Contents

0.1 Liapunov's Second Method

Consider the autonomous system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x,y) \text{ and } \frac{\mathrm{d}x}{\mathrm{dt}} = G(x,y).$$

We will obtain a criterion for concluding asymptotic stability and even determining the basin of attraction.

0.1.1 Example-presenting the method

We return to the damping-free pendulum

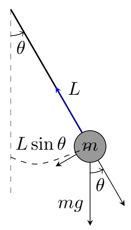

Figure 0.1.1: oscillating pendulum

whose angle θ satisfies the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \frac{mg}{L} sin(\theta) = 0.$$

This is a nonhomogeneous second order equation, but we can also view it as a system of equations by letting $x := \theta$ and $y := \frac{\mathrm{d}\theta}{\mathrm{dt}}$:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\frac{g}{L} sin(x).$$

1. Consider the total energy of the system:

$$\begin{aligned}E(x,y) &= \text{Potential } + \text{ Kinetic} \\ &= U(x,y) + K(x,y) \\ &:= mgL(1 - cos(x)) + \frac{1}{2}mL^2y^2.\end{aligned}$$

2. Since the system is damping-free, the energy is conserved and so we should have:

$$\frac{\mathrm{d}V}{\mathrm{dt}} = 0.$$

Lets prove this:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{dt}} &= \frac{\mathrm{d}}{\mathrm{dt}}[mgL(1 - cos(x)) + \frac{1}{2}mL^2y^2] \\ &= mgLsin(x)\frac{\mathrm{d}x}{\mathrm{dt}} + mL^2y\frac{\mathrm{d}y}{\mathrm{dt}}\end{aligned}$$

using the equations

$$\begin{aligned}&= mgLsin(x)y + mL^2y(-\frac{g}{L}sin(x)) \\ &= 0.\end{aligned}$$

3. Therefore, we obtain an implicit solution:

$$mgL(1 - cos(x)) + \frac{1}{2}mL^2y^2 = constant.$$

4. Next we consider the case where there is damping i.e. θ satisfies the equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \gamma\frac{\mathrm{d}\theta}{\mathrm{dt}} + \omega^2 sin(\theta) = 0$$

and so the system is:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \frac{g}{L}sin(x).$$

5. By computing the time derivative of the total energy we obtain:

$$\frac{\mathrm{d}V}{\mathrm{dt}} = -mL^2\gamma y^2 = -mL^2\gamma(\dot{\theta})^2 \leq 0.$$

6. Physically this means that the energy will be decreasing over time as the damping force keeps slowing down the pendulum. Therefore, we expect that the system will be asymptotically stable towards the origin, where both the angle and the velocity are zero.

0.1.2 Method formal steps

For the autonomous system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x,y) \text{ and } \frac{\mathrm{d}x}{\mathrm{dt}} = G(x,y)$$

let (x_0, y_0) denote a critical point. We will need the following definitions:

Definition 0.1.1. A point (x_0, y_0) is *Lyapunov-stable* if a solution that starts close to it, then it will stay close to that critical point for all future time. Given any desired $\varepsilon > 0$ we can find $\delta > 0$ s.t. if we start $\delta-$close

$$\|\mathbf{x}(0) - (x_0, y_0)\| \leq \delta,$$

then we stay $\varepsilon-$close:

$$\|\mathbf{x}(t) - (x_0, y_0)\| \leq \varepsilon, \forall t > 0.$$

A point (x_0, y_0) is *asymptotically stable* if a solution that starts close to it converges to that critical point:

$$\|\mathbf{x}(t) - (x_0, y_0)\| \to 0.$$

1. Find the "total energy" V of the autonomous system by solving:

$$\begin{aligned}\frac{\mathrm{d}V(x(t), y(t))}{\mathrm{ddt}} &= V_x \frac{\mathrm{d}x}{\mathrm{dt}} + V_y \frac{\mathrm{d}y}{\mathrm{dt}} \\ &= V_x F(x(t), y(t)) + V_y G(x(t), y(t)).\end{aligned}$$

