Outline

1. Lyapunov method
2. Periodic solutions

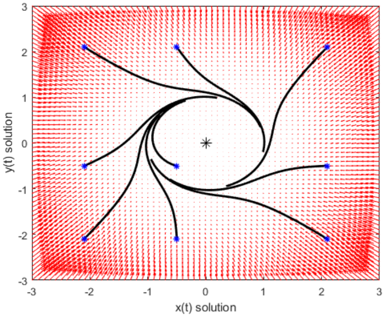

Figure: Entire curves can be sinks

We return to the damping-free pendulum system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\frac{g}{L} sin(x).$$

Consider the total energy of the system:

$$\begin{aligned} E(x,y) &= \text{Potential} \ + \ \text{Kinetic} \\ &= U(x,y) + K(x,y) \\ &:= mgL(1 - cos(x)) + \frac{1}{2} mL^2 y^2. \end{aligned}$$

But close to $(\pi, 0)$ change to

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = \frac{g}{L} sin(x)$$

and Lyapunov function $V = ysin(x)$

Stability criterion

We have

$$\dot{V} = V_x \dot{x} + V_y \dot{y} = \nabla V \cdot T = |\nabla V||T| cos(\theta).$$

Stability criterion

We have

$$\dot{V} = V_x \dot{x} + V_y \dot{y} = \nabla V \cdot T = |\nabla V||T| cos(\theta).$$

So if $\dot{V} \leq 0$ then $cos(\theta) \leq 0 \Leftrightarrow \theta \in [\frac{\pi}{2}, \pi]$.

Stability criterion

We have

$$\dot{V} = V_x \dot{x} + V_y \dot{y} = \nabla V \cdot T = |\nabla V||T| cos(\theta).$$

So if $\dot{V} \leq 0$ then $cos(\theta) \leq 0 \Leftrightarrow \theta \in [\frac{\pi}{2}, \pi]$.
Basin of attraction is the region bounded by the largest level set $\{V = c\}$ s.t. we still have $\dot{V} < 0$.

Lyapunov criterion

If V satisfies $V(x_0, y_0) = 0$ and $V(x, y) > 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0) then

Lyapunov criterion

If V satisfies $V(x_0, y_0) = 0$ and $V(x, y) > 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0) then

- If $\frac{\mathrm{d}V}{\mathrm{dt}} \leq 0$ for $(x, y) \neq (x_0, y_0)$ then (x_0, y_0) is Lyapunov-stable.
- If $\frac{\mathrm{d}V}{\mathrm{dt}} < 0$ for $(x, y) \neq (x_0, y_0)$ then (x_0, y_0) is asymptotically stable.

If $V(x, y) > 0$ for at least one point close to (x_0, y_0) and $\frac{\mathrm{d}V}{\mathrm{dt}} > 0$ for $(x, y) \neq (x_0, y_0)$ then (x_0, y_0) is unstable.

Presenting the method

Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \mathbf{x},$$

for Lyapunov function $V = \frac{1}{2}x^2 - \frac{2}{3}xy + \frac{7}{12}y^2$.

Presenting the method

1. At the origin we indeed have V(0,0)=0.
2. Next we prove that $V > 0$. The goal is to complete the square. A quick formula for any monomial is

$$x^2 + bx + c = (x + \frac{1}{2}b)^2 + c - \frac{b^2}{4}.$$

   So if we have $k > 0$ we are done. Indeed

$$c - \frac{b^2}{4} = y^2(\frac{7}{6} - \frac{4}{9}) > 0.$$

3. Next we check the sign of $\dot{V}$. We have

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = x^2 + y^2 > 0.$$

4. So it says that the system is unstable. Indeed its eigenvalues are $1, 2$ and so the origin is a source (i.e. the initial data matters because for initial data it stays trapped in the origin).

In class example

Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{x},$$

for Lyapunov function $V = \frac{1}{2}x^2 - xy + \frac{3}{2}y^2$.

In class example

Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -x + 2y + y^4, \frac{\mathrm{d}x}{\mathrm{dt}} = -y + x^4$$

for Lyapunov function $V = \frac{1}{2}x^2 + xy + \frac{3}{2}y^2$.

In class example

Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -x + 2y + y^4, \frac{\mathrm{d}x}{\mathrm{dt}} = 2y - 2x + x^4$$

for Lyapunov function $V = 4x^2 - 3xy + \frac{7}{4}y^2$.

Consider the system

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix}.$$

1. First we find the critical points:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 0, \frac{\mathrm{d}y}{\mathrm{dt}} = 0.$$

   If we assume that $x \neq 0, y \neq 0$ we get the contradiction $x^2 + y^2 = 0$ which implies that the origin $(x, y) = (0, 0)$ is the only critical point.
2. We linearize around the origin to obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

   which has complex eigenvalues $1 \pm i$.
3. Since $Re(\lambda) > 0$ we have that locally the phase portrait is a spiral source. But interestingly the behaviour changes globally. Here the stable sink trajectory will be the unit circle centered at the origin.

limit cycle

1. First we change to polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ to obtain the system:

$$r\frac{\mathrm{d}r}{\mathrm{dt}} = r^2(1-r^2) \text{ and } \frac{\mathrm{d}\theta}{\mathrm{dt}} = -1.$$

2. These equations are now decoupled and can be solved by separation of variables:

$$\begin{aligned} r\frac{\mathrm{d}r}{\mathrm{dt}} &= r^2(1-r^2) \Rightarrow \\ \int \frac{-1}{r(1-r^2)}\mathrm{d}r &= \int \mathrm{d}t \Rightarrow \\ \frac{1}{2}\ln(1-r^2) - \ln(r) &= t + c \Rightarrow \\ r &= \frac{1}{\sqrt{1+c_1 e^{-2t}}}. \end{aligned}$$

Similarly, for θ we obtain

limit cycle

1. We observe a couple of things. First, as $t \to +\infty$, the radius $r(t)$ of the solution converges to 1 irrespective of the constants c_1, c_2
2. These constants encode the initial data and ,in particular, the initial radius and angle because at time 0 we have

$$r(0) = \frac{1}{\sqrt{1 + c_1}} \text{ and } \theta = c_2.$$

3. So if we happen to start outside the unit circle $r(0) > 1$ (eg. for $c_1 = -1/2$) or inside the unit circle $r(0) < 1$ (eg. for $c_1 = 1$), the solution will converge to the unit circle regardless.

The End