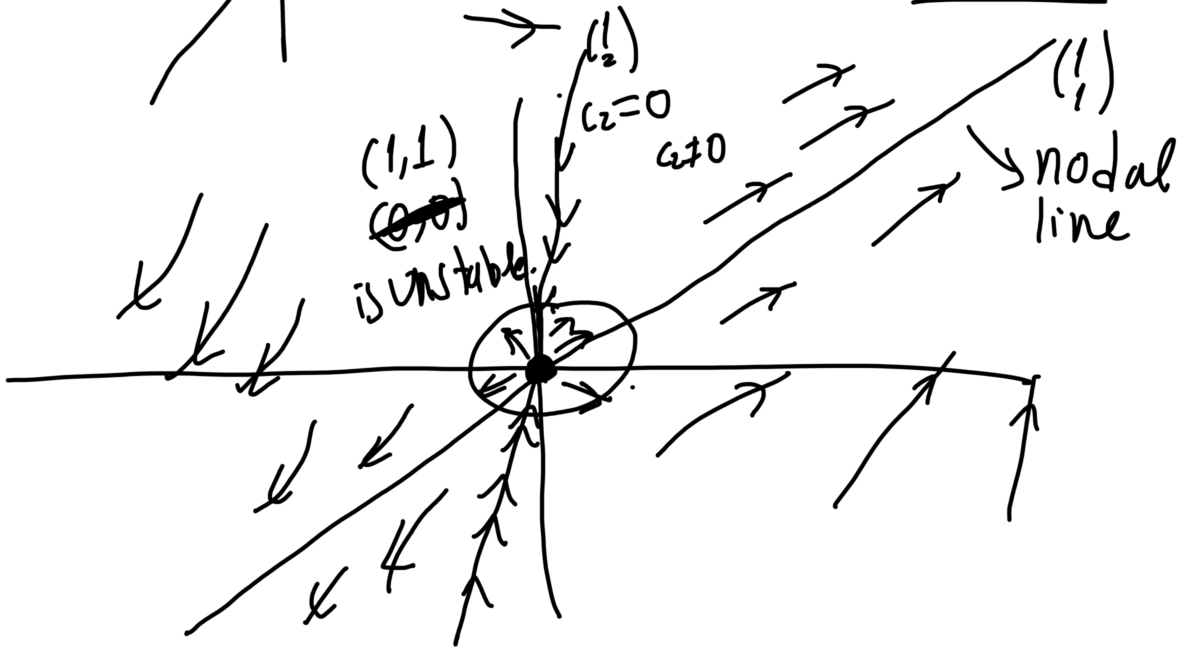
$\Rightarrow$ so solution $X(t)$ is on the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$\lambda_1 = -5 < 0$

$\lambda_2 = 1 > 0$

$\Rightarrow$ Saddle



$$\underline{x' = Ax + V}$$

1) $x_1' = Ax_1 \Rightarrow x_1 = ?$

2) Assume that $x_2' = 0 \Rightarrow Ax_2 + V = 0$

$$\Rightarrow x_2 = -A^{-1}V$$
$$= -\frac{1}{\lambda}V$$

3) $x = x_1 + x_2$

$$(x_1 + x_2)' - A(x_1 + x_2) - V$$
$$= \underline{x_1' - Ax_1} + \cancel{x_2'}^{\to 0} - (\underbrace{Ax_2 + V}_{= x_2' = 0})$$
$$= x_1' - Ax_1 = 0 .$$

$$x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

1) Find the eigenvalues

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4\det(A)}$$
$$= \frac{2-2}{2} \pm \frac{1}{2}\sqrt{(2-2)^2 - 4(2\cdot(-2) - (-1)\cdot 3)}$$
$$= 0 \pm \frac{1}{2}\sqrt{0 - 4(-4+3)} = \pm 1$$

$$V_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} = \begin{pmatrix} 1-(-2) \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$V_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix} = \begin{pmatrix} -1-(-2) \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