2. If $\frac{\mathrm{d}V(x(t),y(t))}{\mathrm{ddt}} = 0$, then use to find the implicit solutions:

$$constant = V(x(t), y(t)).$$

3. If V satisfies the following conditions:

   - $V(x_0, y_0) = 0$,
   - $V(x, y) > 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0),
   - and it is nondecreasing in time $\frac{\mathrm{d}V}{\mathrm{dt}} \leq 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0),
     then
     $$(x_0, y_0) \text{ is a Lyapunov-stable critical point.}$$
   - If it is even a strictly non-decreasing function in time $\frac{\mathrm{d}V}{\mathrm{dt}} < 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0),
     then
     $$(x_0, y_0) \text{ is a stable critical point.}$$

4. If instead we have

   - $V(x_0, y_0) = 0$,
   - $V(p) > 0$ for at least one point $p \neq (x_0, y_0)$ in a disk around (x_0, y_0),
   - and strictly decreasing energy $\frac{\mathrm{d}V}{\mathrm{dt}} > 0$ for all other $(x, y) \neq (x_0, y_0)$,

   then
   $$(x_0, y_0) \text{ is an asymptotically unstable critical point.}$$

0.1.3 Finding the Lyapunov function

So clearly finding such a scalar function is the first serious obstacle. Here are some ideas and heuristics on guessing such a function:

Physical systems For physical systems the energy/Hamiltonian is a good guess. For example, if there is no new energy input, then the energy function will decay over time or remain constant.

Lur'e type systems Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -y - h_1(x) \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = h_2(x),$$

h_1 is differentiable and h_2 integrable. We will obtain a Lyapunov function for this type of system.By taking t-derivative of the first equation and using the second one we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}^2\mathrm{t}} = -h_2(x) - h_1'(x)\frac{\mathrm{d}x}{\mathrm{dt}}$$

multiplying by $\frac{\mathrm{d}x}{\mathrm{dt}}$ we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}^2\mathrm{t}}\frac{\mathrm{d}x}{\mathrm{dt}} = -h_2(x)\frac{\mathrm{d}x}{\mathrm{dt}} - h_1'(x)(\frac{\mathrm{d}x}{\mathrm{dt}})^2 \Rightarrow$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(\frac{(\frac{\mathrm{d}x}{\mathrm{dt}})^2}{2} + \int_0^x h_2(s)ds\Big) = -h_1'(x)(\frac{\mathrm{d}x}{\mathrm{dt}})^2.$$

Thus, if we have $h_1'(x) > 0$ in a neighbourhood of x_0, then the function

$$V(x) := \frac{(\frac{\mathrm{d}x}{\mathrm{dt}})^2}{2} + \int_0^x h_2(s)ds$$

has a strictly negative derivative. Moreover, if $\int_0^x h_2(x)$

Cost functions Distance and cost functions with respect to the critical point (x_0, y_0) are also good guesses because close to the critical point the derivatives $\frac{\mathrm{d}x}{\mathrm{dt}}, \frac{\mathrm{d}y}{\mathrm{dt}}$ are decaying to zero and so we might indeed have:

$$V_x\frac{\mathrm{d}x}{\mathrm{dt}} + V_y\frac{\mathrm{d}y}{\mathrm{dt}} \leq 0.$$

Linear systems When the system is linear i.e. $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ then a candidate Lyapunov function is

$$V(\mathbf{x}) = \int_0^\infty \left\|e^{At}\mathbf{x}\right\|dt = \mathbf{x}^T\Big(\int_0^\infty e^{(A^T+A)t}dt\Big).$$

This is indeed a Lyapunov function for *exponentially stable systems* (see more details in the converse theorems 0.1.3)

Polynomial systems When f_1, f_2 are polynomials of highest degree m, then there are many algorithms for generating the corresponding Lyapunov functions of the form

$$V(x,y) = \sum_{j,k}^{m+1} c_{j,k}x^jy^k,$$

by optimizing over the coefficients (see [**giesl2015review**] for the sum-of-squares (SOS) theory).

