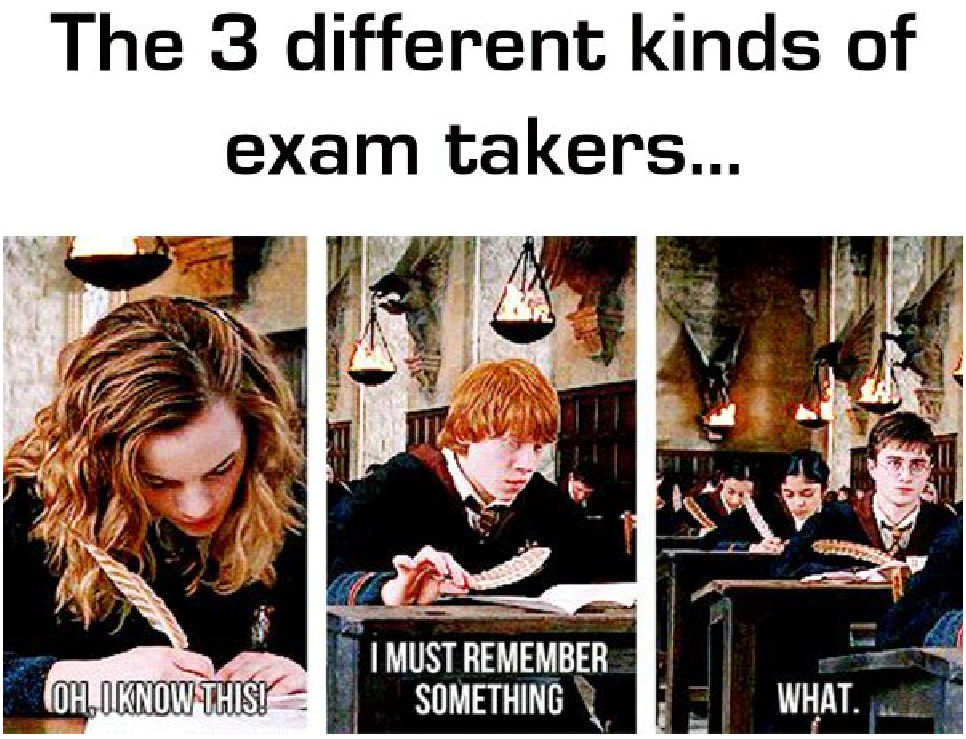

Figure 0.1: Oh I know this

1 Second order

second order and system: real Consider the equation

$$\theta'' - 2\theta' - 3\theta = e^{-t} + 3$$

and its related system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ e^{-t} + 3 \end{pmatrix}.$$

Proof.

1. First we solve the second order equation. The homogeneous part is

$$x_h = c_1 e^{3t} + c_2 e^{-t}.$$

2. Then we apply our guess: $x_{nh} = at^s e^{-t}$. Since -1 is a simple root, we have $s = 1$ and so it remains to find a:

$$a(-2e^{-t} + te^{-t} - 2e^{-t} + 2te^{-t} - 3te^{-t}) = e^{-t} \Rightarrow a = \frac{-1}{4}.$$

3. So the solution is:

$$x_{gen} = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{4} te^{-t} + 1.$$

4. Next we solve the corresponding nonhomogeneous system. The eigenpairs are $(3, \binom{1}{3}), (-1, \binom{-1}{1})$ and so the homogeneous part is:

$$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

5. So the change of basis is matrix $\mathbf{T} = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$ and we obtain:

$$\mathbf{h}(t) = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}.$$

6. Thus, we obtain

$$\begin{aligned} y_1(t) =& e^{3t} \int e^{-3s} (\frac{e^{-s}}{4}) ds \\ =& \frac{e^{-t}}{-16} \end{aligned}$$

and

$$\begin{aligned} y_2(t) =& e^{-t} \int e^{s} (\frac{e^{-s}}{4}) ds \\ =& \frac{e^{-t} t}{4}. \end{aligned}$$

7. Therefore, the nonhomogeneous part is

$$\mathbf{x}_{nh,1} = \mathbf{T}\mathbf{y} = \frac{e^{-t}}{4} \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4} \\ t \end{pmatrix} = \frac{e^{-t}}{4} \begin{pmatrix} -\frac{1}{4} \\ \frac{-3}{4} \end{pmatrix} + \frac{e^{-t} t}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

8. For the system $\mathbf{x}' = Ax + \binom{0}{3}$, we have

$$\mathbf{x}_{nh,2} = A^{-1} \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

9. The general solution will be:

$$\mathbf{x}_{gen} = c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{-e^{-t}}{16} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{e^{-t} t}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

10. Indeed the $x = \theta$ component is of the form:

$$\theta = x = c_1 e^{3t} + c_2 e^{-t} - \frac{e^{-t} t}{4} + 1,$$

which agrees with our second order solution.

