Hamiltonian Systems and Modeling of Mechanical Systems

MAT244 Group Assignment

1 Spring-Mass System

A A Hamiltonian System has No Spiral Sources/Sinks

A system will have spiral sources/sinks if its eigenvalues evaluated at the critical points are complex with non-zero real parts. We will show this does not occur for Hamiltonian systems.

Proof. We assume a general Hamiltonian system $\mathbb{H}$ with:

$$\frac{dy}{dt} = f(y,p), \frac{dp}{dt} = g(y,p)$$

Then by existence of $\mathbb{H}$ we have:

$$\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial p} \tag{1}$$

Assume f and g are C^2 s.t. the system is locally linear in a neighborhood of its critical points. We take its Jacobian evaluated at the critical points to find the corresponding linear system:

$$J(y,p) = \begin{bmatrix} f_y & f_p \\ g_y & g_p \end{bmatrix} \begin{pmatrix} y - y_0 \\ p - p_0 \end{pmatrix}$$

Let f_{y_0}, g_{p_0} be the partial derivatives $\frac{\partial f}{\partial y}, \frac{\partial g}{\partial p}$ evaluated at arbitrary critical points (y_0, p_0) and compute the eigenvalues:

$$det(J - \lambda I) = \begin{vmatrix} f_{y_0} - \lambda & f_{p_0} \\ g_{y_0} & g_{p_0} - \Lambda \end{vmatrix} = \lambda^2 - (f_{y_0} + g_{p_0})\lambda + (f_{y_0} g_{p_0} - f_{p_0} g_{y_0})$$

Applying (1), $(f_{y_0} + g_{p_0})\lambda$ vanishes and we have $\lambda = \pm\sqrt{F_{y_0}^2 + F_{p_0} G_{y_0}}$. We evaluate the cases:

- For $F_{y_0}^2 + F_{p_0} G_{y_0} = 0$, the eigenvalue is 0 with multiplicity 2.
- For $F_{y_0}^2 > F_{p_0} G_{y_0}$, the eigenvalues are real, and diverse with opposite signs (the critical point is a saddle point).
- For $F_{y_0}^2 < F_{p_0} G_{y_0}$, the eigenvalues are purely imaginary.

As there are no cases where the eigenvalues will be complex with non-zero real components, a Hamiltonian system cannot have spiral sources or sinks.

□

B The Hamiltonian Nature of the Spring Force

The system is Hamiltonian if the forces acting on the system depend only on position; equivalently, it obeys conservation of energy. More generally, it has a Hamiltonian if it obeys Equation (1).

If we let $f(y,p) = \frac{p}{m}$ and $g(y,p) = -ky$ then we have $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial p} = 0$ and thus the system is Hamiltonian.

While the above is a sufficient condition for demonstrating that the system is Hamiltonian, we also show the necessary condition that for a force $F(y)$ to be conservative, there exists a potential $V(y)$ s.t. $F(y) = \frac{-dV(y)}{dy}$, a function of position only. We first note that $-ky$ depends only on y, producing the following separable equation:

$$\frac{-dV(y)}{dy} = -ky$$

We solve to find $V(y) = k\frac{1}{2}(y^2 + c)$, $c \in \mathbb{R}$. From the definition of the Hamiltonian, we have:

$$H(y,p) = \frac{p^2}{2m} + V(y) \tag{2}$$

and take the following derivatives to find Hamilton's equations:

$$\frac{dy}{dt} = \frac{dH}{dp} = \frac{p}{m} \tag{3}$$

$$\frac{dp}{dt} = -\frac{dH}{dy} = -\frac{dV(y)}{dy} = -ky \tag{4}$$

C Show the System's Phase Plane Trajectories are Constant

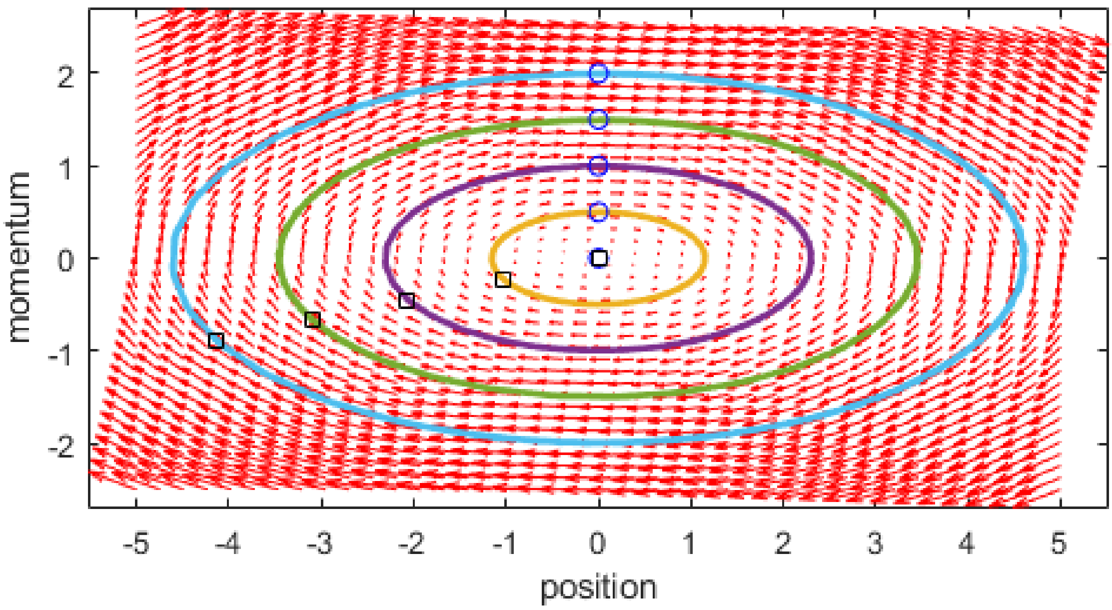
Taking the ratio of (3) and (4), we have a separable equation:

$$\frac{dy}{dp} = -\frac{p}{mky}$$

Solving, we find. $\frac{p^2}{2m} + k\frac{y^2}{2} = c$, as desired.

D Mass-Spring System ODE Plot

Below we plot the results of the previous section, with initial position 0 and the set of initial momentum $\{0, 0.5, 1, 1.5, 2\}$, with initial points on a given curve indicated by circles and final points indicated by boxes. All required constants have been set to 1:



E Linearization of Spring-Mass System at Critical Points

There is only a single critical point in this system, and it occurs at the origin. The Jacobian is:

$$J(0,0) = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix}$$

The trace of this linearized system is 0 and its determinant is positive. The eigenvalues will be purely imaginary, producing concentric circles around the origin; this agrees with our Matlab direction field.

2 A Pendulum System With Damping

Please note that all Matlab plots have had all required constants set to 1 unless otherwise indicated. Plots include direction fields and full trajectories for a set of random initial values of position and momentum.

A Hamiltonian of Undamped System

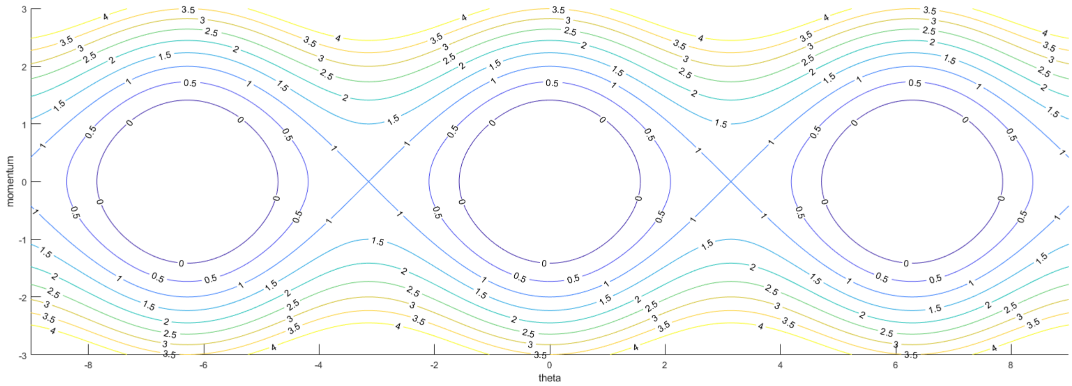
The Hamiltonian $H(\theta, p)$ of the undamped system can be obtained by reading off the value of $V(\theta)$ from the definition of the system, and plugging it into the definition of the Hamiltonian. Taking $V(\theta)$ to be $-lmg \cdot cos(\theta)$ where $l, m, g > 0 \in \mathbb{R}$, we have:

$$\mathbb{H}(\theta, p) = \frac{p^2}{2m} - l \cdot m \cdot g \cdot cos(\theta) \tag{5}$$

We obtain sketches by setting $\mathbb{H} = C_i$ and solving for p, with C_i a finite set of constant solutions indexed over i. Those solutions which produce negative discriminants do not have real-valued solutions. As we provide Matlab plots in the next subsection, we omit the hand sketch.