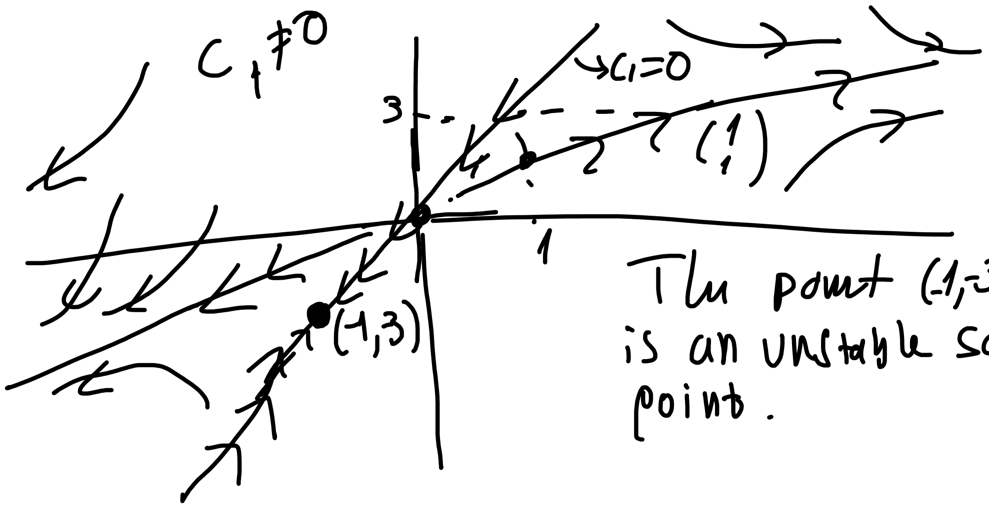
3) $X' = 0 \Rightarrow Ax_2 + \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0$

$(*)\ A^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow X_2 = -A^{-1}\begin{pmatrix} -1 \\ -3 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \overset{(*)}{=} -\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$X = X_1 + X_2 = C_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \underline{\begin{pmatrix} -1 \\ -3 \end{pmatrix}}$

4) Dominating term

$C_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



The point (-1,-3) is an unstable saddle point.

Complex

$x' = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} x$

1) Find the eigvl

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{a^2 - 4 \cdot 1}$$

$$= \frac{a}{2} \pm \frac{1}{2}\underline{\sqrt{a^2 - 4}}$$

If $a^2 < 4 \Rightarrow a \in (-2, 2)$

2) $V_1 = \begin{pmatrix} \lambda_1 \\ c \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{a}{2} + \frac{i}{2}\sqrt{|a^2-4|} \\ -1 \end{pmatrix}$

$AV_1 = \lambda_1 V_1$

$$x = c_1 V_1 e^{(\lambda + i\mu)t} + c_2 V_2 e^{(\lambda - i\mu)t}$$

$$V_1 e^{(\lambda + i\mu)t} = (\underbrace{a + ib}_{= V_1}) e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= e^{\lambda t}\Big[\big(a\cos(\mu t) + i^2 b\sin(\mu t)\big) + i\big(b\cos(\mu t) + a\sin(\mu t)\big)\Big]$$

$$\overset{i^2=-1}{=} \underbrace{e^{\lambda t}(a\cos(\mu t) - b\sin\mu t)}_{\xi_1} + i\,\underbrace{e^{\lambda t}(b\cos(\mu t) + a\sin(\mu t))}_{\xi_2}$$

$$W(\xi_1, \xi_2) \neq 0$$

$$\begin{bmatrix} \xi_1 & \xi_2 \\ \xi_1' & \xi_2' \end{bmatrix} = \Big[e^{\lambda t}(b$$

$$\xi_2' = \lambda e^{\lambda t}(\ldots) + e^{\lambda t}(-b\cdot\mu\sin(\mu t) + a\cos(\mu t)\mu)$$

$$\xi_1' = \lambda e^{\lambda t}(\ldots) + e^{\lambda t}(-a\mu\sin(\mu t) - b\mu\cos(\mu t))$$

Ex: $W(\xi_1, \xi_2) \neq 0$.

$$X = \cancel{e^{\lambda t}\left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right]}$$

$$= e^{\lambda t} c_1\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\cos(\mu t) - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\sin(\mu t)\right)$$

$$+ e^{\lambda t} c_2\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\cos(\mu t) + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\sin(\mu t)\right)$$

eigenvalu $\tilde{\lambda} = \lambda + i\mu,\ \lambda - i\mu$

eigenvector $v_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i\begin{pmatrix} b_1 \\ b_2 \end{pmatrix},\ v_2 = \overline{v_1}$

$= a + ib \qquad = a - ib$

Back to

$$X' = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

$$X = c_1 e^{(\lambda + i\mu)t} v_1 + c_2 e^{(\lambda - i\mu)t} v_2$$

$$\lambda \pm i\mu = \frac{a}{2} \pm \frac{i}{2}\sqrt{|a^2 - 4|} = \mu$$

$$v = (\lambda \pm i\mu) = (\lambda \ldots) + i(\mu \ldots)$$

$$v_{\pm} = \begin{pmatrix} \lambda \pm i\mu \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ -1 \end{pmatrix} \pm i \begin{pmatrix} \mu \\ 0 \end{pmatrix}$$

$$x = e^{a/2}\left(c_1\left(\begin{pmatrix} \lambda \\ -1 \end{pmatrix}\cos(\mu t) - \begin{pmatrix} \mu \\ 0 \end{pmatrix}\sin(\mu t)\right) + c_2\left(\begin{pmatrix} \lambda \\ -1 \end{pmatrix}\sin(\mu t) + \begin{pmatrix} \mu \\ 0 \end{pmatrix}\cos \mu t\right)\right)$$