□

second order and system: complex Consider the equation

$$\theta'' + 2\theta' + 2\theta = 3 + e^{-t} sin(t)$$

and its related system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 3 + e^{-t} sin(t) \end{pmatrix}.$$

Proof. 1. First we solve the second order equation. The homogeneous part is

$$x_h = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}.$$

2. Then we apply our guess: $x_{nh} = t^s e^{-t}(a\cos(t) + b\sin(t))$. Since -1+i is a simple root, we have $s = 1$ and so it remains to find a,b:

$$\begin{aligned}
&2(e^{-t} - e^{-t}t)(b\cos(t) - a\sin(t)) \\
&+ e^{-t}t(-a\cos(t) - b\sin(t)) + 2e^{-t}t(a\cos(t) + b\sin(t)) \\
&+ (e^{-t}t - 2e^{-t})(a\cos(t) + b\sin(t)) \\
&+ 2(e^{-t}t(b\cos(t) - a\sin(t)) \\
&+ e^{-t}(a\cos(t) + b\sin(t)) - e^{-t}t(a\cos(t) + b\sin(t))) \\
&= e^{-t}sin(t).
\end{aligned}$$

Isolating the term $e^{-t}sin(t)$ we obtain

$$-2a - b = 1 \Rightarrow a = -\frac{1}{2} \text{ and } b = 0$$

and so the nonhomogeneous part is

$$x_{nh} = te^{-t}(-\frac{1}{2}\cos(t)).$$

3. So the solution is:

$$x_{gen} = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} + te^{-t}(-\frac{1}{2}\cos(t)) + \frac{3}{2}.$$

4. Next we solve the corresponding nonhomogeneous system. The eigenpairs are $(-1 + i, \binom{-1-i}{2}), (-1+i, \binom{1-i}{-2})$ and so the homogeneous part is:

$$\mathbf{x} = c_1 e^{(-1+i)t}\begin{pmatrix} -1-i \\ 2 \end{pmatrix} + c_2 e^{(-1-i)t}\begin{pmatrix} -1+i \\ 2 \end{pmatrix}.$$

5. So the change of basis is matrix $\mathbf{T} = \begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix}$ and we obtain:

$$\mathbf{h}(t) = \begin{pmatrix} -i-1 & i-1 \\ 2 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ e^{-t}sin(t) \end{pmatrix} = \frac{e^{-t}sin(t)}{4}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.$$

6. Thus, we obtain

$$\begin{aligned}
y_1(t) =& e^{(-1+i)t}\int e^{-(-1+i)s}(\frac{e^{-s}\sin(s)(1+i)}{4})ds \\
=& \frac{e^{(-1+i)t}(1+i)}{4}(\frac{-it}{2} - \frac{e^{-2it}}{4}) \\
=& \frac{e^{(-1+i)t}(-1-i)}{8}(it + \frac{e^{-2it}}{2})
\end{aligned}$$

and

$$\begin{aligned}
y_2(t) =& e^{(-1-i)t}\int e^{-(-1-i)s}(\frac{e^{-s}\sin(s)(1-i)}{4})ds \\
=& \frac{e^{(-1-i)t}(1-i)}{8}(it - \frac{e^{2it}}{2}).
\end{aligned}$$

7. Therefore, the nonhomogeneous part is

$$\mathbf{x}_{nh,1} = \mathbf{T}\mathbf{y} = \begin{pmatrix} \frac{ie^{(-1-i)t}}{4}(it - \frac{e^{2it}}{2}) + \frac{e^{(-1+i)t}}{4}(it + \frac{e^{-2it}}{2}) \\ (1+i)\frac{e^{(-1+i)t}}{4}(-it - \frac{e^{-2it}}{2}) + (1-i)\frac{e^{(-1-i)t}}{4}(it - \frac{e^{2it}}{2}) \end{pmatrix}$$

using the Euler formula we obtain

$$= \begin{pmatrix} \frac{-e^{-t}}{4}t2\cos(t) + i\frac{ie^{-t}}{4}\sin(t) \\ \frac{e^{-t}}{4}(2t-1)(\sin(t)+\cos(t)) \end{pmatrix}$$

$$= \frac{e^{-t}}{4}t2\cos(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{e^{-t}}{4}\sin(t)\begin{pmatrix} i \\ -1 \end{pmatrix} + \frac{e^{-t}}{4}(2t\sin(t) - \cos(t))\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

8. For the system $\mathbf{x}' = Ax + \binom{0}{3}$, we have

$$\mathbf{x}_{nh,2} = A^{-1}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$

9. The general solution will be:

$$\begin{aligned} \mathbf{x}_{gen} =& c_1 e^{(-1+i)t}\begin{pmatrix} i+1 \\ -2 \end{pmatrix} + c_2 e^{(-1-i)t}\begin{pmatrix} i-1 \\ -2 \end{pmatrix} \\ &+ \frac{e^{-t}}{4}t2\cos(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &+ \frac{e^{-t}}{4}\sin(t)\begin{pmatrix} i \\ -1 \end{pmatrix} + \frac{e^{-t}}{4}(2t\sin(t) - \cos(t))\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}. \end{aligned}$$

10. Indeed the $x = \theta$ component is of the form:

$$\theta = x = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} + te^{-t}(-\frac{1}{2}\cos(t)) + \frac{3}{2},$$

which agrees with our second order solution.