B Matlab Plot: Hamiltonian System

We plot several level curves of the above Hamiltonian:



There are no spiral sinks or sources here, agreeing with previous results regarding behaviour of systems which obey conservation laws.

C Linearized System Around Arbitrary Point

We first define the Hamiltonian equations given by the problem:

$$\frac{\partial \mathbb{H}}{\partial p} = \frac{d\theta}{dt} = \frac{p}{m} := F \tag{6}$$

$$\frac{\partial \mathbb{H}}{\partial \theta} = \frac{dp}{dt} = -lmg \cdot sin(\theta) - b\frac{p}{ml^2} := G \tag{7}$$

We linearize the damped system by finding the Jacobian matrix:

$$J(\theta, p) = \begin{bmatrix} F_\theta & F_p \\ G_\theta & G_p \end{bmatrix}$$

Then linearize the damped system by taking appropriate derivatives of equations 6 and 7:

$$J_{damped} := \begin{bmatrix} 0 & \frac{1}{m} \\ -lmg \cdot cos(\theta) & \frac{-b}{ml^2} \end{bmatrix}$$

Finally we obtain the linearization of the undamped system by taking J_{damped} and setting $b = 0$:

$$J_{undamped} := \begin{bmatrix} 0 & \frac{1}{m} \\ -lmg \cdot cos(\theta) & 0 \end{bmatrix}$$

D Linearized Undamped System Around Critical Points

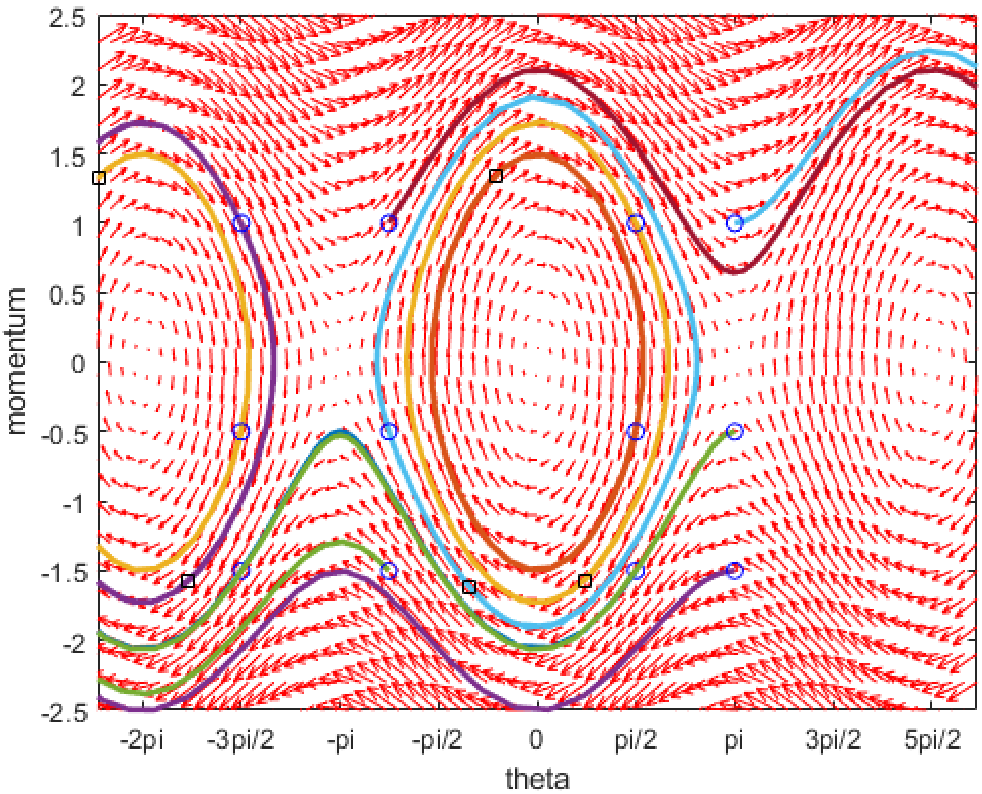
For the undamped system, we note that equation 6 vanishes when $p = 0$. Equation 7 vanishes at $n\pi$, $n \in \mathbb{Z}$. The critical points occur at $(p, \theta) = (0, n\pi)$, and the behavior depends on the parity of n.