If $a < 0 \to$ spiral sink
$a > 0 \to$ spiral source
$a = 0 \to$ concentric circles

$$x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x = c_1 e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix} + c_2 e^{(-1-2i)t}\begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 2i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$\begin{cases} 1 = 2ic_1 + (-2i)c_2 \\ 1 = c_1 + c_2 \end{cases}$$

Fact CV: If $a + ib = c + id$
$\Rightarrow a = c, \ b = d.$

$a + ib \Longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\Rightarrow \begin{matrix} a = c \\ b = d. \end{matrix}$

$\Rightarrow \quad b = d$

$$\begin{cases} 1 + i \cdot 0 = (c_1 \cdot 2 - c_2 \cdot 2) i \\ 1 = c_1 + c_2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 \cdot 2 - 2 \cdot c_2 = 0 \\ 1 = c_1 + c_2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = c_2 \\ 1 = 2c_1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{cases}$$

$$x = \frac{1}{2} e^{-t} \left(e^{2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix} + e^{-2it} \begin{pmatrix} -2i \\ 1 \end{pmatrix} \right)$$

$$x = e^{at} \left(c_1 e^{it} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} -i \\ 1 \end{pmatrix} \right)$$

$a > 0 \longrightarrow$ spiral source

$a < 0 \longrightarrow$ spiral sink

$a = 0 \longrightarrow$ concentric circles.

In second order with repeated eigenvalues we had the ansatz $x_2 = t \cdot e^{\lambda t}$

Lets try $x_2 = t \cdot e^{\lambda t} \cdot \xi$ $\rightarrow$ eigvt

$$x_2' = Ax_2 \Rightarrow e^{\lambda t} \xi + t \cdot \lambda e^{\lambda t} \xi = t e^{\lambda t} A \xi$$

$$x_2' = Ax_2 \Rightarrow e^{\lambda t}\xi + t \cdot \lambda e^{\lambda t}\xi = te^{\lambda t}A\xi$$
$$= \lambda t e^{\lambda t}\xi$$

$$\Rightarrow e^{\lambda t}\xi = 0 \Rightarrow \xi = \begin{pmatrix}0\\0\end{pmatrix} \text{ ↯}$$

Instead assume $x_2 = te^{\lambda t}\xi + \eta \cdot e^{\lambda t}$

$\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$

$$x_2' = Ax_2 \Rightarrow e^{\lambda t}\xi + \lambda t e^{\lambda t}\xi + \lambda e^{\lambda t}\eta$$
$$= Ae^{\lambda t}t\xi + Ae^{\lambda t}\eta$$

$$\Rightarrow e^{\lambda t}\xi + \lambda e^{\lambda t}\eta = Ae^{\lambda t}\eta$$

$$\Rightarrow (A - \lambda I_2)\eta = \xi$$

$\Rightarrow$ we'll obtain η

1) $\xi = \begin{pmatrix}1\\-1\end{pmatrix}$, $\lambda = 2$

$$(A - \lambda I)\eta = \xi \Rightarrow$$

$$\begin{pmatrix}1-2 & -1\\ 1 & 3-2\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow \begin{cases} -\eta_1 - \eta_2 = 1 \\ \eta_1 + \eta_2 = -1 \end{cases}$$

$$\eta_2 = -1 - k, \quad k \in \mathbb{R}$$

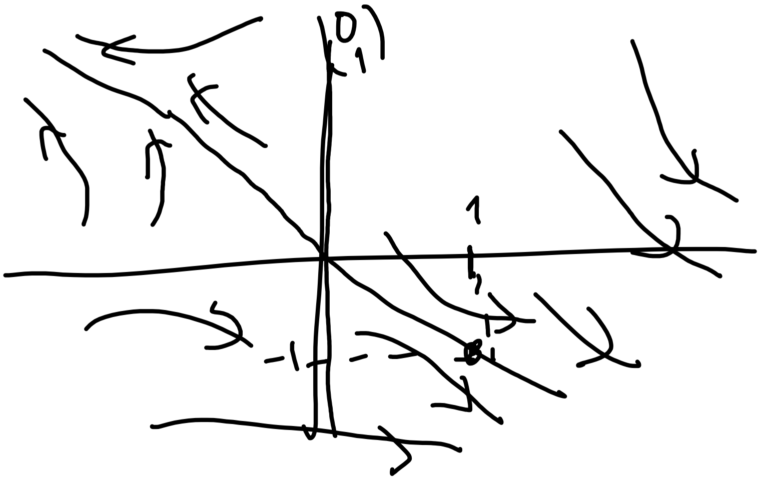
$$\Rightarrow \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = c_1 X_1 + c_2 X_2, \quad X_1 = \xi e^{\lambda t}$$

