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Chapter 1

First order

1.1 First order odes basic concepts

We first study order one ODEs:

$$y' = f(x, y).$$

For nice enough functions f we have existence and uniqueness. Here are some examples:

- For any continuous function, f, that depends *only* on t gives a solution y.
  - For $f(t, y) = t$ we get $y' = t \Rightarrow y(t) = \frac{t^2}{2} + c$, where c is some constant from integrating.
  - For $f(t, y) = \cos(t)$ we get $y' = \cos(t) \Rightarrow y(t) = \sin(t) + c$.
- For linear $y' = f(t, y) = ay$ we obtain the solution $y(t) = ce^{a \cdot t}$ (integrating factors).
- For $y' = f(t, y) = \frac{t^2}{1-y^2}$, $|y| \neq 1$ we obtain $-t^3 + 3y - y^3 = c$ (separable equations).
- For $y' = f(t, y) = -\frac{2t+y^2}{2ty}$ we obtain $t^2 + ty^2 = c$ (exact equations).

1.1.1 Direction fields: Vector field interpretation

A useful tool will be the vector field interpretation of $y' = f(t, y)$, for continuous f. A solution curve $\gamma(t) := (t, y(t))$ has slope $\gamma'(t_0) = (1, y'(t_0)) = (1, f(t_0, y(t_0)))$ at the point $(t_0, y(t_0))$. Also, recall that the vector $\gamma'(t)$ is tangent to the curve γ. So if we plot the vector $\gamma'(t) = (1, f(t, y(t)))$ we obtain qualitative behaviour of the solution. For example, for $f(t, y) = t$ we have the following linear vector field $(1, f(t, y)) = (1, t)$ depicted in Figure 1.1.1:

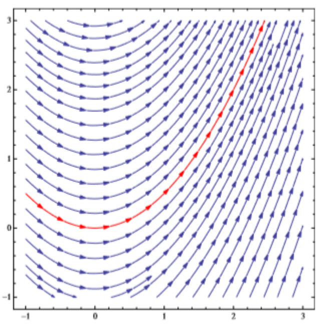


Figure 1.1.1: the horizontal axis is t and the vertical is $y(t)$.

Every solution curve corresponds to a distinct function of the form $y_c(t) = \frac{t^2}{2} + c$ and the red curve is for $y_0(t) = \frac{t^2}{2}$ (i.e when $c = 0$). The arrows are pointing in the direction $(1, t)$. The power of this method is that for ODEs for which we do not have an explicit solution we are still able to plot the vector field to determine behaviour of the solution function. For example, for the competing species equations (9.4, ex.1)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x - y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{4}(3 - 4y - 2x)$$

we don't have an exact solution but we do have a vector field diagram that we will keep returning to (see Figure 1.1.2):

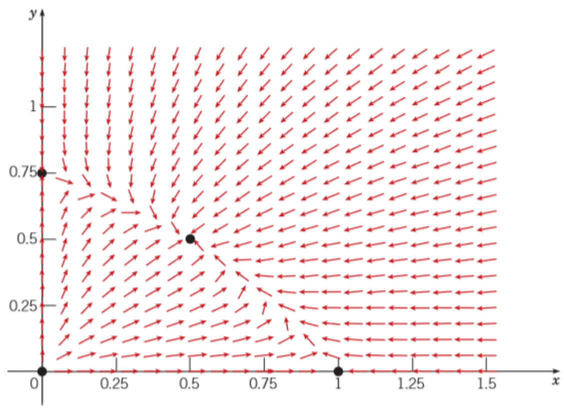


Figure 1.1.2: the horizontal axis is $x(t)$ and the vertical is $y(t)$.

Method formal steps

1. First draw the curves $(x, y(x))$ where $f(x, y) = 0$ (called the **equilibrium solutions** or **critical points**).

2. These will separate the regions where $f(x, y) > 0$ and $f(x, y) < 0$ and in turn where arrows point up and down respectively. Evaluate f at a point to decide what sign it has.