0.1.4 General result:

Lyapunov's second method

Theorem 0.1.2. *Consider system*

$$\frac{\mathrm{d}x}{\mathrm{dt}} = f_1(x, y) \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = f_2(x, y),$$

and (x_0, y_0) a particular isolated equilibrium point. If we can find a continuously differentiable function $V : U_{(x_0,y_0)} \to \mathbb{R}$ around some neighbourhood $U_{(x_0,y_0)}$ of the critical point (x_0, y_0) with the following properties:

1. $V(x_0, y_0) = 0,$
2. $V(x, y) > 0$ *for* $(x, y) \in U_{(x_0,y_0)} \setminus \{(x_0, y_0)\},$
3. *and* $\frac{\mathrm{d}V}{\mathrm{dt}} := \frac{\mathrm{d}V}{\mathrm{dx}} f_1 + \frac{\mathrm{d}V}{\mathrm{dy}} f_2 \leq 0$ *in punctured neighbourhood* $U_{(x_0,y_0)} \setminus \{(x_0, y_0)\}$

then the critical point (x_0, y_0) is stable. In fact if we replace the last condition by strict inequality:

$$\frac{\mathrm{d}V}{\mathrm{dx}} f_1 + \frac{\mathrm{d}V}{\mathrm{dy}} f_2 < 0$$

then critical point (x_0, y_0) is asymptotically stable.

Converse theorems: existence of Lyapunov function

Theorem 0.1.3. *Suppose that the linearization around an equilibrium point is*

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

s.t. $det(A) = ad - bc > 0$ and $a + d < 0$. Then the function

$$V(x, y) = Ax^2 + Bxy + Cy^2,$$

is a Lyapunov function for this system with $\dot{V} < 0$ if

$$A = -\frac{c^2 + d^2 + det(A)}{2Tr(A)det(A)}$$
$$B = \frac{bd + ac}{Tr(A)det(A)}$$
$$C = -\frac{a^2 + b^2 + det(A)}{2Tr(A)det(A)}$$

because then $\dot{V}(x, y) = -x^2 - y^2$ and the matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

is positive definite.

0.1.5 Examples

- Consider the system
$$\frac{\mathrm{d}x}{\mathrm{dt}} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = -x - y.$$
  1. First we find that the equilibrium point is only the origin $(0, 0)$.
  2. We take the Euclidean distance function as a guess:
  $$V(x, y) = \frac{1}{2}(x^2 + y^2).$$
  3. Next we check each of the properties
     - $V(0, 0) = 0$,
     - $V(x, y) > 0$ for $(x, y) \neq 0$,
     - and
     $$\frac{\mathrm{d}V}{\mathrm{dt}} = V_x f_1 + V_y f_2 = (x)y + (y)(-x - y) = -y^2 < 0$$
     for (x, y) in a punctured disk centered at the origin.
  4. Therefore, $V(x, y)$ is a Lyapunov function and so the point (x_0, y_0) is asymptotically stable.

- Consider the linear harmonic oscillator
$$\frac{\mathrm{d}x}{\mathrm{dt}} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = -kx,$$
for $k > 0$, with $V = Pot + Kin = \frac{1}{2}kx^2 + \frac{1}{2}y^2$ as its candidate Lyapunov function.
  1. First, we check that we indeed have a Lyapunov function.
     - We indeed have $V(0, 0) = \frac{1}{2}k0 + \frac{1}{2}0 = 0$.
     - We have $V(x, y) = \frac{1}{2}kx^2 + \frac{1}{2}y^2 > 0$ for $(x, y) \neq (0, 0)$.
     - Finally, we have
     $$\frac{\mathrm{d}V}{\mathrm{dt}} = V_x\dot{x} + V_y\dot{y} = kxy - kxy = 0.$$
  2. So the origin will be an asymptotically stable point (solutions that start close, remain close).
  3. Indeed from linearization around the origin we obtain:
  $$J_F = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix},$$
  which has eigenvalues $\lambda = \pm i\sqrt{k}$ and so the phase portrait will be concentric circles centered at the origin. This agrees with our Lyapunov behaviour because solutions that start in a circle close to the origin, stay on that circle for all future time.

