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1 First order odes basic concepts

We first study order one ODEs:

$$y' = f(x, y).$$

For nice enough functions f we have existence and uniqueness. Here are some examples:

- For any continuous function, f, that depends *only* on t gives a solution y.
  - For $f(t, y) = t$ we get $y' = t \Rightarrow y(t) = \frac{t^2}{2} + c$, where c is some constant from integrating.
  - For $f(t, y) = \cos(t)$ we get $y' = \cos(t) \Rightarrow y(t) = \sin(t) + c$.
- For linear $y' = f(t, y) = ay$ we obtain the solution $y(t) = ce^{a \cdot t}$ (integrating factors).
- For $y' = f(t, y) = \frac{t^2}{1-y^2}$, $|y| \neq 1$ we obtain $-t^3 + 3y - y^3 = c$ (separable equations).
- For $y' = f(t, y) = -\frac{2t+y^2}{2ty}$ we obtain $t^2 + ty^2 = c$ (exact equations).

1.1 Direction fields: Vector field interpretation

A useful tool will be the vector field interpretation of $y' = f(t, y)$, for continuous f. A solution curve $\gamma(t) := (t, y(t))$ has slope $\gamma'(t_0) = (1, y'(t_0)) = (1, f(t_0, y(t_0)))$ at the point $(t_0, y(t_0))$. Also, recall that the vector $\gamma'(t)$ is tangent to the curve γ. So if we plot the vector $\gamma'(t) = (1, f(t, y(t)))$ we obtain qualitative behaviour of the solution. For example, for $f(t, y) = t$ we have the following linear vector field $(1, f(t, y)) = (1, t)$ depicted in Figure 1.1:

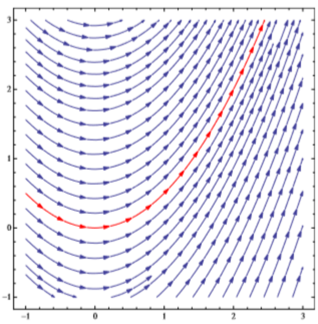

Figure 1.1: the horizontal axis is t and the vertical is $y(t)$.

Every solution curve corresponds to a distinct function of the form $y_c(t) = \frac{t^2}{2} + c$ and the red curve is for $y_0(t) = \frac{t^2}{2}$ (i.e when $c = 0$). The arrows are pointing in the direction $(1, t)$. The power of this method is that for ODEs for which we do not have an explicit solution we are still able to plot the vector field to determine behaviour of the solution function. For example, for the competing species equations (9.4, ex.1)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x - y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{4}(3 - 4y - 2x)$$

we don't have an exact solution but we do have a vector field diagram that we will keep returning to (see Figure 1.2):

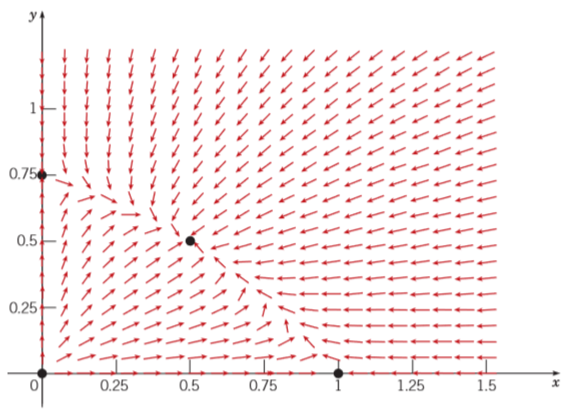

Figure 1.2: the horizontal axis is $x(t)$ and the vertical is $y(t)$.

Method formal steps

1. First draw the curves $(x, y(x))$ where $f(x, y) = 0$ (called the **equilibrium solutions** or **critical points**).
2. These will separate the regions where $f(x, y) > 0$ and $f(x, y) < 0$ and in turn where arrows point up and down respectively. Evaluate f at a point to decide what sign it has.
3. Next draw the curves where $f(x, y) = c$ for fixed constant c.

Examples

- Consider the equation $y' = f(x, y) = y(x - y)$. We illustrate the above formal steps to obtain the diagram depicted in Figure 1.3

  1. We have that $y(x - y) = 0 \Rightarrow$ the curves are the $y = 0$ (x-axis) and the line $x = y$.
  2. We have $y(x - y) > 0$ when $y > 0$ and $x > y$ or $y < 0$ and $x < y$. Therefore, the arrows are pointing up: below the $x = y$ line in $y > 0$ and above $x = y$ in $y < 0$.
  3. So we observe that if the initial condition satisfies $y_0(x_0 - y_0) > 0$ and $y_0 > 0$ then $y(x) \to +\infty$ and similarly for the other sign.

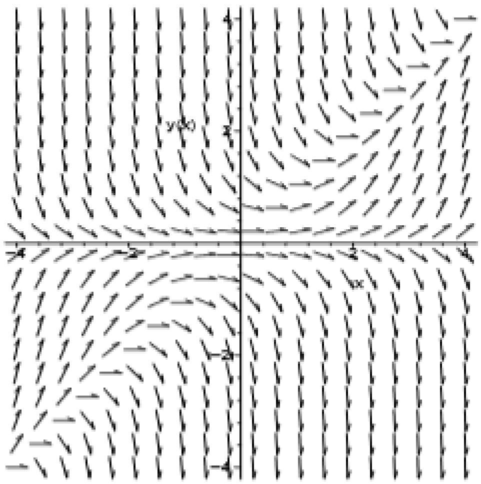

Figure 1.3: Direction field of $y' = y(x - y)$.

  4. En route to studying the competing species we will need the logistic equation (2.5):

$$\frac{dy}{dt} = r\Big(1 - \frac{y}{K}\Big)y,$$

  where $r > 0$ is called the intrinsic growth rate and K is the saturation level.

  (a) First we find the equilibrium solutions: $r\big(1 - \frac{y}{K}\big)y = 0 \Rightarrow y = K$ or $y = 0$. So the equilibrium solutions are $\{\varphi_1(t), \varphi_2(t)\} = \{0, K\}$.

  (b) We have $\big(1 - \frac{y}{K}\big)y > 0$ when $K > y$ and $y > 0$ or $K < y$ and $y < 0$.

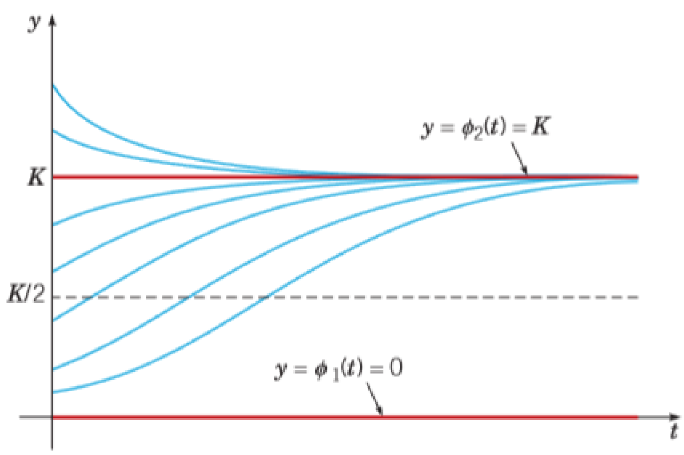

Figure 1.4: Direction field for the logistic equation.

(c) So we observe that if initially $y_0 > 0$ then $y \to K$ (i.e. will limit to the saturation level.)

2 Methods for first order

Given a specific first-order differential equation to be solved, we can attack it by means of the following steps:

1. Is it separable? If so, separate the variables and integrate.
2. Is it linear? That is, can it be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = Q(t)?$$

   If so, multiply by the integrating factor $\mu(t) = exp\{\int P(s)\mathrm{d}s\}$.

3. Is it exact? That is, can the equation be written in the form:

$$M(t,y)\mathrm{d}t + N(t,y)\mathrm{d}y = 0$$

   with $M_y = N_t$?

4. If the equation is not exact, do we have either $\frac{M_y - N_x}{N}$ or $\frac{M_y - N_x}{M}$ being a function of only x, y respectively? If so then it can be made exact.
5. If the equation as it stands is not separable, linear, or exact, is there a plausible substitution that will make it so?

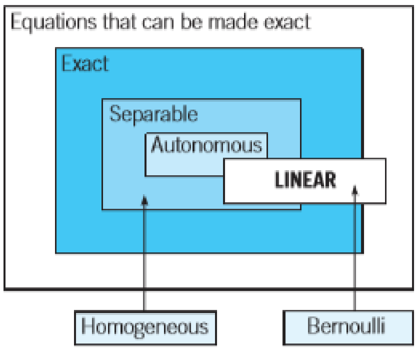

Figure 2.1: Summarizing diagram of all the methods

2.1 Method 1: Separable equations

An equation is called **separable** if we can factor $f(t,y) = f_1(t) \cdot f_2(y)$ for some functions f_1 and f_2 that only depend on t and y respectively. Assuming $f_2(y)$ is not 0 we obtain:

$$y' = f_1(x) \cdot f_2(y) \Leftrightarrow M(t) + N(y)y' = 0$$

where $M(t) := -f_1(t), N(y) := \frac{1}{f_2(y)}$.

Method formal steps

1. Separate variables to either side

$$M(t) + N(y)\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow N(y)\mathrm{d}y = -M(t)\mathrm{d}t$$

2. Integrate both sides

$$\int N(y)\mathrm{d}y = -\int M(t)\mathrm{d}t.$$

Example-presenting the method

Let $P(t)$ be the number of dollars in a savings account at time t and suppose that the interest is compounded continuously at an annual interest rate r(t), that varies in time. Then, after Δt units of time, we expect, provided the interest rate does not change too much over a small period of time, that we have obtained $\frac{\Delta t}{1}$ of the total amount of annual interest, $r(0)$, which updates the amount in the savings account at time Δt to be:

$$P(\Delta t) = (\Delta t \cdot r(0))P(0).$$

In particular, through similar reasoning we see that for any time t we have

$$P(t+\Delta t) - P(t) = (t+\Delta t)\cdot r(t)P(t) - t\cdot r(t)P(t) = (\Delta t)r(t)P(t)$$

provided $r(t)$ does not change too much over the time period $[t, t+\Delta t]$. In short, we obtain $\Delta P = r(t)P(t)\Delta t$ or in continuously updated time:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = r(t)P(t).$$

Example:

Assume that $r(t) = t^2$ and $P(0) = \$\,10^3$.

1. We separate

$$\frac{\mathrm{d}P}{\mathrm{d}t} = r(t)P \Rightarrow \frac{1}{P}\mathrm{d}P = r(t)\mathrm{d}t$$

2. We integrate

$$\int \frac{1}{P}\mathrm{d}P = \int t^2\mathrm{d}t \Rightarrow$$
$$\ln|P| = \frac{t^3}{3} + c$$

for some constant c. Since $P \geq 0$ we obtain

$$P = \$\,exp\left\{\frac{t^3}{3} + c\right\}.$$

3. Plugging in the initial condition we get

$$P(t) = \$\, 10^3 \cdot exp\left\{\frac{t^3}{3}\right\}.$$

General result:

If M, N are continuous, we can obtain an implicit solution by a clever use of chain-rule (§2.1). First lets recall the definition of an implicit solution as well as the implicit function theorem. An **implicit equation** is of the form $f(t, y(t)) = 0$ for some function f, and $y(t)$ is the **implicit solution**. There is an existence result for such equations called the **Implicit function theorem** which we state here in a simple form:

Theorem 2.1. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable (i.e. both f and its derivatives are continuous) and $\partial_{x_2} f(x_1, x_2) \neq 0$, then there exists function $y : \mathbb{R} \to \mathbb{R}$ s.t.*

$$f(t, y(t)) = 0.$$

Separable equation

The equations $M(t) + N(y)y' = 0$ with $y(t_0) = y_0$, where M and N are continuous, have implicit solutions of the form

$$\int_{t_0}^{t} M(s)\mathrm{d}s + \int_{y_0}^{y} N(x)\mathrm{d}x = 0.$$

Proof.