3. Next draw the curves where $f(x, y) = c$ for fixed constant c.

Examples

- Consider the equation $y' = f(x, y) = y(x - y)$. We illustrate the above formal steps to obtain the diagram depicted in Figure 1.1.3

  1. We have that $y(x - y) = 0 \Rightarrow$ the curves are the $y = 0$ (x-axis) and the line $x = y$.
  2. We have $y(x - y) > 0$ when $y > 0$ and $x > y$ or $y < 0$ and $x < y$. Therefore, the arrows are pointing up: below the $x = y$ line in $y > 0$ and above $x = y$ in $y < 0$.
  3. So we observe that if the initial condition satisfies $y_0(x_0 - y_0) > 0$ and $y_0 > 0$ then $y(x) \to +\infty$ and similarly for the other sign.

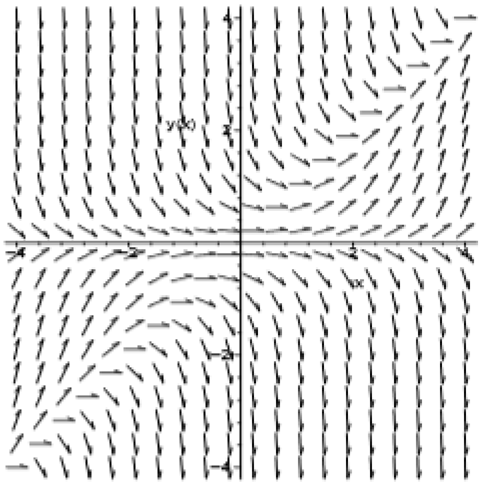


Figure 1.1.3: Direction field of $y' = y(x - y)$.

4. En route to studying the competing species we will need the logistic equation (2.5):

$$\frac{dy}{dt} = r\Big(1 - \frac{y}{K}\Big)y,$$

where $r > 0$ is called the intrinsic growth rate and K is the saturation level.

(a) First we find the equilibrium solutions: $r\big(1 - \frac{y}{K}\big)y = 0 \Rightarrow y = K$ or $y = 0$. So the equilibrium solutions are $\{\varphi_1(t), \varphi_2(t)\} = \{0, K\}$.

(b) We have $\big(1 - \frac{y}{K}\big)y > 0$ when $K > y$ and $y > 0$ or $K < y$ and $y < 0$.

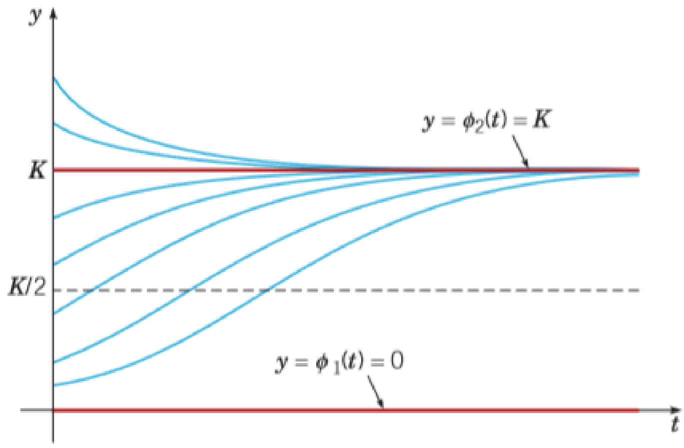


Figure 1.1.4: Direction field for the logistic equation.

(c) So we observe that if initially $y_0 > 0$ then $y \to K$ (i.e. will limit to the saturation level.)

1.2 Methods for first order

Given a specific first-order differential equation to be solved, we can attack it by means of the following steps:

1. Is it separable? If so, separate the variables and integrate.

2. Is it linear? That is, can it be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = Q(t)?$$

If so, multiply by the integrating factor $\mu(t) = exp\{\int P(s)\mathrm{d}s\}$.

3. Is it exact? That is, can the equation be written in the form:

$$M(t, y)\mathrm{d}t + N(t, y)\mathrm{d}y = 0$$

with $M_y = N_t$?

4. If the equation is not exact, do we have either $\frac{M_y - N_x}{N}$ or $\frac{M_y - N_x}{M}$ being a function of only x, y respectively? If so then it can be made exact.

5. If the equation as it stands is not separable, linear, or exact, is there a plausible substitution that will make it so?