- Consider the linear harmonic oscillator with damping
$$\frac{\mathrm{d}x}{\mathrm{dt}} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = -kx - \alpha y^3(1 + x^2),$$
for $k > 0$, with the same $V = Pot + Kin = \frac{1}{2}kx^2 + \frac{1}{2}y^2$ as its candidate Lyapunov function.
  1. First, we check that we indeed have a Lyapunov function.

- The first two conditions are the same.
- Finally, we have

$$\frac{\mathrm{d}V}{\mathrm{dt}} = V_x\dot{x} + V_y\dot{y} = kxy - kxy - \alpha y^4(1+x^2) = -\alpha y^4(1+x^2).$$

2. If $\alpha > 0$ (i.e. the is positive damping removing energy from the system), then $\frac{\mathrm{d}V}{\mathrm{dt}} \leq 0$ and so the origin will be an asymptotically stable point (solutions that start close, remain close).
3. The linearization around the origin is:

$$J_F = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix},$$

which again has eigenvalues $\lambda = \pm i\sqrt{k}$ and so the phase portrait will be concentric circles centered at the origin.
4. For $\alpha > 0$ the behaviour is more complicated and we will explain it later.

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -x + 4y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = -x - y^3,$$

with $V = ax^2 + by^2$ as its candidate Lyapunov function.

1. First, we check that we indeed have a Lyapunov function.
   - We indeed have $V(0,0) = 0$ and $V(x,y) > 0$ for $(x,y) \neq (0,0)$ if $a, b > 0$.
   - We have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{dt}} &= V_x\dot{x} + V_y\dot{y} \\ &= 2ax(-x+4y) + 2by(-x-y^3) \\ &= -2ax^2 + xy(8a-2b) - 2by^4,\end{aligned}$$

to make this strictly negative we set $a = 1, b = 4$ to get

$$= -2x^2 - 8y^4 < 0.$$

2. So the origin will be a stable point (solutions converge to the origin).
3. Indeed from linearization we obtain:

$$J_F = \begin{pmatrix} -1 & 4 \\ -1 & 0 \end{pmatrix},$$

which has repeated eigenvalue $\lambda = -1$ and so the solution will converge to the origin for any initial data.

- Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\mathbf{x},$$

for Lyapunov function $V = \frac{1}{2}x^2 - \frac{2}{3}xy + \frac{7}{12}y^2$.

1. At the origin we indeed have V(0,0)=0.
2. Next we prove that $V > 0$. The goal is to complete the square. A quick formula for any monomial is

$$x^2 + bx + c = (x + \frac{1}{2}b)^2 + c - \frac{b^2}{4}.$$

So if we have $k > 0$ we are done. Indeed

$$c - \frac{b^2}{4} = y^2(\frac{7}{6} - \frac{4}{9}) > 0.$$

3. Next we check the sign of $\dot{V}$. We have

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = x^2 + y^2 > 0.$$

4. So it says that the system is unstable. Indeed its eigenvalues are $1, 2$ and so the origin is a source (i.e. the initial data matters because for initial data it stays trapped in the origin).

- Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{x},$$

for Lyapunov function $V = \frac{1}{2}x^2 - xy + \frac{3}{2}y^2$.

1. As above in order to complete the square we find the sign of

$$c - \frac{b^2}{4} = y^2(\frac{1}{2} - \frac{1}{4}) = \frac{y^2}{4}.$$

2. Next we find the sign of $\dot{V}$:

$$\begin{aligned}\dot{V} &= V_x\dot{x} + V_y\dot{y} \\ &= (x-y)(x+2y) + (-x+3y)y \\ &= x^2 + xy(2-1-1) + y^2 \qquad\qquad = x^2 + y^2 > 0.\end{aligned}$$

3. So it says that the origin is unstable. Indeed the eigenvalues are 1,1 and so the origin is a source (i.e. for zero initial data it gets trapped whereas for nonzero initial data it moves to infinity and so the initial data is relevant).

- Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 2y + y^4, \frac{\mathrm{d}x}{\mathrm{d}t} = -y + x^4$$

for Lyapunov function $V = \frac{1}{2}x^2 + xy + \frac{3}{2}y^2$.