1. Let functions H_M, H_N be antiderivatives of M, N respectively (i.e. $H_M'(t) = M(t)$, $H_N'(y) = N(y)$). Then we can rewrite our ODE as

$$H_M'(t) + H_N'(y)\frac{dy}{dt} = 0.$$

2. For the second term we note that by chain rule we have $\frac{d}{dt}(H_N(y)) = H_N'(y)\frac{dy}{dt}$ and so we can rewrite the equation as:

$$0 = H_M'(t) + H_N'(y)\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(H_M(t) + H_N(y)).$$

3. This implies that $H_M(t) + H_N(y) = c$ for some constant c. Therefore, by the fundamental theorem of calculus we have

$$\int_{t_0}^{t} M(s)\mathrm{d}s + \int_{y_0}^{y} N(x)\mathrm{d}x = H_M(t) - H_M(t_0) + H_N(y) - H_N(y_0) = 0$$

where the last equation follows by using $y(t_0) = y_0$ to give $c = H_M(t_0) + H_N(y_0)$. □

Examples

- It can even tackle non-linear equations:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t-5}{y^2}, \quad y(0) = 1.$$

We first separate and integrate

$$\int y^2 \mathrm{d}y = \int (t-5)\mathrm{d}t.$$

This gives

$$\frac{y^3}{3} = \frac{t^2}{2} - 5t + c$$

and using that $y(0) = 1$ we obtain

$$y(t) = \left(\frac{3t^2}{2} - 15t + 1\right)^{1/3}.$$

- Restricted solution:

$$y' = \frac{2x-3}{y}, \quad y(0) = 2$$

We separate and integrate to get

$$\int y\,\mathrm{d}y = \int (2x-3)\mathrm{d}x \quad \Rightarrow \quad \frac{y^2}{2} = x^2 - 3x + c.$$

Using the initial condition we obtain

$$y = \sqrt{2(x^2 - 3x + 2)} = \sqrt{2(x-1)(x-2)}$$

and since the square root is only defined for positive numbers, we require

$$x > 2 \text{ or } x < 1.$$

- Asymptotic solution:

$$\frac{dy}{dt} = t(1 + b \cdot y), \quad y(0) = 0$$

for $b \neq 0$. We separate and integrate

$$\frac{1}{b} \ln|1 + b \cdot y| = \frac{t^2}{2} + c$$

using the initial condition we obtain $c = 0$ which tells us that

$$\frac{1}{b} \ln|1 + b \cdot y| = \frac{t^2}{2} \geq 0$$

which tells us that $|1 + b \cdot y| \geq 1 > 0$ and so $1 + b \cdot y$ does not change sign. Observe that by the initial condition we have $1 + b \cdot y \geq 1$ and so we obtain

$$\frac{1}{b} \ln(1 + b \cdot y) = \frac{t^2}{2} + c$$

and after solving for y we obtain

$$y = \frac{1}{b}\left(exp\left\{b\left(\frac{t^2}{2}+c\right)\right\}-1\right).$$

So the asymptotic behaviour, as $t \to \pm\infty$, depends on b:

$$\begin{aligned} y &\to +\infty \text{ if } b > 0 \\ y &\to \frac{-1}{b} \text{ if } b < 0 \ . \end{aligned}$$

- An example with implicit solution:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t^2+1}{\cos(y)+e^y}, \quad y(0) = \pi.$$

We separate and integrate

$$\begin{aligned} \int (\cos(y)+e^y)\mathrm{d}y &= \int \left(t^2+1\right)\mathrm{d}t \\ \Rightarrow \sin(y)+e^y &= \frac{t^3}{3}+t+c. \end{aligned}$$

Using $y(0) = \pi$ we get

$$\sin(y)+e^y = \frac{t^3}{3}+t+e^\pi.$$

This equation cannot be directly solved in terms of y. One only has an implicit solution that is obtained numerically.

Applied example

- Let $X(t)$ denote the national product, $K(t)$ the capital stock and $L(t)$ the number of workers in a country at time t. We assume the following relations:

$$X = AK^{1-\alpha}L^\alpha, \quad K' = sX, \quad L = L_0 e^{\lambda t},$$

where A, s, L_0, λ are positive constants and $0 < \alpha < 1$ is called elasticity. The first equation is the Cobb-Douglas production model. The second equation says that aggregate investment is proportional to output. The third equation says that the labour forces grows exponentially. Using these three we obtain the equation

$$K' = sX = ce^{\alpha\lambda t}K^{1-\alpha},$$

where $c := AsL_0^\alpha$. We also let K_0 denote the initial capital stock (the stock at time $t = 0$). We first separate variables and integrate:

$$\int K^{\alpha-1}\mathrm{d}K = \int ce^{\alpha\lambda t}\mathrm{d}t.$$

This gives:

$$\frac{K^\alpha}{\alpha} = c\frac{e^{\alpha\lambda t}}{\alpha\lambda}+C.$$

We find the constant C by plugging in the initial condition, so we get $C := \frac{K_0^\alpha}{\alpha} - \frac{sAL_0^\alpha}{\alpha\lambda}$

and in turn

$$K = \left[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(e^{\alpha\lambda t} - 1)\right]^{1/\alpha}.$$

Next we study the asymptotic behaviour of the ratio $\frac{K}{L}$ (also called the capital-labor ratio) as $t \to +\infty$:

$$\frac{K}{L} = \frac{1}{L_0 e^{\lambda t}}\left[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(e^{\alpha\lambda t} - 1)\right]^{1/\alpha}.$$

For simplicity we will first compute the asymptotic behaviour of

$$\left(\frac{K}{L}\right)^\alpha = \frac{1}{L_0^\alpha e^{\lambda\alpha t}}\left[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(e^{\alpha\lambda t} - 1)\right].$$

We note that the only surviving term as t tends to $+\infty$ is the following:

$$\left(\frac{K}{L}\right)^\alpha \approx \frac{1}{L_0^\alpha e^{\lambda\alpha t}}\frac{sAL_0^\alpha}{\lambda}e^{\alpha\lambda t} = \frac{sA}{\lambda}.$$

Therefore, the capital-labor ratio converges to

$$\lim_{t\to\infty}\frac{K}{L} = \left(\frac{sA}{\lambda}\right)^{1/\alpha}.$$

This means that in the long term the national product per worker will be approximately constant

$$\frac{X}{L} = \frac{AK^{1-\alpha}L^\alpha}{L} = A\left(\frac{K}{L}\right)^{1-\alpha} \approx A\left(\frac{sA}{\lambda}\right)^{\frac{1-\alpha}{\alpha}} = A\left(\frac{sA}{\lambda}\right)^{\frac{1}{\alpha}-1}.$$

2.2 Method 2: Linear and Integrating factor

An equation is called linear if it is of the form:

$$y' + p(t)y = g(t)$$

for continuous functions p, g where p is assumed to not change signs.

Method formal steps

1. Starting from $y' + p(t)y = g(t)$, we multiply both sides by a function $\mu(t)$ that we will determine later:

$$\mu(t)\cdot y' + \mu(t)p(t)\cdot y = \mu(t)g(t).$$

2. If we had $\mu' = \mu(t)p(t)$ then observe that we can use the product rule:

$$\mu\cdot y' + \mu'\cdot y = \mu g(t) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = \mu g(t).$$

3. Therefore,

$$y(t) = \frac{1}{\mu(t)}\int_0^t \mu(s)g(s)\mathrm{d}s + c.$$

4. To find the desired function $\mu(t)$ we use the imposed condition, $\mu'(t) = \mu(t)p(t)$, to get:

$$\mu(t) = exp\left\{\int_0^t p(s)\mathrm{d}s\right\}.$$

Note that we ignore the constant of integration in step 4 since we just want to find some function satisfying $\mu'(t) = \mu(t)p(t)$ and not all such functions.

For linear equations we can also draw a direction field to obtain qualitative behaviour when an exact solution is not possible. The idea is to draw the curves along which $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and then study the regions where $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$.

1. As explained above we first draw the curves along which

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) = 0 \quad\Rightarrow\quad p(t)y = g(t) \quad\Rightarrow\quad y = \frac{g(t)}{p(t)}$$

provided that $p(t)$ is not 0.

2. Then we identify the regions where

$$p(t)y - g(t) > 0, p(t)y - g(t) < 0 \quad Rightarrow \;\; y > \frac{g(t)}{p(t)}, \;\; y < \frac{g(t)}{p(t)}$$

respectively, provided that $p(t) > 0$ (the inequalities are reversed otherwise).

Example-presenting the method

A rock contains two radioactive isotopes R_1, R_2 with R_1 decaying into R_2 with rate $5e^{-10t}$ kg/sec. So if $y(t)$ is the total mass of R_2, we obtain:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \text{rate of creation of } R_2 \text{ - rate of decay of } R_2\\ &= 5e^{-10t} - ky(t),\end{aligned}$$

where $k > 0$ is the decay constant for R_2. Also assume that $y(0) = 40\,\mathrm{kg}$. Lets start by drawing the direction field to guess the solution:

1. The tangent is zero along the curve (equilibrium solution) so setting $\frac{dy}{dt} = 0$ and solving for y gives:

$$y(t) = \frac{5e^{-10t}}{k}$$

2. From this we see that $5e^{-10t} - ky(t) = \frac{dy}{dt} > 0$ when $\frac{5}{k}e^{-10t} > y$. Similarly we see that $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$ when $\frac{5}{k}e^{-10t} < y$.