The eigenvalues of $J_{undamped}$ for even values of n are $\pm i\sqrt{lg}$. These are purely imaginary, and solutions in a neighborhood of these points will be concentric circles.

For odd values of n, the eigenvalues are $\pm\sqrt{lg}$. These are real and distinct, producing a saddle point.

E Matlab Plot: Linearized Undamped System

We plot the direction field and iterate through several values of initial position/momentum. The results agree with our Hamiltonian plot.



F Linearized Damped System Around Critical Points

Note that as $p = 0$ makes the damping term vanish, the critical points do not actually change: they continue to occur at $(p, \theta) = (0, n\pi)$. However, the Jacobian matrix now includes the damping term, producing new phase trajectories.

Evaluating J_{damped} for even values of n, we find the eigenvalues are given by:

$$\lambda_{1even}, \lambda_{2even} = \frac{1}{2}\left[\frac{-b}{ml^2} \pm \sqrt{\left(\frac{-b}{ml^2}\right)^2 - 4lg}\right] \tag{8}$$

Assume l, m, g are positive, real, and fixed. The value of b then produces three cases for the behaviour in a neighbourhood of the critical point:

- For $b = 2lm\sqrt{gl}$, we have the repeated eigenvalue $\frac{-b}{2ml^2}$ with $Det(J_{damped}) < 0$, a degenerate sink
- For $b < 2lm\sqrt{gl}$, we have complex eigenvalues with a negative real part, producing a spiral sink
- For $b > 2lm\sqrt{gl}$, we have real and distinct eigenvalues which are always negative, producing a stable sink

Evaluating J_{damped} for odd values of n, we find that the eigenvalues are given by:

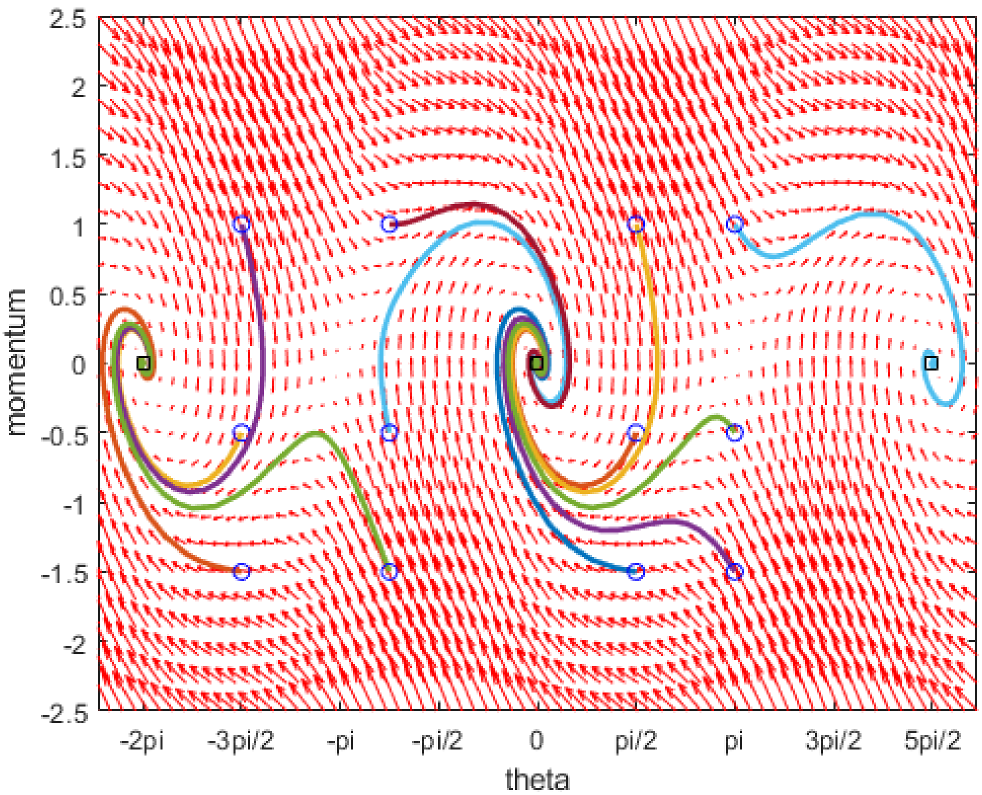
$$\lambda_{1odd}, \lambda_{2odd} = \frac{1}{2}\left[\frac{-b}{ml^2} \pm \sqrt{\left(\frac{-b}{ml^2}\right)^2 + 4lg}\right] \tag{9}$$