□

Repeated eigenvalues system and sketch Consider the equation

$$\theta'' - 2\theta' + \theta = 3 + e^{-t}$$

and its related system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ 3 + e^{-t} \end{pmatrix}.$$

Proof. 1. First we solve the second order equation. The homogeneous part is

$$x_h = c_1 e^t + c_2 e^t t.$$

2. Then we apply our guess: $x_{nh} = at^s e^{-t}$. Since -1 is not a root, we have $s = 0$ and so it remains to find a:
$$a(e^{-t} + 2e^{-t} + e^{-t}) = e^{-t} \Rightarrow a = \frac{1}{4}.$$

3. So the solution is:
$$x_{gen} = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{4}e^{-t} + 3.$$

4. Next we solve the corresponding nonhomogeneous system. The eigenpair is $(1, \binom{1}{1})$ and $\eta = k\binom{1}{1} + \binom{0}{1}$ and so the homogeneous part is:
$$\mathbf{x} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t (t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

5. So the fundamental matrix is
$$\mathbf{\Psi} = \begin{pmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{pmatrix}.$$

6. The system becomes
$$\begin{cases} x_{1,1}(t)v_1' + x_{1,2}(t)v_2' = g_1(t) \\ x_{2,1}(t)v_1' + x_{2,2}(t)v_2' = g_2(t) \end{cases} \Rightarrow$$
$$\begin{cases} e^t v_1' + te^t v_2' = 0 \\ e^t v_1' + (t+1)e^t v_2' = e^{-t} \end{cases} \Rightarrow$$
$$\begin{cases} v_1' = -te^{-2t} \\ v_2' = e^{-2t} \end{cases} \Rightarrow$$
$$\begin{cases} v_1 = \frac{1}{4}e^{-2t}(2t+1) \\ v_2 = \frac{e^{-2t}}{-2} \end{cases}$$

7. Therefore, the nonhomogeneous part is
$$\mathbf{x}_{nh,1} = \mathbf{\Psi}\mathbf{v} = \frac{e^{-t}}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

8. For the system $\mathbf{x}' = Ax + \binom{0}{3}$, we have
$$\mathbf{x}_{nh,2} = A^{-1}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

9. The general solution will be:
$$\mathbf{x}_{gen} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t (t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) + \frac{e^{-t}}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

10. Indeed the $x = \theta$ component is of the form:
$$\theta = x = c_1 e^t + c_2 te^t + \frac{e^{-t}}{4} + 3,$$
which agrees with our second order solution.

$\square$

2 Autonomous systems and Linearization

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = x + y^2, \frac{\mathrm{d}y}{\mathrm{dt}} = x + y.$$

- nullcline,
- linearize and global chart,
- Lyapunov.

Proof. 1. First we find the critical points:

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{dt}} = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = 0 \Rightarrow \\
&x = -y \text{ and } y(y-1) = 0 \Rightarrow (x, y) = (0,0), (-1,1).
\end{aligned}$$

2. We linearize around the origin:

$$\begin{aligned}
\mathbf{x}' &= \begin{pmatrix} 1 & 2y \\ 1 & 1 \end{pmatrix}|_{(0,0)}\mathbf{x} \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{x}
\end{aligned}$$

3. The eigenpair is $(1, \binom{0}{1})$ and $\eta = k\binom{0}{1} + \binom{1}{0}$ and so the solution is:

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^t (t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

4. The Lyapunov function is $V(x, y) = \frac{3}{4}x^2 - \frac{1}{2}xy + \frac{1}{2}y^2$.
5. We linearize around the (-1,1):

$$\begin{aligned}
\mathbf{x}' &= \begin{pmatrix} 1 & 2y \\ 1 & 1 \end{pmatrix}|_{(-1,1)}(\mathbf{x} - \begin{pmatrix} -1 \\ 1 \end{pmatrix}) \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\end{aligned}$$

6. The eigenpairs are $(1 + \sqrt{2}, \binom{\sqrt{2}}{1}), (1 - \sqrt{2}, \binom{-\sqrt{2}}{1})$ and so the solution is:

$$\mathbf{x} = c_1 e^{(1+\sqrt{2})t} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} + c_2 e^{(1-\sqrt{2})t} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