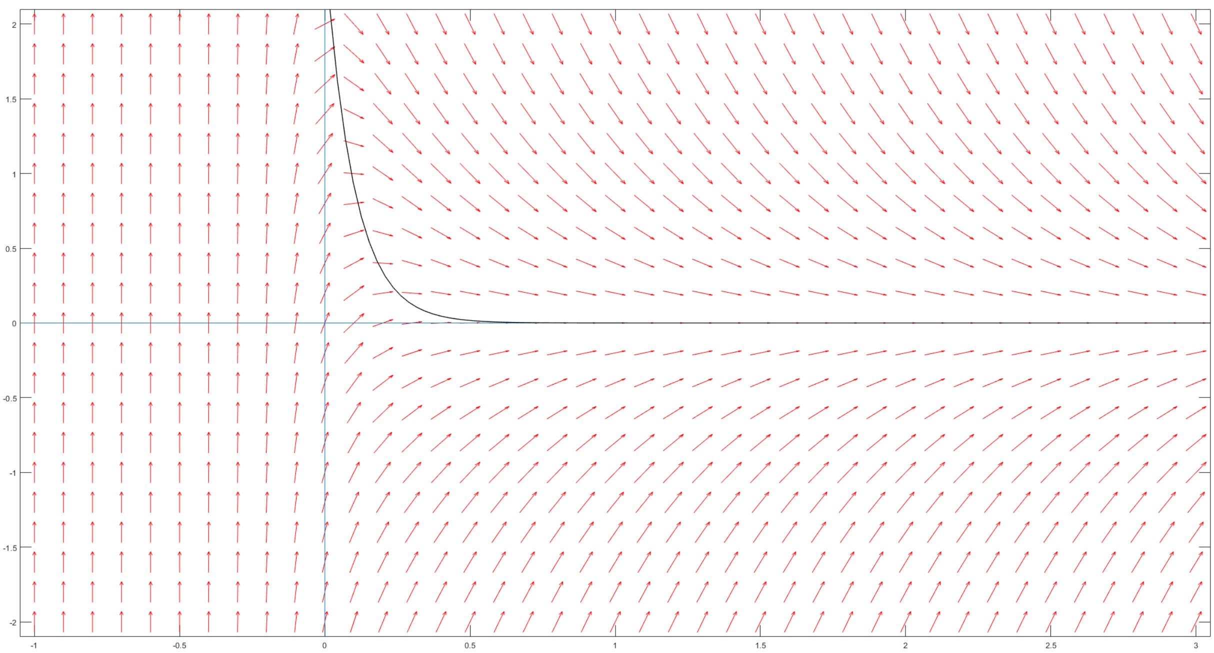

Figure 2.2: The black line is the equilibrium solution

3. So we observe that if $t > 0$, the solution $y \to 0$ as $t \to +\infty$.

Next we find the explicit solution:

1. We first multiply by the function $\mu(t)$, which we will determine specifically in the next step,
$$\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \mu(t)ky(t) = \mu(t)5e^{-10t}.$$
2. We require $\mu'(t) = \mu(t)k$, which can be easily solved to give:
$$\mu(t) = e^{kt}.$$
3. Then by product rule we have
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{kt}y(t)) = 5e^{(k-10)t} \Rightarrow y(t) = 5e^{-10t} + e^{-kt}c = 5e^{-10t} + 35e^{-kt}$$
where we used the initial condition $y(0) = 40\,\mathrm{kg}$ to determine that $c = 35$.
4. Therefore, we indeed obtain that $y \to 0$ as $t \to +\infty$.

General result

We can obtain a solution by a clever use of product-rule (§2.1).

Integrating factor

The $y' + p(t)y = g(t)$ have solutions of the form

$$y(t) = \frac{1}{\mu(t)} \int_0^t \mu(s)g(s)ds + c,$$

where $\mu(t) = exp\Big\{\int_0^t p(s)ds\Big\}$, provided that g and p are continuous.

Proof.

$$\mu(t) \cdot (y' + p(t)y) = \mu(t) \cdot g(t) \Rightarrow \mu(t) \cdot y' + \mu(t)p(t) \cdot y = \mu(t)g(t).$$

We note that if we pick $\mu(t)$ so that $\mu'(t) = \mu(t)p(t)$, we can then rewrite

$$\mu \cdot y' + \mu' \cdot y = \mu(t)g(t) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = \mu(t)g(t).$$

Therefore,

$$y(t) = \frac{1}{\mu(t)} \int_0^t \mu(s)g(s)\mathrm{d}s + c.$$

To determine $\mu(t)$ we use $\mu'(t) = \mu(t)p(t)$

$$\begin{aligned} \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} &= \mu(t)p(t) \\ \Rightarrow \frac{1}{\mu(t)}\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} &= p(t) \end{aligned}$$

we integrate both sides

$$\begin{aligned} \int_0^t \frac{1}{\mu(s)}\mathrm{d}\mu(s) &= \int_0^t p(s)\mathrm{d}s \\ \Rightarrow \ln|\mu(t)| - \ln|\mu(0)| &= \int_0^t p(s)\mathrm{d}s + c' \\ \Rightarrow \ln\left(\frac{|\mu(t)|}{|\mu(0)|}\right) &= \int_0^t p(s)\mathrm{d}s + c' \\ \Rightarrow \frac{|\mu(t)|}{|\mu(0)|} &= exp\Big\{\int_0^t p(s)\mathrm{d}s + c'\Big\} \\ \Rightarrow |\mu(t)| &= |\mu(0)|exp\Big\{\int_0^t p(s)\mathrm{d}s + c'\Big\} \end{aligned}$$

where, for simplicity, we set $\mu(0) = 1$ as well as $c' = 0$. Observe that since the exponential function is non-negative then the sign of $\mu(t)$ is determined by $\mu(0)$ (by continuity). Since we chose $\mu(0) > 0$ then we obtain

$$\mu(t) = exp\Big\{\int_0^t p(s)\mathrm{d}s\Big\}$$

$\square$

Examples

- Consider the equation
$$y' - 2y = t^2 e^{2t}.$$
First we look at the equilibrium solution:

  1. We have equilibrium solution $y' = 0 \Rightarrow y = -\frac{1}{2}t^2 e^{2t}$.
  2. We have positive/growing region $y' > 0 \Rightarrow t^2 e^{2t} > -2y$

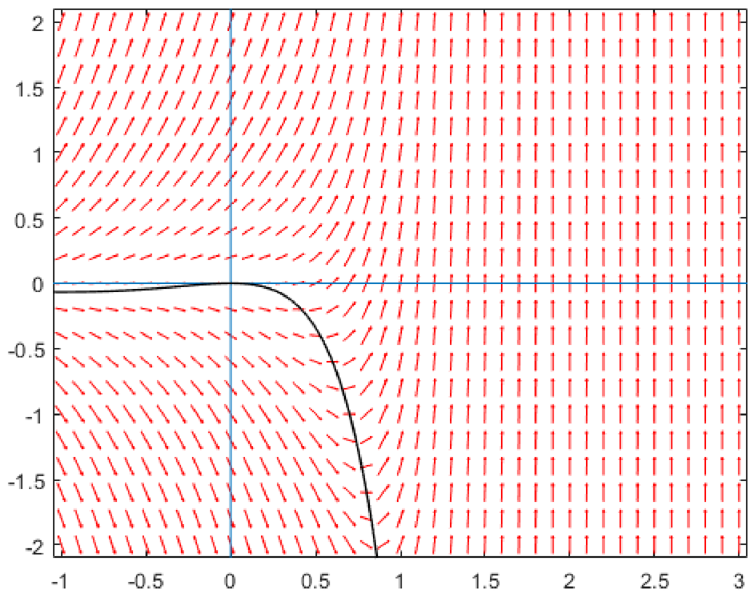

Figure 2.3: The black line is the equilibrium solution

Next we obtain the solutions (notice that there is no initial condition so we anticipate that we will have many solutions): 1^{st}: we multiply by $\mu(t)$
$$\mu y' - 2\mu y = \mu t^2 e^{2t}.$$
2^{nd}: We solve (and choose the constant of integration to be 0)
$$\mu'(t) = -2\mu(t) \Rightarrow \mu = exp\{-2t\}.$$
3^{rd}: We obtain
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-2t}y) = e^{-2t}t^2 e^{2t} = t^2$$
we integrate both sides to get
$$y(t) = e^{2t}\left(\frac{t^3}{3} + c\right).$$

- Consider equation (§2.1)
$$y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}.$$
We start with the 1^{st} step of multiplying by $\mu(t)$ of our choice
$$\mu(t)y' + \mu(t)\frac{1}{2}y = \mu(t)\frac{1}{2}e^{t/3}.$$

2^{nd} step: We observe that to make use of product rule we need

$$\begin{aligned}\mu'(t) &= \frac{1}{2}\mu(t)\\ \Rightarrow \ln|\mu(t)| &= \frac{1}{2}t + c'\\ \Rightarrow \mu(t) &= e^{t/2}\end{aligned}$$

by setting $c' = 0$ and assuming $\mu(t) \geq 0$ to simplify.
3^{rd} step: by product rule we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)\cdot y) &= \mu(t)\frac{1}{2}e^{t/3} = \frac{1}{2}e^{\frac{5t}{6}}\\ \Rightarrow y(t) &= e^{-t/2}(\frac{3}{5}e^{\frac{5t}{6}} + c)\\ \Rightarrow y(t) &= \frac{3}{5}e^{\frac{t}{3}} + ce^{-\frac{t}{2}}.\end{aligned}$$

Applied example

- Returning to the compounded interest example, suppose that we also have deposits and withdrawals with rates $d(t), w(t)$ respectively. Then the equation will be

$$P' = r(t)P + d(t) - w(t).$$

To see that this is the expected differential equation observe that at time t, for a small amount of time Δt, we expect

$$P(t+\Delta t) \quad \approx \quad \underbrace{P(t)}_{\substack{\text{amount at}\\ \text{time } t}} \; + \; \underbrace{r(t)(\Delta t)P(t)}_{\substack{\text{interest earned}\\ \text{at time } t}} \; + \; \underbrace{d(t)\Delta t}_{\substack{\text{deposit}\\ \text{amount}}} \; - \; \underbrace{w(t)\Delta t}_{\substack{\text{withdrawal}\\ \text{amount}}}$$

where we have assumed that the interest rate, the deposit rate, and withdrawal rate do not change too much in the interval $[t, t+\Delta t]$. Notice that the presence of Δt in $d(t)\Delta t$ and $w(t)\Delta t$ is because $d(t)$ and $w(t)$ represent the *rates* for which we are depositing and withdrawing at time t respectively. Subtracting $P(t)$ and dividing by Δt gives

$$\frac{\Delta P}{\Delta t} \approx r(t)P(t) + d(t) - w(t)$$

and so letting Δt tend to 0 leads to the differential equation.

1. We first multiply by unknown factor μ

$$\mu P' - \mu r(t)P = \mu(d(t) - w(t)).$$

2. Then we obtain μ

$$\mu' = -\mu r(t) \Rightarrow \mu = exp\Big\{-\int_0^t r(s)\mathrm{d}s\Big\}.$$

3. Therefore, by product rule we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(P\mu) = exp\bigg\{-\int_0^t r(s)\mathrm{d}s\bigg\}(d(t) - w(t))$$

$$\Rightarrow P(t)exp\bigg\{-\int_0^t r(s)\mathrm{d}s\bigg\} = P(0) + \int_0^t exp\bigg\{-\int_0^x r(s)\mathrm{d}s\bigg\}(d(x) - w(x))\mathrm{d}x$$

$$\Rightarrow P(t) = exp\bigg\{\int_0^t r(s)\mathrm{d}s\bigg\}\bigg(P(0) + \int_0^t exp\bigg\{-\int_0^x r(s)\mathrm{d}s\bigg\}(d(x) - w(x))\mathrm{d}x\bigg)$$

4. This equation states that the present discounted value on the left, is the sum of the initial assets $P(0)$ plus the present discounted value of deposits minus withdrawals.

- When the price of a good is p, the total demand is $D(p) = a - bp$ and the total supply is $S(p) = \alpha + \beta p$, where a, b, α, and β are positive constants. Observe that the slope of the linear function respresenting D is negative while the slope of the linear function representing S is positive. This matches the expectation that when price is increased consumers are less interested in goods (making goods expensive prevents some from purchasing the item) and that suppliers are more inclined to produce that product (it is advantageous to sell items that generate more profit). When demand exceeds supply, price rises (people are willing to pay more for scarce items), and when supply exceeds demand it falls (common goods are easy to obtain, so consumers will simply look for lower prices). We assume that the speed at which the price changes is proportional to the difference between supply and demand. Specifically

$$p' = \lambda(D(p) - S(p))$$

for $\lambda > 0$.