In all cases, the discriminant of Equation 9 is strictly positive, and the eigenvalues will be real and distinct: these critical points will be saddle points.

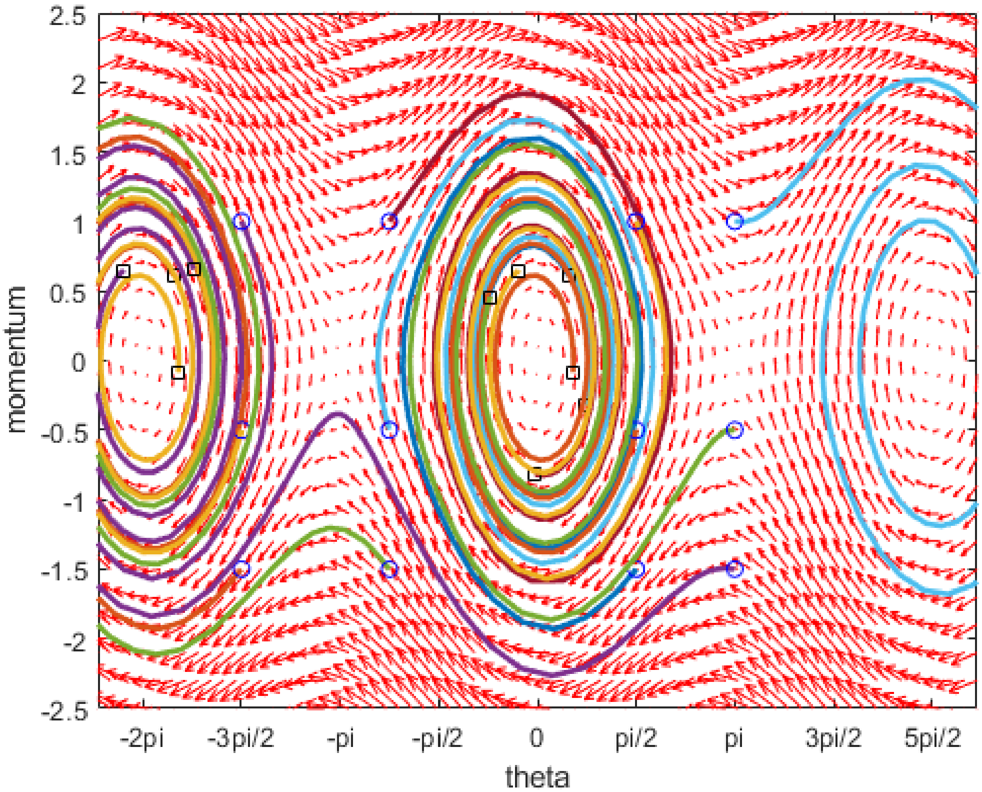
As the damping parameter b becomes smaller, the trajectories take longer to sink into one of the basins of attraction. They may even drift for a very long time into basins "far away."

G Matlab Plot: Linearized Damped System

We provide two plots of the direction field and iterate through several values of initial position/momentum. The first plot has damping parameter $b = 0.75$:



The second plot holds all initial values and elapsed time of the ODE solver constant, while reducing b to 0.1:



In the 2nd plot, we see that the trajectories will take longer to reach the spiral sink critical points, and that initial values sufficiently far from a sink may drift for a while before being "captured" by the sink.

As the damping parameter goes to zero, solution curves will more closely approximate the undamped system. These plots agrees with our stability analysis.