7. The Lyapunov function is $V(x, y) = \frac{3}{4}x^2 - \frac{1}{2}xy + \frac{1}{2}y^2$.

□

Nullcline and linearize Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 1 - xy, \frac{\mathrm{d}y}{\mathrm{dt}} = x - y^3.$$

1. First we find the critical points:

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{dt}} = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = 0 \Rightarrow \\ &1 = xy \text{ and } x = y^3 \Rightarrow (x, y) = (-1, -1), (1, 1).\end{aligned}$$

2. We linearize around the critical point (-1,-1):

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix} -y & -x \\ 1 & 3y^2 \end{pmatrix}|_{(-1,-1)} \left(\mathbf{x} - \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}\end{aligned}$$

3. For solutions of the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{v}$ we have

$$\mathbf{x}_{nh} = \mathbf{A}^{-1}(-\mathbf{v}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

4. The eigenpair is $(2 + \sqrt{2}, \binom{-1+\sqrt{2}}{1}), (2 - \sqrt{2}, \binom{-1-\sqrt{2}}{1})$ and so the general solution is:

$$\mathbf{x} = c_1 e^{(2+\sqrt{2})t} \begin{pmatrix} -1 + \sqrt{2} \\ 1 \end{pmatrix} + c_2 e^{(2-\sqrt{2})t} \begin{pmatrix} -1 - \sqrt{2} \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

5. We linearize around the critical point (1,1):

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix} -y & -x \\ 1 & 3y^2 \end{pmatrix}|_{(1,1)} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}\end{aligned}$$

6. The eigenpair is $(1 + \sqrt{3}, \binom{-2+\sqrt{3}}{1}), (1 - \sqrt{3}, \binom{-2-\sqrt{3}}{1})$ and so the general solution is:

$$\mathbf{x} = c_1 e^{(1+\sqrt{3})t} \begin{pmatrix} -2 + \sqrt{3} \\ 1 \end{pmatrix} + c_2 e^{(1-\sqrt{3})t} \begin{pmatrix} -2 - \sqrt{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3 Laplace transform

repeated and forcing Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ e^{-(t-1)}(1 - f_{step}(t, 1)) \end{pmatrix}.$$

1. First we take the Laplace transform forcing term:

$$\mathcal{L}\Big\{\begin{pmatrix} 0 \\ e^{-(t-1)}(1 - f_{step}(t,1)) \end{pmatrix}\Big\} = \begin{pmatrix} 0 \\ \frac{e^{-s}}{s+1} \end{pmatrix}.$$

2. Therefore,

$$\begin{aligned}\mathcal{L}\Big\{\mathbf{x}\Big\} &= (s\mathbf{I} - \mathbf{A})^{-1}\begin{pmatrix} 0 \\ \frac{e^{-s}}{s+1} \end{pmatrix} \\ &= \frac{1}{(s-1)^2}\begin{pmatrix} s-2 & 1 \\ -1 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{e^{-s}}{s+1} \\ &= \frac{1}{(s-1)^2}\begin{pmatrix} -1 \\ s \end{pmatrix}\frac{e^{-s}}{s+1}.\end{aligned}$$

Using the cover-up method we compute the partial fraction for the first component:

$$-e^{-s}\left(\frac{1}{2(s-1)^2} - \frac{1}{4(s-1)} + \frac{1}{4(s+1)}\right)$$

and for the second component

$$-e^{-s}\left(\frac{1}{2(s-1)^2} + \frac{1}{4(s-1)} - \frac{1}{4(s+1)}\right).$$

Next we invert the first component:

$$\begin{aligned} x &= \mathcal{L}^{-1}\Big\{-e^{-s}\left(\frac{1}{2(s-1)^2} - \frac{1}{4(s-1)} + \frac{1}{4(s+1)}\right)\Big\} \\ &= -\frac{e^{t-1}(t-1)}{2}f_{heavy}(t,1) + \frac{e^{t-1}}{4}f_{heavy}(t,1) - \frac{e^{1-t}}{4}f_{heavy}(t,1)\end{aligned}$$

and the second component

$$y = -\frac{e^{t-1}(t-1)}{2}f_{heavy}(t,1) - \frac{e^{t-1}}{4}f_{heavy}(t,1) + \frac{e^{1-t}}{4}f_{heavy}(t,1).$$

3. So in vector notation we have the general solution:

$$\begin{aligned}\mathbf{x}_{gen} =& c_1 e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t}\left(t\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ & - \frac{e^{t-1}(t-1)}{2}f_{heavy}(t,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & + \left[\frac{e^{t-1}}{4}f_{heavy}(t,1) - \frac{e^{1-t}}{4}f_{heavy}(t,1)\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix}.\end{aligned}$$