1. We multiple by μ to get

$$\mu p' + \mu\lambda(b + \beta)p = \lambda\mu(a - \alpha).$$

2. We obtain μ

$$\mu' = \lambda(b + \beta)\mu \Rightarrow \mu = exp\{\lambda(b + \beta)t\}.$$

3. Therefore, by product rule we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu p) = \lambda(a - \alpha)exp\{\lambda(b + \beta)t\} \Rightarrow p(t) = c \cdot exp\{-\lambda(b + \beta)t\} + \frac{(a - \alpha)}{(b + \beta)}.$$

4. So as $t \to +\infty$ the price of this good converges to $\frac{(a-\alpha)}{(b+\beta)}$ (This is the equilibrium position for which demand is equal to supply. That is, the point for which the lines intersect.).

2.3 Method 3: Exact equations

Suppose the function $F(x, y)$ represents some physical quantity, such as temperature, in a region of the xy-plane. Then the level curves of F, where $F(x, y) =$ constant, could be interpreted as isotherms on a weather map (i.e curves on a weather map representing constant temperatures).

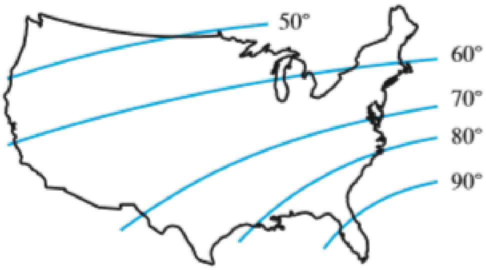

Figure 2.4: Level sets of constant temperature across the US.

Along one of these curves, $\gamma(x) = (x, y(x))$, of constant temperature we have, by Chain rule and the fact that the temperature, F, is constant on these curves:

$$0 = \frac{\mathrm{d}F(\gamma(x))}{\mathrm{d}x} = F_x + F_y \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Multiplying through by $\mathrm{d}x$ we obtain

$$0 = F_x \mathrm{d}x + F_y \mathrm{d}y.$$

Therefore, if we were not given the original function F but only an equation of the form:

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0,$$

we could set $F_x := M(x, y), F_y := N(x, y)$ and then by integrating figure out the original F.

1. First ensure that there is such an F, by checking the *exactness-condition*:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

   This is because if there was such an F, then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

   where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ simply denote the partial derivatives with respect to the variables x and y respectively (where we hold the other variable constant while taking the derivative).

2. Second, integrate M, N with respect to x, y respectively:

$$\int M(x, y) dx = \int F_x(x, y) dx = F(x, y) + a(y)$$
$$\int N(x, y) dy = \int F_y(x, y) dy = F(x, y) + b(x)$$

   for some unknown functions a, b (these play the role of constant of integration when you integrate with respect to a single variable). So to obtain F it remains to determine either a or b.

3. Equate the above two formulas for $F(x, y)$:

$$\int M(x,y)\mathrm{d}x + a(y) = F(x,y) = \int N(x,y)\mathrm{d}y + b(x).$$

4. Since to find F it suffices to determine a or b, pick the integral that is easier to evaluate. Suppose that $\int M(x,y)\mathrm{d}x$ is easier to evaluate. To obtain $a(y)$ we differentiate both expression for F in y (for fixed x):

$$a'(y) = -\int M_y(x,y)\mathrm{d}x + N(x,y)$$

and then integrate in y:

$$a(y) = \int \left[-\int M_y(x,y)dx + N(x,y)\right]\mathrm{d}y + c.$$

Observe that a is only a function of y since if we differentiate the expression we found for a and use step 1 we find that

$$\begin{aligned}\frac{\partial a}{\partial x} &= \int \frac{\partial}{\partial x}\left[-\int M_y(x,y)\mathrm{d}x + N(x,y)\right]\mathrm{d}y\\ &= \int [-M_y(x,y) + N_x(x,y)]\mathrm{d}x\\ &= \int 0\mathrm{d}x\\ &= 0\end{aligned}$$

Example-presenting the method

We measured the velocity field (u,v) of a two-dimensional incompressible flow and the curves are given by $\psi(x,y) = c$ for some yet unknown potential function ψ.

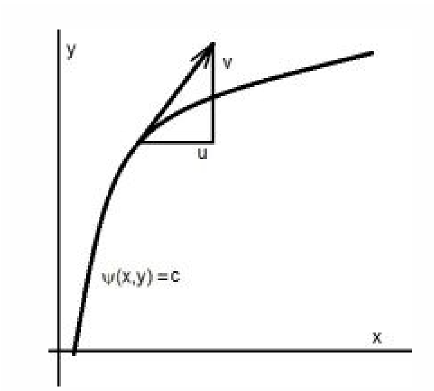

Figure 2.5: Velocity field of two dimensional flow.

For example suppose that $v(x,y) := -\frac{y}{x^2+y^2}, u(x,y) := \frac{x}{x^2+y^2}$ then by incompressibility our equation is:

$$udx - vdy = 0.$$

1. First we check the exactness condition:

$$-v_x = -\left(-\frac{y}{x^2+y^2}\right)_x = -\frac{2xy}{x^2+y^2}$$

$$u_y = -\frac{2xy}{x^2+y^2}.$$

2. Next we integrate u, v in y, x respectively:

$$\psi(x,y) = \int u(x,y)\mathrm{d}x + a(y) = \int \frac{x}{x^2+y^2}dx + a(y) = \frac{1}{2}\ln(x^2+y^2) + a(y)$$
$$\psi(x,y) = \int -v(x,y)dy + b(x) = \frac{1}{2}\ln(x^2+y^2) + b(x)$$

3. Next to obtain a(y) we equate the two ψ-formulas to obtain

$$\frac{1}{2}\ln(x^2+y^2) + a(y) = \frac{1}{2}\ln(x^2+y^2) + b(x)$$
$$\Rightarrow a(y) = b(x)$$
$$\Rightarrow a'(y) = 0$$

   and so $a(y) \equiv$ constant. Since a is constant then b is constant as well.

4. Therefore, the potential field is

$$\psi(x,y) = \frac{1}{2}\ln(x^2+y^2) + c.$$

5. Finally, we obtain the solution $(x, y(x))$ along which $\psi(x, y(x)) = \text{constant} = C$:

$$\frac{1}{2}\ln(x^2+y(x)^2) = C - c \Rightarrow y(x) = \pm\sqrt{exp\{2(C-c)\} - x^2}.$$

6. For example, if we knew that $\psi(0,1) = 0$ then we would obtain $c = 0$. From this we see that the solution of the level set, say for $\psi = 60 = C$, is

$$x^2 + y^2 = exp\{2 \cdot 60\} \Leftrightarrow y(x) = \pm\sqrt{exp\{2 \cdot 60\} - x^2}.$$

   So the level curves are simply concentric circles.

General result:

Exact equation

If the equations $M(x,y) + N(x,y)y' = 0$ satisfy

1. M, N, M_y, N_x are continuous
2. the exactness condition:
$$M_y(x,y) = N_x(x,y)$$

then there exists a function, ψ, such that

$$\psi_x = M \text{ and } \psi_y = N.$$

Thus, along the solutions $\gamma(x) := (x, y(x))$ of the above equation we have:

$$\frac{d\psi(x,y)}{dx} = M(x,y)dx + N(x,y)dy = 0$$

and in turn we obtain the implicit solution $(x, y_c(x))$ for the following level set:

$$\psi(x, y_c(x)) = c.$$

Examples

- Consider the equation

$$(1 + e^x y + xe^x y)\mathrm{d}x + (xe^x + 2)\mathrm{d}y = 0.$$

First we check the exactness

$$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x}.$$

Second we integrate N(x,y)

$$F(x, y) = \int N(x, y)\mathrm{d}y + b(x) = (xe^x + 2)y + b(x).$$

Third, it remains to obtain b(x). We differentiate F in x

$$\begin{aligned} 1 + e^x y + xe^x y &= M = F_x = (xe^x + e^x)y + b'(x) \\ \Rightarrow b(x) &= \int [(1 + e^x y + xe^x y) - (xe^x + e^x)y]\mathrm{d}x + c = x + c. \end{aligned}$$

Therefore,

$$F(x, y) = (xe^x + 2)y + x + c.$$

So for a fixed level set $F \equiv C + c$ we obtain the solution:

$$y(x) = \frac{C - x}{xe^x + 2}.$$

- Consider the equation

$$(6xy - y^3)\mathrm{d}x + (4y + 3x^2 - 3xy^2)\mathrm{d}y = 0$$

First we check exactness

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}.$$

Second, we integrate $M(x, y)$

$$F(x, y) = \int M(x, y)\mathrm{d}x + a(y) = 3x^2 y - y^3 x + a(y).$$

Third we obtain $a(y)$:

$$\begin{aligned} a'(y) &= F_y - (3x^2 y - y^3 x)_y \\ &= 4y + 3x^2 - 3xy^2 - 3x^2 + 3y^2 x \\ &= 4y \\ \Rightarrow a(y) &= 2y^2. \end{aligned}$$

Therefore,

$$F(x,y) = 3x^2y - y^3x + 2y^2 + c.$$

So for a fixed level set $F \equiv C + c$ we obtain the implicit solution:

$$C = 3x^2y - y^3x + 2y^2.$$

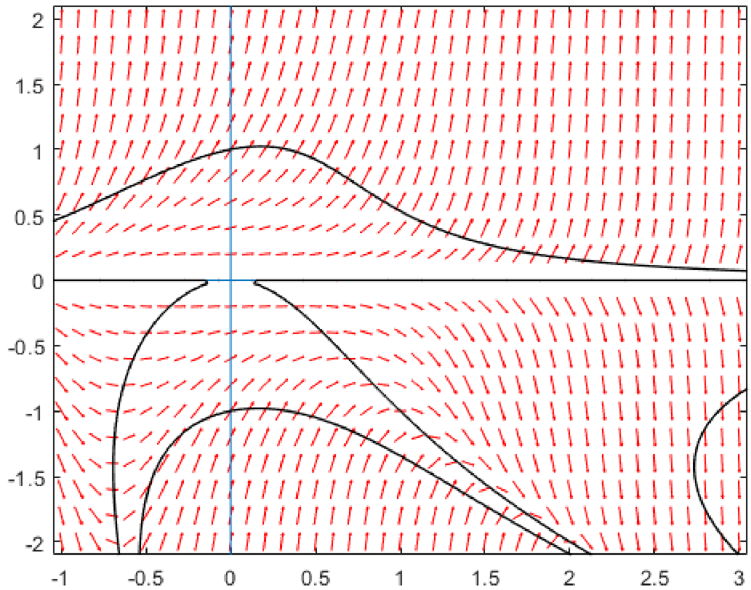

Figure 2.6: Direction field for $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$.