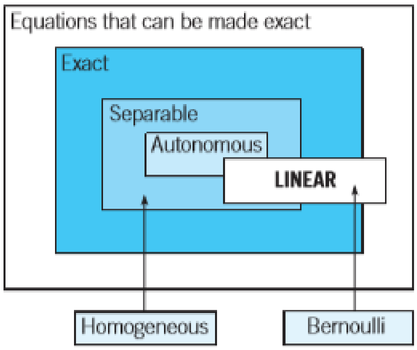


Figure 1.2.1: Summarizing diagram of all the methods

1.2.1 Method 1: Separable equations

An equation is called **separable** if we can factor $f(t, y) = f_1(t) \cdot f_2(y)$ for some functions f_1 and f_2 that only depend on t and y respectively. Assuming $f_2(y)$ is not 0 we obtain:

$$y' = f_1(x) \cdot f_2(y) \Leftrightarrow M(t) + N(y)y' = 0$$

where $M(t) := -f_1(t), N(y) := \frac{1}{f_2(y)}$.

Method formal steps

1. Separate variables to either side

$$M(t) + N(y)\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow N(y)\mathrm{d}y = -M(t)\mathrm{d}t$$

2. Integrate both sides

$$\int N(y)\mathrm{d}y = -\int M(t)\mathrm{d}t.$$

Example-presenting the method

Let $P(t)$ be the number of dollars in a savings account at time t and suppose that the interest is compounded continuously at an annual interest rate r(t), that varies in time. Then, after Δt units of time, we expect, provided the interest rate does not change too much over a small period of time, that we have obtained $\frac{\Delta t}{1}$ of the total amount of annual interest, $r(0)$, which updates the amount in the savings account at time Δt to be:

$$P(\Delta t) = (\Delta t \cdot r(0))P(0).$$

In particular, through similar reasoning we see that for any time t we have

$$P(t + \Delta t) - P(t) = (t + \Delta t) \cdot r(t)P(t) - t \cdot r(t)P(t) = (\Delta t)r(t)P(t)$$

provided $r(t)$ does not change too much over the time period $[t, t + \Delta t]$. In short, we obtain $\Delta P = r(t)P(t)\Delta t$ or in continuously updated time:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = r(t)P(t).$$

Example:

Assume that $r(t) = t^2$ and $P(0) = \$\,10^3$.

1. We separate
$$\frac{\mathrm{d}P}{\mathrm{d}t} = r(t)P \Rightarrow \frac{1}{P}\mathrm{d}P = r(t)\mathrm{d}t$$

2. We integrate
$$\int \frac{1}{P}\mathrm{d}P = \int t^2 \mathrm{d}t \Rightarrow$$
$$\ln|P| = \frac{t^3}{3} + c$$
for some constant c. Since $P \geq 0$ we obtain
$$P = \$\,exp\left\{\frac{t^3}{3} + c\right\}.$$

3. Plugging in the initial condition we get
$$P(t) = \$\,10^3 \cdot exp\left\{\frac{t^3}{3}\right\}.$$

General result:

If M, N are continuous, we can obtain an implicit solution by a clever use of chain-rule (§2.1). First lets recall the definition of an implicit solution as well as the implicit function theorem. An **implicit equation** is of the form $f(t, y(t)) = 0$ for some function f, and $y(t)$ is the **implicit solution**. There is an existence result for such equations called the **Implicit function theorem** which we state here in a simple form:

Theorem 1.2.1. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable (i.e. both f and its derivatives are continuous) and $\partial_{x_2} f(x_1, x_2) \neq 0$, then there exists function $y : \mathbb{R} \to \mathbb{R}$ s.t.*

$$f(t, y(t)) = 0.$$

Separable equation

The equations $M(t) + N(y)y' = 0$ with $y(t_0) = y_0$, where M and N are continuous, have implicit solutions of the form

$$\int_{t_0}^{t} M(s)\mathrm{d}s + \int_{y_0}^{y} N(x)\mathrm{d}x = 0.$$

Proof.