$$X_2 = \underline{t \cdot e^{\lambda t}}\, \xi + \eta \cdot e^{\lambda t}$$

$$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(t e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left(k \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{\lambda t} \right)$$

$$= \boxed{c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 t e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$



$$x' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\xi_2 = 0$

1) $\lambda = 2$, $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2) $(A - \lambda I)\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} \eta_2 = 1 \\ \eta_1 = k \end{array}$$

$$\Rightarrow \eta = \begin{pmatrix} k \\ 1 \end{pmatrix} = \underline{k \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3) $$x = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 t e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \eta e^{2t}$$

$$= \cdots + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$c_1 \qquad = e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 t e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e^{2t} \left(\underline{(c_1 + c_2 t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

dominates.

Constant term: Assum $x_2' = 0 \Rightarrow A x_2 - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$

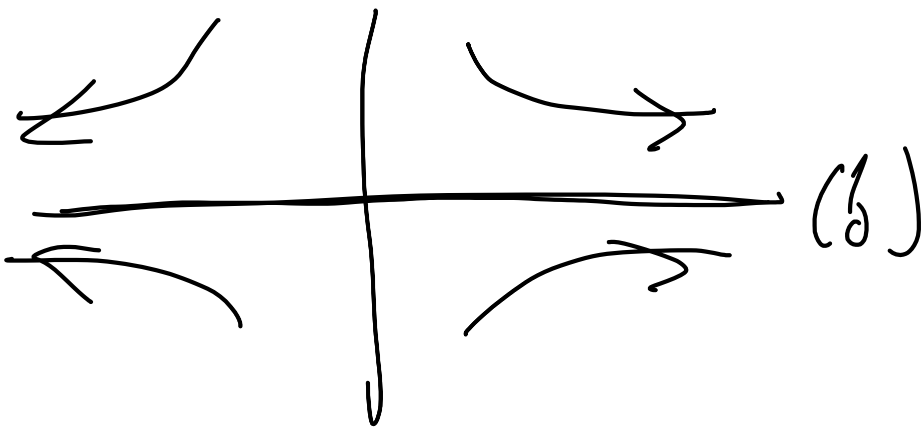
$$\Rightarrow A x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x_2 = ($$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

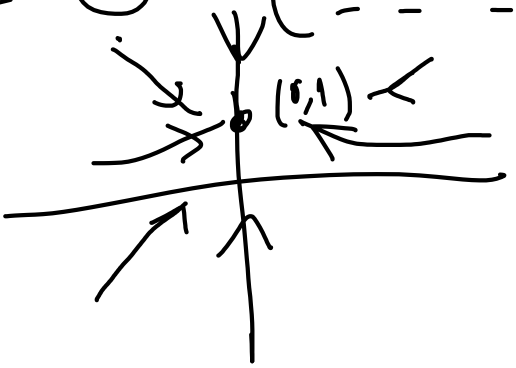
$$\Rightarrow \begin{cases} 2v_1 + v_2 = 1 \\ 2v_2 = 2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = e^{2t}\left((c_1 + c_2 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \underline{+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$



$$X = e^{-2t}\left(\ldots \right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Nonhomogeneous

$x' = Ax + g(x)$.

Assume $\underline{x = Ty}$, when $T = \left(\xi_1 \ \xi_2 \ \cdots\right)$

$(Ty)' = ATy + g(Ty)$ $\quad T^{-1}$ exists

$\Rightarrow Ty' = ATy + g(Ty)$

$\Rightarrow y' = T^{-1}ATy + T^{-1}g(t)$

$= Dy + T^{-1}g(t)$

$$\begin{cases} y_1' = \lambda_1 y_1 + (T^{-1}g(t))_1 \\ y_2' = \lambda_2 y_2 + (T^{-1}g(t))_2 \end{cases} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Integrating factor

$y' = \lambda y + q(t)$

$\mu y' - \mu \lambda y = \mu q(t)$ $\quad -\lambda t$

$$\Rightarrow \mu' = -\mu\lambda \Rightarrow \mu = e^{-\lambda t}$$

$$y_i(t) = e^{\lambda_i t}\left(\int_0^t q_i(s)\, e^{-\lambda_i s}\, ds + C\right)$$

$$q_i(s) = (T^{-1} g(s))_i$$