Applied examples

A geometric problem occurring often in engineering is that of finding a family of curves (orthogonal trajectories) that intersects a given family of curves orthogonally at each point. For example, we may be given the lines of force of an electric field and want to find the equation for the equipotential curves. Consider the level sets $F(x,y) = k$ then their slope is given by

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = -\frac{F_x}{F_y}.$$

Recall that two curves y_1, y_2 are perpendicular if their derivatives multiply to -1 (This definition comes from the fact that two lines are perpendicular if their slopes multiply to be -1. Hence, if we multiply the tangent vectors (slopes of tangent lines) then perpendicularity should be that the derivatives vectors multiply to be -1.) and so:

$$\frac{dy_2}{dx} = \frac{-1}{\frac{\mathrm{d}y_1}{\mathrm{d}x}} = \frac{F_y}{F_x} \Rightarrow F_y dx - F_x dy = 0$$

. Thus, by solving this equation we will obtain the implicit solution for the perpendicular curve. Consider $F(x,y) = xy$, then $xy = k$ are hyperbolas. We will show that $x^2 - y^2 = k$ are the curves perpendicular to them. First we check exactness for the equation

$$0 = F_y \mathrm{d}x - F_x \mathrm{d}y = x\mathrm{d}x - y\mathrm{d}y$$

we indeed have $M_y = 0 = N_x$. Second we evaluate F by integrating M

$$F(x, y) = \int M \mathrm{d}x + a(y) = \frac{x^2}{2} + a(y).$$

Third we differentiate $a(y)$

$$a'(y) = -y \Rightarrow a(y) = -\frac{y^2}{2} + c.$$

Therefore, for $F(x, y) = k$ we obtain

$$k = \frac{x^2}{2} - \frac{y^2}{2}.$$

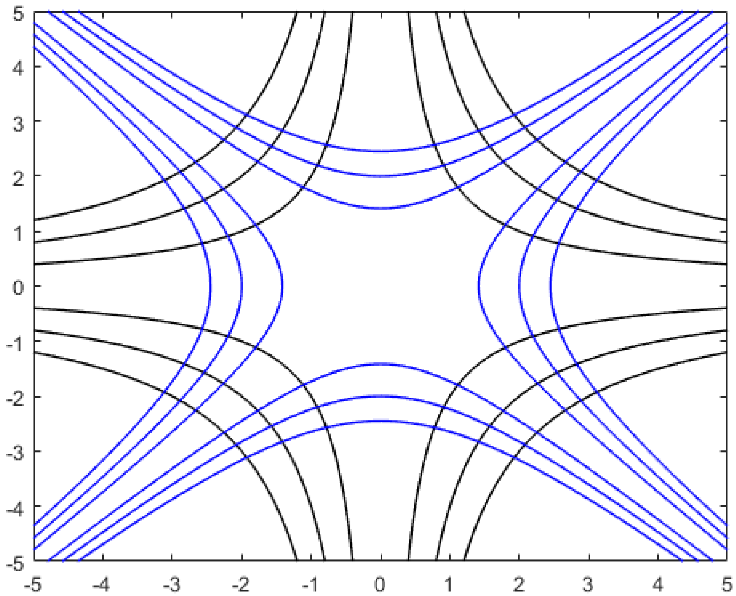

Figure 2.7: Hyperbolas perpendicular to each other

2.4 Method 4: Integrating factor and Exact

Some equations are close to being exact (they become exact after multiplying by an integrating factor). For example, the equation

$$(2y - 6x)\mathrm{d}x + (3x - 4x^2y^{-1})\mathrm{d}y = 0$$

is not exact but if we multiply both sides by $\mu(x, y) = xy^2$ we obtain

$$xy^2(2y - 6x)\mathrm{d}x + xy^2(3x - 4x^2y^{-1})\mathrm{d}y = 0$$

an exact equation. You can check that $F(x, y) = x^2y^3 - 2x^3y^2 + c$ solves the modified problem.

Method formal steps

1. If the equation is not exact, check whether

$$\frac{M_y - N_x}{N}$$

is a function of only x or

$$\frac{N_x - M_x}{M}$$

is a function of only y.

2. Then in the first case the correct integrating factor is

$$\mu(x) = exp\left\{\int \frac{M_y - N_x}{N}\mathrm{d}x\right\}$$

and in the second case

$$\mu(y) = exp\left\{\int \frac{N_x - M_y}{M}\mathrm{d}y\right\}$$

Example-presenting the method

In the context of the above example on perpendicular trajectories, suppose that we want to find the curves perpendicular to concentric circles. Then as explained above they satisfy the equation:

$$2y\mathrm{d}x - 2x\mathrm{d}y = 0.$$

This equation is not exact because $M_y = 2 \neq -2 = N_x$.

1. First we check the ratio

$$\frac{M_y - N_x}{N} = \frac{2+2}{-2x} = -\frac{2}{x}$$

and so indeed this ratio only depends on x and it is continuous away from the origin.

2. We integrate to obtain μ

$$\mu(x) = exp\left\{\int \frac{-2}{x}\mathrm{d}x\right\} = exp\left\{\ln\left(x^{-2}\right)\right\} = x^{-2}.$$

3. Therefore, by multiplying by μ we obtain an exact equation:

$$2yx^{-2}\mathrm{d}x - 2x^{-1}\mathrm{d}y = 0.$$

4. Next we carry out the exact equation steps. First we obtain F

$$F(x,y) = \int M\mathrm{d}x + a(y) = -2yx^{-1} + a(y).$$

5. Second we obtain the function a(y) by differentiating in y

$$-2x^{-1} + a'(y) = F_y = N(x,y) = -2x^{-1} \Rightarrow a(y) = c.$$

6. Therefore, we obtain F

$$F(x,y) = -2yx^{-1} + c$$

and note that for its level sets $F(x,y) = k$ we have

$$y = \frac{(c-k)}{2}x = m \cdot x.$$

In other words, lines going through the origin are perpendicular to concentric circles (as depicted in Figure 2.8).

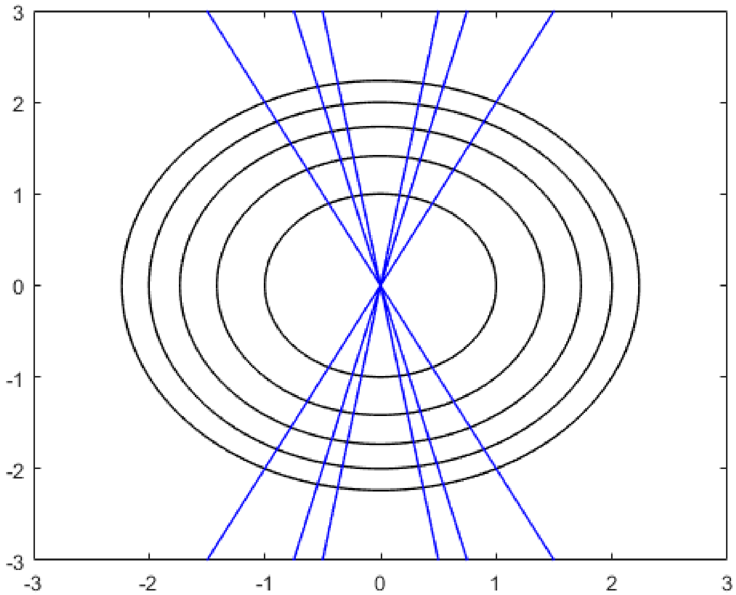

Figure 2.8: Lines perpendicular to circles

General result:

The equations $M(x,y)+N(x,y)y'=0$ can be made exact if either

$$\frac{M_y-N_x}{N}$$

is continuous and depends only on x or

$$\frac{N_x-M_x}{M}$$

is continuous and depends only on y.

Proof. We are searching for a factor $\mu(x,y)$ that will satisfy the exact condition

$$\frac{\partial}{\partial y}(\mu(x,y)M(x,y)) = \frac{\partial}{\partial x}(\mu(x,y)N(x,y))$$
$$\Rightarrow M\mu_y - N\mu_x = (N_x - M_y)\mu.$$

This equation can be simplified if we assume that μ and $\frac{M_y-N_x}{N}$ are functions of only x:

$$0-\mu_x(x) = \frac{N_x-M_y}{N}\mu(x).$$

Therefore, we obtain

$$\mu(x) = exp\left\{\int \frac{M_y-N_x}{N}\mathrm{d}x\right\}.$$

$\square$

Examples

- Consider the equation

$$(2x^2 + y)\mathrm{d}x + (x^2y - x)\mathrm{d}y = 0.$$

We first see that exactness is not satisfied

$$M_y = 1 \neq 2xy - 1 = N_x.$$

First we check if the first ratio only depends on x:

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2y - x} = \frac{-2}{x}.$$

Indeed it does, and so we can multiply by

$$\mu(x) = exp\left\{\int \frac{-2}{x}dx\right\} = x^{-2}.$$

This gives us the exact equation

$$(2 + yx^{-2})\mathrm{d}x + (y - x^{-1})\mathrm{d}y = 0.$$

Then we repeat the steps for exact equations. First we obtain F

$$F(x,y) = \int M dx + a(y) = 2x - yx^{-1} + a(y).$$

Then we need to obtain $a(y)$:

$$-x^{-1} + a'(y) = F_y = N = y - x^{-1} \Rightarrow a(y) = \frac{y^2}{2} + c$$

and so F is

$$F(x,y) = 2x - yx^{-1} + \frac{y^2}{2} + c.$$

Therefore, for $F \equiv C$ we obtain the implicit solution:

$$2x - yx^{-1} + \frac{y^2}{2} = C - c.$$

2.5 Method 5: Substitution methods

If an equation is of the form $y' = f(t,y)$ with f being nonlinear, then one can try to figure out a change of variables $(t,y) \mapsto z(t)$ that gives $z' = g(t,z)$ where now g is linear and thus the above methods apply.

2.5.1 Homogeneous equations

A function $f(t,y)$ is homogeneous of order α (or $\alpha-$homogeneous) if it satisfies:

$$f(\lambda t, \lambda y) = \lambda^\alpha f(t,y)$$

for all $\lambda \in \mathbb{R}$. Equivalently, such functions are of the form:

$$f(t,y) = f\Big(t\cdot 1, t\cdot \frac{y}{t}\Big) = t^{\alpha}\underbrace{f\Big(1,\frac{y}{t}\Big)}_{h\left(\frac{y}{t}\right)} = t^{\alpha}h\Big(\frac{y}{t}\Big) \text{ or}$$

$$f(t,y) = y^{\alpha}f\Big(\frac{t}{y},1\Big) = y^{\alpha}\underbrace{f\Bigg(\frac{1}{\left(\frac{y}{t}\right)},1\Bigg)}_{g\left(\frac{y}{t}\right)} = y^{\alpha}g\Big(\frac{y}{t}\Big).$$

For example, $f(t,y) = t^{\alpha}e^{\frac{y}{t}}$ is homogeneous of order α. If an equation is of the form $M(t,y) + N(t,y)y' = 0$, for continuous functions M, N that are both $\alpha-$homogeneous, we can obtain a solution by a clever use of chain-rule (§2.1).

Homogeneous equations

The equations $M(t,y) + N(t,y)y' = 0$,for $\alpha-$homogeneous M, N, have implicit solutions of the form

$$\int_{z(0)}^{z}\frac{1}{h(x)-x}dx = \ln|t| + c,$$

where $z(t) := \frac{y}{t}$ and $h\big(\frac{y}{t}\big) := -\frac{M(t,y)}{N(t,y)}$.

Proof. 1. We rewrite the equation as

$$y' = -\frac{M(t,y)}{N(t,y)} =: f(t,y)$$

where we note that $f(t,y))$ is $0-$homogeneous due to the ratio i.e. $-\frac{M(t,y)}{N(t,y)} =: h\big(\frac{y}{t}\big)$.

2. Therefore we can rewrite the equation as

$$y' = h\Big(\frac{y}{t}\Big).$$

3. By change of variables $z(t) := \frac{y(t)}{t}$ we can rewrite the equation as

$$t\frac{dz}{dt} + z = y' = h\Big(\frac{y}{t}\Big) = h(z).$$

By rearranging it we get

$$\frac{dz}{dt} = \frac{h(z)-z}{t}$$
$$\Rightarrow \int_{z_0}^{z}\frac{1}{h(x)-x}\mathrm{d}x = \int_{t_0}^{t}\frac{1}{t}\mathrm{d}t + c$$

where $z(t_0) = z_0$. Let $\Phi(z) := LHS = \int_{z_0}^{z}\frac{1}{h(x)-x}\mathrm{d}x$ then we obtain the implicit solution:

$$\Phi(z) = \ln|t| + c. \qquad \square$$

2.5.2 Bernoulli equation

The non-linear equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

are called **Bernoulli equations**. By substituting $v = y^{1-n}$ we obtain the linear equation:

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)P(x)v = (1-n)Q(x)$$

which we can solve by means of an integrating factor.