1. First we check the sign of V. In completing the square we find the sign of

$$c - \frac{b^4}{2} = y^2\frac{1}{4} > 0.$$

2. Next we check the sign of $\dot{V}$:

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = -x^2 - y^2 - (x+y)y^4 - (\frac{3x}{2} - 2y)x^4.$$

   For (x, y) close to zero we have $x^4 << x^2, y^4 << y^2$ and so we indeed have $\dot{V} < 0$.

3. This agrees with the linearization since the eigenvalues will be the repeated -1, which makes the origin a source.

- Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 2y + y^4, \frac{\mathrm{d}x}{\mathrm{d}t} = 2y - 2x + x^4$$

for Lyapunov function $V = 4x^2 - 3xy + \frac{7}{4}y^2 = 4(x^2 - \frac{3}{4}xy + \frac{7}{16}y^2)$.

1. First we check the sign of V. In completing the square we find the sign of

$$c - \frac{b^4}{2} = y^2(\frac{7}{16} - \frac{9}{4*16}) > 0.$$

2. Next we check the sign of $\dot{V}$:

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = x^2 + y^2 - (8x - 3y)y^4 - (-3x + \frac{7}{2}y)x^4.$$

For (x, y) close to zero we have $x^4 << x^2, y^4 << y^2$ and so we indeed have $\dot{V} > 0$.

3. This agrees with the linearization since the eigenvalues will be the repeated $-1, 2$, which makes the origin an unstable saddle.

0.1.6 Applied examples

Walras's law and the *ttonnement* mechanism

Here, we consider the question of stability of a pure exchange, competitive equilibrium with an adjustment mechanism known as t^atonnement and directly inspired by the work of $LonWalras$ (1874), one of the founding fathers of mathematical economics.
The basic idea behind the t^atonnement mechanism is the same assumed in the rudimentary price adjustment mechanism models, namely that prices of commodities rise and fall in response to discrepancies between demand and supply (the so-called 'law of demand and supply').
In the present case, demand is determined by individual economic agents maximising a utility function subject to a budget constraint, given a certain initial distribution of stocks of commodities. The model can be described schematically as follows.

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{f}(\mathbf{p}) = \begin{pmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \end{pmatrix},$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions with all their derivatives continuous as well.

1. A price point $\mathbf{p}_0$ is called an equilibrium if

$$\begin{aligned} &f_i(\mathbf{p}_0) \leq 0, p_i \geq 0,\text{and } p_j > 0\text{for some j} \\ &\text{or } f_i(\mathbf{p}_0) < 0, \mathbf{p}_0 = \mathbf{0}. \end{aligned}$$

   The first case makes economic sense (i.e. at least one price is nonzero) and so by *equilibrium point* we will mean the first case.

2. (Hypothesis **H**)The hypothesis that agents maximise utility is that the functions $f_i(\mathbf{p})$ are **homogeneous of degree zero**, namely $f_i(\lambda\mathbf{p}) = \lambda^0 f_i(\mathbf{p}) = f_i(\mathbf{p})$ for any $\lambda > 0$.

3. (**Walras's law**)Consider that the budget constraint for each individual k takes the form

$$\sum_{i=1}^{2} p_i f_i^k(\mathbf{p}) = p_1 f_1^k(\mathbf{p}) + p_2 f_2^k(\mathbf{p}) = 0,$$

where f_i^k denotes the excess demand by the kth economic agent for the ith commodity, i.e., the difference between the agent's demand for, and the agent's initial endowment of, that commodity. In general for m commodities by summing over all N economic agents we have:

$$\sum_{k=1}^{N}\sum_{i=1}^{m} p_i f_i^k(\mathbf{p}) = \sum_{i} p_{i=1}^{m} f_i(\mathbf{p}) = 0.$$

This law states that, in view of the budget constraints, for any set of semipositive prices p (not necessarily equilibrium prices), the value of aggregate excess demand, evaluated at those prices, must be zero.

The Jacobian matrix for $\mathbf{f}$ is

$$D\mathbf{f}(\mathbf{p}_0) = \begin{pmatrix} \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_2} \\ \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_2} \end{pmatrix}.$$