2.5.3 Riccati Equation

The non-linear equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A(x)y^2 + B(x)y + C(x)$$

are called **Riccati equations**. By substituting $y = f + \frac{1}{v}$, where f is a particular solution of the above, we obtain the linear equation:

$$\frac{dv}{dx} + (B + 2fA)v = -A.$$

2.6 Summary

Given a specific first-order differential equation to be solved, we can attack it by means of the following steps:

1. Is it separable? If so, separate the variables and integrate.
2. Is it linear? That is, can it be written in the form:

   $$\frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = Q(t)?$$

   If so, multiply by the integrating factor $\mu(t) = exp\{\int P(s)\mathrm{d}s\}$.
3. Is it exact? That is,when the equation is written in the form:

   $$M(t,y) + N(t,y)\mathrm{d}y = 0$$

   with $M_y = N_t$.
4. If the equation is not exact, do we have either $\frac{M_y - N_t}{N}$ or $\frac{M_y - N_t}{M}$ being a function of only t, y respectively? If so then it can be made exact.
5. If the equation as it stands is not separable, linear, or exact, is there a plausible substitution that will make it so? For instance, is it homogeneous?

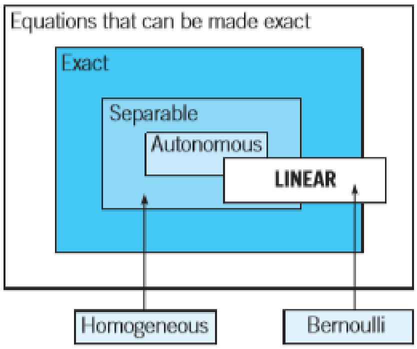

Figure 2.9: Summarizing diagram of all the methods

3 Existence and Uniqueness

If the first order equation $y' = f(t, y)$ satisfies that f, f_y are continuous as functions of (t, y) for $|t| < a, |y| < b$, then there is some interval $t \in [0, h]$ in which there exists a unique solution $y = \varphi(t)$.

3.1 Method Formal steps: Picard's method

Observe that by integrating both sides we obtain by fundamental theorem of calculus:

$$y(t) - y_0 = \int_{t_0}^{t} f(s, y(s))\mathrm{d}s,$$

where $y_0 = y(t_0)$. We will approximate the solution y as follows

1. Let $\varphi_0(t)$ be an initial guess to the solution eg. $\varphi_0(t) \equiv y_0$, then we define the next step as:

$$\varphi_1(t) := y_0 + \int_{t_0}^{t} f(s, \varphi_0(s))\mathrm{d}s.$$

2. Similarly, given φ_n we define

$$\varphi_{n+1}(t) := y_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\mathrm{d}s.$$

Example-Presenting the method

Consider the initial value problem:

$$y' = y, y(0) = 1$$

1. Let $\varphi_0 \equiv 1$ then

$$\varphi_1(t) := 1 + \int_{0}^{t} f(s, \varphi_0(s))\mathrm{d}s = 1 + \int_{0}^{t} 1\mathrm{d}s = 1 + t.$$

2. We repeat

$$\varphi_2(t) := 1 + \int_0^t f(s, \varphi_1(s))\mathrm{d}s = 1 + \int_0^t (1+s)\mathrm{d}s = 1 + t + \frac{t^2}{2}.$$

3. Assume that $\varphi_n(t) := 1 + t + ... + \frac{t^n}{n!}$ then

$$\varphi_{n+1}(t) := 1 + \int_0^t f(s, \varphi_n(s))\mathrm{d}s = 1 + \int_0^t \sum_{k=0}^{n} \frac{t^k}{k!}\mathrm{d}s = 1 + t + ... + \frac{t^{n+1}}{(n+1)!}.$$

4. Therefore, as expected we obtained the approximation to $y(t) = e^t$, which is indeed the solution of the above initial value problem.

3.2 Picard-Lindelöf Theorem

In this section we will prove the Picard-Lindel**ö**f Theorem. To do this we will first reformulate the differential equation into an "integral equation". The reason we do this is because integral are more lenient with respect to regularity (think of how shabby a Riemann integrable function can look) while derivatives are not. This reformulation is the goal of the following lemma:

Integral equation reformulation

Suppose $f(x,t)$ is a continuous function on a domain $(a,b) \times (c,d)$. Then if x is a solution to

$$x'(t) = f(x(t), t), \quad \text{for } t \in (c,d) \quad \text{such that } x(t_0) = x_0$$

whose derivative is continuous and for $t_0 \in (c,d)$, then x satisfies

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)\mathrm{d}s$$

for all $t \in (c,d)$. Conversely, suppose x is a continuous function that satisfies

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)\mathrm{d}s$$

for all $t \in (c,d)$ then x is differentiable withcontinuous derivative and x solves

$$x'(t) = f(x(t), t), \quad \text{for } t \in (c,d) \quad \text{such that } x(t_0) = x_0.$$

Proof. First suppose that x solves

$$x'(t) = f(x(t), t), \quad \text{for } t \in (c,d) \quad \text{such that } x(t_0) = x_0.$$

and has a continuos derivative. Since x is continuous then $f(x(t), t)$ is a continuous function of t. Integrating both sides from t_0 to any $t \in (c,d)$ gives, by the Fundamental Theorem of Calculus,

$$x(t) - x_0 = x(t) - x(t_0) = \int_{t_0}^t x(s)\mathrm{d}s = \int_{t_0}^t f(x(s), s)\mathrm{d}s.$$

Since t was arbitrary we conclude that for all $t \in (c,d)$ we have

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)\mathrm{d}s.$$

Conversely, suppose x is a continuous function that satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s.$$

Observe that since x is continuous then $f(x(s), s)$ is continuous. By the Fundamental Theorem of Calculus we conclude that the map

$$t \mapsto \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$$

is differentiable for all t. Since constant functions are differentiable we conclude that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$$

is differentiable for all t. By the Fundamental Theorem of Calculus we have

$$x'(t) = f(x(t), t)$$

for all $t \in (c, d)$. Since $f(x(t,t))$ is continuous then $x'(t)$ is continuous. So x solves the ODE and has continuous derivative. In particular, notice that $x(t_0) = x_0$. $\square$

Now we have exchanged the goal of demonstrating existence and uniqueness of an IVP for demonstrating existence and uniqueness of an integral equation. Notice that this reformulation has x appear on both sides of the desired equation. This allows us to use "fixed point methods" to achieve our goal. We recall, for convenience, the Banach Fixed Point Theorem which is proven in the appendix of analysis results:

Banach Fixed Point Theorem

Suppose X is a non-empty and complete metric space. Suppose also, that $f : X \to X$ satisfies

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ and where $0 \leq k < 1$. Then there exists a unique point $z \in X$ such that $f(z) = z$.

We are now ready to prove the existence and uniqueness theorem.

Picard-Lindelöf Theorem

Suppose $f : [a, b] \times [c, d] \to \mathbb{R}$ satisfies, for each fixed $t \in [c, d]$ and for any $x_1, x_2 \in [a, b]$

$$|f(x_1, t) - f(x_2, t)| \leq M|x_1 - x_2|$$

where $M \geq 0$ and does not depend on t. Then, if $t_0 \in (c, d)$ and $x_0 \in (a, b)$, there exists $\epsilon > 0$ and a continuous function x such that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$$

for $t \in [t_0 - \epsilon, t_0 + \epsilon]$.

Proof. Observe that for any $x, y \in (a,b)$ we have, for $t \geq t_0$

$$\begin{aligned}
\left|\left(x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s\right) - \left(x_0 + \int_{t_0}^{t} f(y,s)\mathrm{d}s\right)\right| &= \left|\int_{t_0}^{t} \big(f(x,s) - f(y,s)\big)\mathrm{d}s\right| \\
&\leq \int_{t_0}^{t} |f(x,s) - f(y,s)|\mathrm{d}s \\
&\leq M(t - t_0)|x - y| \\
&= M|t - t_0||x - y|
\end{aligned}$$

A similar computation holds for $t < t_0$ with the same conclusion. Thus, for all t we have

$$\left|\left(x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s\right) - \left(x_0 + \int_{t_0}^{t} f(y,s)\mathrm{d}s\right)\right| \leq M|t - t_0||x - y|$$

and in particular if we restrict the consideration of the above inequality to $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}]$ then we can further conclude that

$$\left|\left(x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s\right) - \left(x_0 + \int_{t_0}^{t} f(y,s)\mathrm{d}s\right)\right| \leq \frac{1}{2}|x - y|$$

for $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}]$. This constraint shows that the dependence of $x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s$ on x is that of a contraction. Next observe that for $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}]$ we have

$$\left|\left(x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s\right) - x_0\right| = \left|\int_{t_0}^{t} f(x,s)\mathrm{d}s\right| \leq \|f\|_{C([a,b]\times[c,d])}|t - t_0|.$$

Thus, if we choose $t \in \left[t_0 - \frac{\min\{x_0 - a, b - x_0\}}{\|f\|_{C([a,b]\times[c,d])}}, t_0 + \frac{\min\{x_0 - a, b - x_0\}}{\|f\|_{C([a,b]\times[c,d])}}\right]$ then we may conclude that

$$a \leq x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s \leq b.$$

Thus, in this constraint the output of $x_0 + \int_{t_0}^{t} f(x,s)\mathrm{d}s$ is in between a and b. We restrict our attention to $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}] \cap \left[t_0 - \frac{\min\{x_0 - a, b - x_0\}}{\|f\|_{C([a,b]\times[c,d])}}, t_0 + \frac{\min\{x_0 - a, b - x_0\}}{\|f\|_{C([a,b]\times[c,d])}}\right] = I$ so that both of the above described conditions are true. Now we define $\mathcal{F} : C(I; [a,b]) \to C(I; [a,b])$ by

$$\mathcal{F}(x) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s.$$

We verify that $\mathcal{F}$ is well defined and that $\mathcal{F}$ is a contraction. Let $x \in C(I; [a,b])$. Then x is bounded and so

$$\mathcal{F}(x) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s.$$

is a continuous function. In particular, since x is a function that maps to $[a,b]$ the above expression makes sense and is defined for all $t \in I$. By construction, since $t \in I$ and $x(I) \subset [a,b]$ then a previous computation shows that

$$a \leq x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s \leq b.$$

for all $t \in I$. The previous computation still holds since $f(x(s), s) \leq \|f\|_{C([a,b]\times[c,d])}$ for all $s \in I$

remains true. Thus, $\mathcal{F}(x) \in C(I;[a,b])$. Next we show that $\mathcal{F}$ is a contraction. By mimicking a previous computation we have, since $t \in I$ and $x(I), y(I) \subset [a,b]$ for $x, y \in C(I;[a,b])$, that for all $t \in I$

$$\begin{aligned}\left|\left(x_0 + \int_{t_0}^{t} f(x(s),s)\mathrm{d}s\right) - \left(x_0 + \int_{t_0}^{t} f(y(s),s)\mathrm{d}s\right)\right| &\leq M \int_{t_0}^{t} |x(s) - y(s)|\mathrm{d}s \\ &\leq M|t - t_0|\|x - y\|_{C(I;[a,b])} \\ &\leq \frac{1}{2}\|x - y\|_{C(I;[a,b])}\end{aligned}$$

where the last inequality follows from the choice $t \in I$. From this we conclude that

$$|(\mathcal{F}(x))(t) - (\mathcal{F}(y))(t)| \leq \frac{1}{2}\|x - y\|_{C(I;[a,b])}$$

for all $t \in I$. Hence,

$$\|\mathcal{F}(x) - \mathcal{F}(y)\|_{C(I;[a,b])} \leq \frac{1}{2}\|x - y\|_{C(I;[a,b])}$$

which shows that $\mathcal{F}$ is a contraction on $C(I;[a,b])$. By the Banach Fixed Point Theorem we conclude that there is a fixed point $x^* \in C(I;[a,b])$ satisfying

$$x^*(t) = x_0 + \int_{t_0}^{t} f(x^*(s),s)\mathrm{d}s$$

for all $t \in I$. □

3.3 Generalized Existence and Uniqueness*

We prove a general form of existence and uniqueness known as the *Osgood's existence and uniqueness theorem*:

Osgood's existence and uniqueness

Suppose $f(x,t)$ is continuous in both variables and satisfies, for each $x \in [a,b]$,

$$|f(x,t_1) - f(x,t_2)| \leq \omega(|t_1 - t_2|)$$

where $\omega : [0,\infty) \to [0,\infty)$ satisfies $\omega^{-1}(\{0\}) = \{0\}$, $\frac{1}{\omega}$ is Riemann integrable on $[\delta, 1]$ for all $\delta > 0$, and

$$\lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{1}{\omega(s)}\mathrm{d}s = +\infty.$$

Then there exists a unique y that satisfies

$$y'(x) = f(x, y(x))$$

for $x \in [a, a+\epsilon)$ as well as $y(a) = y_0$.

Proof. Since f is continuous then by the *Peano's existence theorem* we have that a solution, y, exists for $x \in [a, a+\epsilon)$ where $\epsilon > 0$. We now demonstrate uniqueness. Suppose not. Then there exist two solutions y_1, y_2 to this initial value problem. Since the solutions are distinct then there is a point x^* such that $y_1(x^*) \neq y_2(x^*)$. In particular, we see that

$$A = \left\{x \in [a, a+\epsilon) \;\middle|\; y_1(x) \neq y_2(x)\right\} \neq \phi.$$

Observe that by continuity that A is also open. Choose any point in A and consider the largest interval containing the chosen point. Suppose this interval has boundary points c and d. I claim that we must have $y_1(c) = y_2(c)$ and $y_1(d) = y_2(d)$. Suppose not. If this fails for c then notice that by continuity there will be a $\delta > 0$ such that $y_1(x) \neq y_2(x)$ for $x \in (c-\delta, c)$. This contradicts how we found the endpoint c. A similar proof works for d. Observe also that we either have $y_1(x) > y_2(x)$ for all $x \in (c, d)$ or $y_2(x) > y_1(x)$ for all $x \in (c, d)$. This is because $(c, d) \subset A$ and if both situations occurred in (c, d) then by the Intermediate Value Theorem there would be a point in $(c, d) \subset A$ where y_1 and y_2 agree. Assume, without loss of generality, that $y_1(x) > y_2(x)$ for all $x \in (c, d)$. Then, for $x \in (c, d)$ we have

$$\begin{aligned} y_1'(x) - y_2'(x) &= f(x, y_1(x)) - f(x, y_2(x)) \leq |f_1(x, y_1(x)) - f(x, y_2(x))| \\ &\leq \omega(|y_1(x) - y_2(x)|) = \omega(y_1(x) - y_2(x)). \end{aligned}$$

Hence, for $x \in (c, d)$

$$\frac{(y_1(x) - y_2(x))'}{\omega(y_1(x) - y_2(x))} \leq 1.$$

By Preiss and Uher's version of the change of variables theorem we have for each $t > c$ and for fixed $t^* > t$ that

$$\int_t^{t^*} \frac{(y_1(x) - y_2(x))'}{\omega(y_1(x) - y_2(x))} \mathrm{d}x = \int_{(y_1-y_2)(t)}^{(y_1-y_2)(t^*)} \frac{1}{\omega(s)} \mathrm{d}s$$

where we choose t^* so that $(y_1 - y_2)(t^*) < 1$ (this is possible since $y_1 - y_2$ is continuous and is 0 at $t = c$. Hence,

$$\int_{(y_1-y_2)(t)}^{(y_1-y_2)(t^*)} \frac{1}{\omega(s)} \mathrm{d}s \leq t^* - t.$$

Letting t tend to c gives

$$+\infty = \lim_{t \to c^+} \int_{(y_1-y_2)(t)}^{(y_1-y_2)(t^*)} \frac{1}{\omega(s)} \mathrm{d}s \leq t^* - c < +\infty$$

which is a contradiction (note that we have used that $(y_1 - y_2)(t)$ tends to 0 as t tends to c). We conclude that the solution is unique. $\square$

3.3.1 Remarks

We remark that the Osgood condition can be weakened to requiring

$$|f(x, t_1) - f(x, t_2)| \leq \omega(|t_1 - t_2|)\varphi(x)$$

for each $x \in [a, b]$ and any t_1, t_2 where ω satisfies the conditions stated in Osgood's existence and uniqueness theorem and $\varphi \geq 0$ is Riemann integrable on $[0, 1]$. This strengthening of Osgood's theorem is referred to as the Montel-Tonelli uniqueness theorem.

4 Autonomous dynamics and Logistic growth

The equations of the form

$$\frac{dy}{dt} = f(y)$$

are called **autonomous**. Such equations might not have explicit solutions, but it is possible to draw qualitative solutions for them.

Method formal steps

1. First we draw the curves $\varphi_i(t) = (t, y(t))$ where $f(y) = 0$ (called the **equilibrium solutions** or **critical points**).
2. These will separate the regions into $y' = f(y) > 0$ and $y' = f(y) < 0$.
3. We classify each φ_i as **asymptotically stable** if for $y(t)$ starting close to φ_i (i.e. $|y_0 - \varphi_i(0)| < \varepsilon$)
$$\lim_{t\to\infty} y(t) = \varphi_i$$
irrespective of whether $y_0 < \varphi_i(0), y_0 > \varphi_i(0)$ and **asymptotically unstable** if solutions that start close to the $\varphi_i(t)$ curve, move away from it.

Example-Presenting the method

En route to studying the competing species we will need the logistic equation (2.5): Let $y(t)$ be the population of a given species at time t then

$$\frac{dy}{dt} = r\Big(1 - \frac{y}{K}\Big)y,$$

where $r > 0$ is called the intrinsic growth rate and K is the saturation level. Since y is a physical quantity, the $y < 0$ is ignored.

1. First we find the equilibrium solutions: $r\big(1 - \frac{y}{K}\big)y = 0 \Rightarrow y = K$ or $y = 0$. So the equilibrium solutions are $\varphi_1(t) \equiv 0, \varphi_2(t) \equiv K$.
2. We have $y' = r(1 - \frac{y}{K})y > 0$ when $K > y$ and $y > 0$ ($y < 0$ is ignored). Therefore, the solutions started from below K will be growing upwards to $y = K$.
3. On the other hand, $y' = r(1 - \frac{y}{K})y < 0$ when $K < y$ and $y > 0$. Therefore, the solutions started from above K will be decaying downwards to $y = K$.

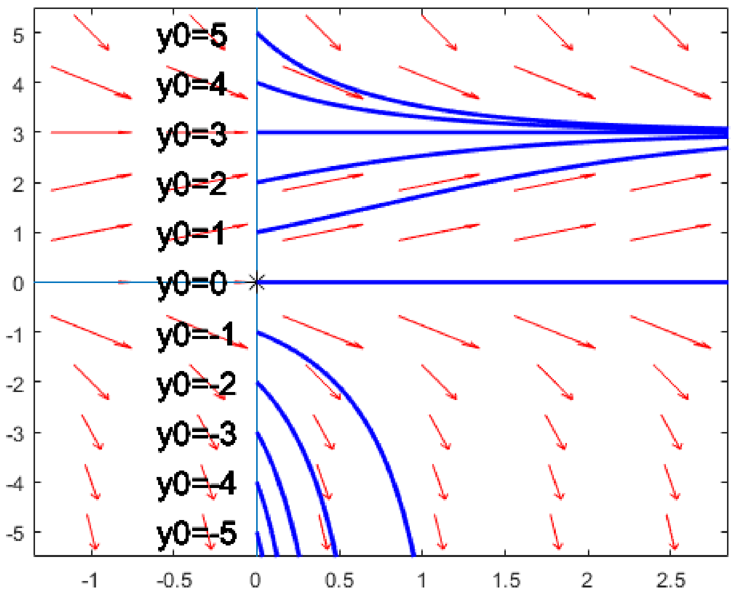

Figure 4.1: Direction field for logistic with $K = 3$.

4. So we observe that irrespective of the initial value the solution converges to the saturation level: $\lim_{t\to\infty} y = K$. Therefore, $\varphi_2(t) \equiv K$ is the asymptotically stable solution.

5. On the other hand, we observe that if y is really small (i.e. close to $\varphi_1 = 0$) but still positive, the solutions still move away from φ_1 and go towards φ_2. Therefore, φ_1 is the asymptotically unstable solution.

6. Physically that the population dynamics will return to the saturation/capacity level K; the most the ecosystem can withhold.

Examples

1. Consider the equation
$$y' = (y-1)(y-2)(y-3).$$
   (a) First we identify the equilibrium solutions:
$$y' = 0 \Rightarrow \varphi_1(t) \equiv 1,\ \varphi_2(t) \equiv 2,\ \text{and } \varphi_3(t) \equiv 3.$$
   (b) Second we identify the sign of y' in each region:
   - Above $y > 3$ we have $y' > 0$ and so the solution will growth to infinity
   - In $y \in [2,3]$ we have $y' < 0$ and so the solution will decay downwards to φ_2.
   - In $y \in [1,2]$ we have $y' > 0$ and so the solution will grow upwards to 2. Therefore, in either case
$$y(t) \to \varphi_2$$
     and so φ_2 is asymptotically stable.
   - In $y \in [0,1]$ we have $y' < 0$ and so the solution will decay downwards to minus infinity.
   - Therefore, φ_1, φ_3 will be asymptotically unstable.

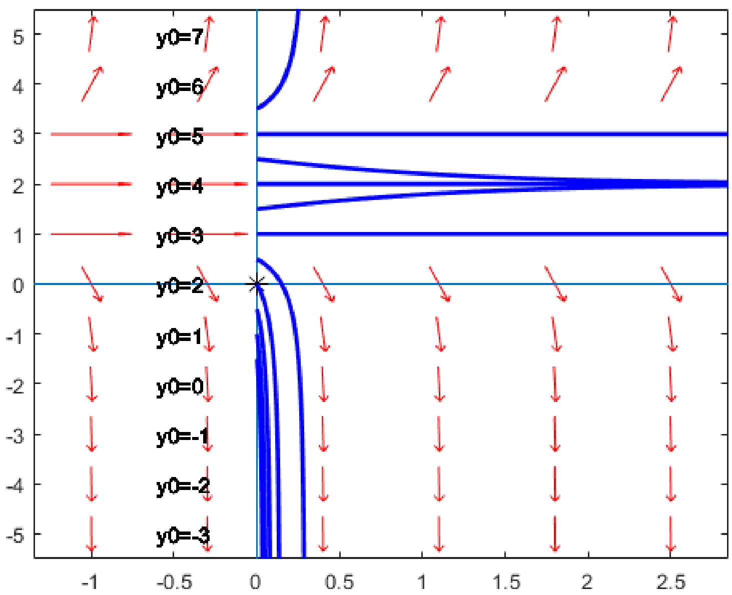

Figure 4.2: Direction field for $y' = (y-1)(y-2)(y-3)$.

2. Consider the equation
$$y' = e^{-y} - 1.$$

(a) First we identify the equilibrium solution/-s:

$$y' = 0 \Rightarrow \varphi(t) = 0.$$

(b) Second we identify the sign of y' in each region:

- Above $y > 0$ we have $y' = e^{-y} - 1 < 0$ and so it will decay downwards to 0.
- Below $y < 0$ we have $y' = e^{-y} - 1 > 0$ and so it will grow upwards to 0. Therefore, φ is asymptotically stable.

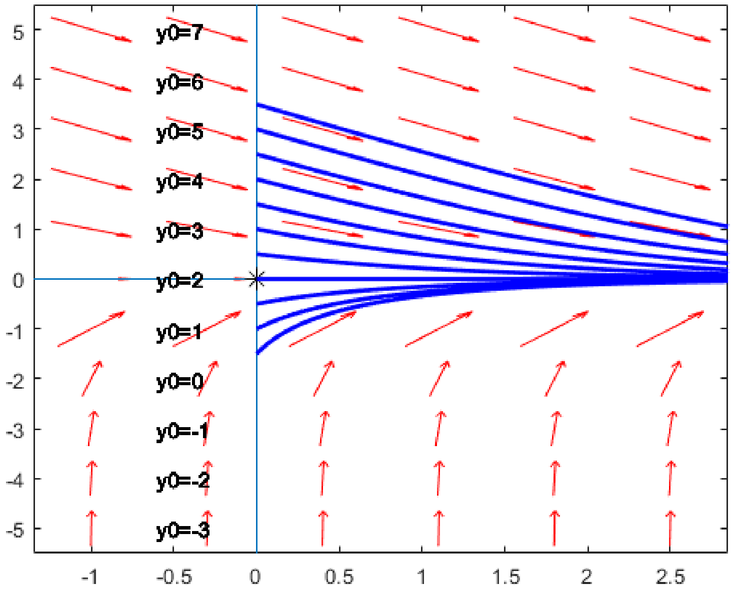

Figure 4.3: Direction field for $y' = e^{-y} - 1$.

4.0.1 Applied examples

- We return to the price of good example from the integrating factor section. When the price of a good is p, the total demand is D(p) = a - bp and the total supply is $S(p) = \alpha + \beta p$, where a, b, α, and β are positive constants. When demand exceeds supply, price rises, and when supply exceeds demand it falls. The speed at which the price changes is proportional to the difference between supply and demand. Specifically

$$p' = \lambda(D(p) - S(p))$$

for $\lambda > 0$.

1. We obtained
$$p(t) = c \cdot exp\{-\lambda(b + \beta)t\} + \frac{(a - \alpha)}{(b + \beta)}.$$
2. So as $t \to +\infty$ the price of this good converges to $\frac{(a-\alpha)}{(b+\beta)}$. Now we will show that $\varphi = \frac{(a-\alpha)}{(b+\beta)}$ in fact an asymptotically stable solution.
3. We first obtain the equilibrium solutions
$$p' = 0 \implies D(p) = S(p) \implies a - bp = \alpha + \beta p \implies p = \frac{(a - \alpha)}{(b + \beta)}.$$

4. Since $D(p) = S(p)$, this is the price where supply will match demand. We have $p' > 0$ when $p < \frac{(a-\alpha)}{(b+\beta)}$ and so the price will grow upwards to the stable price.
5. We have $p' < 0$ when $p > \frac{(a-\alpha)}{(b+\beta)}$ and so the price will decay downards to the stable price.

5 Problems

The questions labeled by (*) are trickier.

- **Linear integrating factor (2.2)**
  - Find the general solution and use it to determine how solutions behave as $t \to +\infty$
    1. $y' + 3y = t + e^{-2t}$
    2. $y' + y = te^{-t} + 1,$
    3. (*)[1] $y' + \frac{1}{t}y = 3\cos(2t), t > 0$
  - Find the general solution and use it to determine the asymptotic behavior for different values of a
    1. $y' - \frac{1}{2}y = 2\cos(t), y(0) = a$ as $t \to +\infty$,
    2. (*) $ty' + (t+1)y = 2te^{-t}, y(1) = a, 0 < t$ as $t \to 0$.
    3. (*) A rock contains two radioactive isotopes R_1, R_2 with R_1 decaying into R_2 with rate $5e^{-20t}$kg/sec. So if $y(t)$ is the total mass of R_2, we obtain:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \text{rate of creation of } R_2 \text{ - rate of decay of } R_2 \\ &= 5e^{-20t} - ky(t),\end{aligned}$$

       where $k > 0$ is the decay constant for R_2. Also assume that $y(0) = 40\,\text{kg}$.

- **Separable (2.1):**
  - For each of the questions (1) – (5) you should:
    (a) find the solution of the given initial value problem
    (b) determine the interval in which the solution is defined:
      1. $y' = (1-2x)y^2, \ \ y(0) = -1/6,$
      2. $y' = (1-2x)/y, \ \ y(1) = -2,$
      3. (*) $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r^2}{\theta}, \ \ r(1) = 2,$
      4. (*) $y' = 2x/(1+2y), \ \ y(2) = 0,$
      5. (*) $y' = y^2 + 1, \ \ y(0) = 0$ (only the interval containing 0).
  - For each of the following questions you should:
    (a) find the solution of the given initial value problem
    (b) determine the behaviour as $t \to +\infty$:
      1. (*) $y' = \cos^2(y), \ \ y(0) = 2,$
      2. (*) $y' = t\frac{y(4-y)}{3}, \ \ y(0) = 1.$
  - (*) Homogeneous equations problem 2.2-(25).

- **Homogeneous equations problem**
  1. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+xy+y^2}{x^2}$

- **Autonomous equations**
  1. Draw the phase lines and identify which solutions are asymptotically stable/unstable.
     (a) $\frac{\mathrm{d}y}{\mathrm{d}t} = ay + by^2, \ \ a > 0, \ \ b > 0, \ \ -\infty < y_0 < \infty$,

[1] We follow the standard book practice of using a (*) to indicate a question is trickier.

(b) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2), \ \ y_0 \geq 0,$

(c) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^y - 1, \ \ -\infty < y_0 < \infty$.

2. (**) Suppose $f : [c, d] \to \mathbb{R}$ and suppose $x : [a, b] \to \mathbb{R}$ is a function which is continuous on $[a, b]$, differentiable on (a, b), $x([a, b]) \subset [c, d]$, and x satisfies $x'(t) = f(x(t))$ for $t \in [a, b]$. Show that x is monotone on $[a, b]$.

   Hints:

   (a) Suppose not. Then, without loss of generality, we can find points $a \leq t_1 < t_2 < t_3 \leq b$ such that $x(t_1) < x(t_2)$ but $x(t_3) < x(t_2)$. Use this to argue that there is a point $t_4 \in (t_1, t_3)$ such that x attains its maximum on $[t_1, t_3]$ at t_4. Then construct $t_5 \in (t_1, t_4)$ and $t_6 \in (t_4, t_3)$ such that $x'(t_5) > 0$, $x'(t_6) < 0$, $x(t_5) < x(t_4)$ and $x(t_6) < x(t_4)$.

   (b) If $x(t_5) = x(t_6)$ use the differential equation satisfied by x to conclude we are done. Otherwise, without loss of generality, we may assume $x(t_5) < x(t_6)$. Show that the collection of $t \in (t_5, t_4)$ such that $x(t) = x(t_6)$ is non-empty.

   (c) Now suppose, for $t \in (t_5, t_4)$, $x'(t) < 0$ whenever $x(t) = x(t_6)$ and consider the set
   $$A = \big\{ t \in (t_5, t_4) \bigm| x(t) \neq x(t_6) \big\}.$$
   Show that if $t_* \in A$ then there exists $\epsilon > 0$ such that $t_* \pm \epsilon \in A$. Finally show that the set $x^{-1}(A)$ has the property that if $s \in x^{-1}(A)$ then $s \pm \delta \in x^{-1}(A)$ for $\delta > 0$ small enough (you will need to use that x is continuous here).

   (d) Conclude that $A = \bigcup_{i=1}^{\infty} (a_i, b_i)$. To see this for note that if a point is in A then nearby points are in A. Thus, we may consider the maximal neighbourhood of a point $x \in A$. Show that if A_x, A_y are maximal neighbourhoods of x and y then either $A_x = A_y$ or $A_x \cap A_y = \phi$. Since rationals are countable conclude that the amount of intervals A_x is countable.

   (e) Pick any maximal interval (a_i, b_i) from A. Observe that $a_i, b_i \notin A$. Conclude, by continuity, that there is $\delta > 0$ such that $x(a_i) > x(t)$ if $0 < t - a_i < \delta$ and $x(t) > x(b_i)$ if $0 < b_i - t < \delta$. Conclude that there is a point $c \in (a_i, b_i)$ such that $x(a_i) = x(c) = x(b_i)$ (they all equal $x(t_6)$). This is a contradiction. Thus, the assumption $x'(t) < 0$ whenever $x(t) = x(t_6)$ is wrong. Conclude that there exists $t_7 \in (t_5, t_4)$ such that $x(t_7) = x(t_6)$ and $x'(t_7) \geq 0$ and use the differential equation to argue that we have a contradiction as in 2b.

3. (**) In this exercise we construct an initial value problem for an autonomous differential equation which has no solution but the function f is continuous at the initial condition.

   (a) Suppose $f : [c, d] \to \mathbb{R}$ is nowhere 0. Show that the differential equation $x'(t) = f(x(t))$ has no constant solutions. That is, if $C \in [c, d]$ is any constant, then $x \equiv C$ is not a solution to $x'(t) = f(x(t))$.[2]

   (b) Suppose $f : [c, d] \to \mathbb{R}$ is continuous on $[c, d]$ and differentiable on (c, d). Then:

      i. if $t, s \in (c, d)$ with $t < s$ and $f'(t) < 0$ and $f'(s) > 0$ then there exists $r \in (t, s)$ such that $f'(r) = 0$.

      ii. if $t, s \in (c, d)$ with $t < s$ and p satisfies $f'(t) < p < f'(s)$ then there exists $r \in (t, s)$ such that $f'(r) = p$.

      Hint: Apply the previous problem to $g(x) = f(x) - xp$.

   (c) Suppose $f : [c, d] \to \mathbb{R}$ and $x : [a, b] \to \mathbb{R}$ satisfies $x([a, b]) \subset [c, d]$, x is continuous on $[a, b]$ but differentiable on (a, b), x satisfies $x'(t) = f(x(t))$ for $t \in (a, b)$. Then if $t, s \in (\min\{x(a), x(b)\}, \max\{x(a), x(b)\})$ with $t < s$, and p is such that $f(t) < p < f(s)$ then there exists $r \in (t, s)$ such that $f(r) = p$.

   Note: You will need to use 2 to conclude.

[2] This is referred to as Darboux's Theorem.

(d) Consider $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \neq \pm\frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 + \frac{1}{n} & x = \pm\frac{1}{n} \end{cases}.$$

Show that the problem $x'(t) = f(x(t))$ with $x(0) = 0$ has no solutions using the previous questions. Also, verify by directly integrating that this initial value problem has no solutions.