1. Let functions H_M, H_N be antiderivatives of M, N respectively (i.e. $H_M'(t) = M(t)$, $H_N'(y) = N(y)$). Then we can rewrite our ODE as

$$H_M'(t) + H_N'(y)\frac{dy}{dt} = 0.$$

2. For the second term we note that by chain rule we have $\frac{d}{dt}(H_N(y)) = H_N'(y)\frac{dy}{dt}$ and so we can rewrite the equation as:

$$0 = H_M'(t) + H_N'(y)\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(H_M(t) + H_N(y)).$$

3. This implies that $H_M(t) + H_N(y) = c$ for some constant c. Therefore, by the fundamental theorem of calculus we have

$$\int_{t_0}^{t} M(s)\mathrm{d}s + \int_{y_0}^{y} N(x)\mathrm{d}x = H_M(t) - H_M(t_0) + H_N(y) - H_N(y_0) = 0$$

where the last equation follows by using $y(t_0) = y_0$ to give $c = H_M(t_0) + H_N(y_0)$. □

Examples

- It can even tackle non-linear equations:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t-5}{y^2}, \quad y(0) = 1.$$

We first separate and integrate

$$\int y^2 \mathrm{d}y = \int (t-5)\mathrm{d}t.$$

This gives

$$\frac{y^3}{3} = \frac{t^2}{2} - 5t + c$$

and using that $y(0) = 1$ we obtain

$$y(t) = \left(\frac{3t^2}{2} - 15t + 1\right)^{1/3}.$$

- Restricted solution:

$$y' = \frac{2x-3}{y}, \quad y(0) = 2$$

We separate and integrate to get

$$\int y\,\mathrm{d}y = \int (2x-3)\mathrm{d}x \quad \Rightarrow \quad \frac{y^2}{2} = x^2 - 3x + c.$$

Using the initial condition we obtain

$$y = \sqrt{2(x^2 - 3x + 2)} = \sqrt{2(x-1)(x-2)}$$

and since the square root is only defined for positive numbers, we require

$$x > 2 \;\text{ or }\; x < 1.$$

- Asymptotic solution:

$$\frac{dy}{dt} = t(1 + b \cdot y), \quad y(0) = 0$$

for $b \neq 0$. We separate and integrate

$$\frac{1}{b}\ln|1 + b \cdot y| = \frac{t^2}{2} + c$$

using the initial condition we obtain $c = 0$ which tells us that

$$\frac{1}{b}\ln|1 + b \cdot y| = \frac{t^2}{2} \geq 0$$

which tells us that $|1 + b \cdot y| \geq 1 > 0$ and so $1 + b \cdot y$ does not change sign. Observe that by the initial condition we have $1 + b \cdot y \geq 1$ and so we obtain

$$\frac{1}{b}\ln(1 + b \cdot y) = \frac{t^2}{2} + c$$

and after solving for y we obtain

$$y = \frac{1}{b}\left(exp\left\{b\left(\frac{t^2}{2} + c\right)\right\} - 1\right).$$

So the asymptotic behaviour, as $t \to \pm\infty$, depends on b:

$$y \to +\infty \text{ if } b > 0$$
$$y \to \frac{-1}{b} \text{ if } b < 0\ .$$

- An example with implicit solution:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t^2 + 1}{\cos(y) + e^y}, \quad y(0) = \pi.$$

We separate and integrate

$$\int (\cos(y) + e^y)\mathrm{d}y = \int \left(t^2 + 1\right)\mathrm{d}t$$
$$\Rightarrow \sin(y) + e^y = \frac{t^3}{3} + t + c.$$

Using $y(0) = \pi$ we get

$$\sin(y) + e^y = \frac{t^3}{3} + t + e^\pi.$$

This equation cannot be directly solved in terms of y. One only has an implicit solution that is obtained numerically.

Applied example

- Let $X(t)$ denote the national product, $K(t)$ the capital stock and $L(t)$ the number of workers in a country at time t. We assume the following relations:

$$X = AK^{1-\alpha}L^\alpha, \quad K' = sX, \quad L = L_0 e^{\lambda t},$$

where A, s, L_0, λ are positive constants and $0 < \alpha < 1$ is called elasticity. The first equation is the Cobb-Douglas production model. The second equation says that aggregate investment is proportional to output. The third equation says that the labour forces grows exponentially. Using these three we obtain the equation

$$K' = sX = ce^{\alpha\lambda t}K^{1-\alpha},$$

where $c := AsL_0^\alpha$. We also let K_0 denote the initial capital stock (the stock at time $t = 0$). We first separate variables and integrate:

$$\int K^{\alpha-1}\mathrm{d}K = \int ce^{\alpha\lambda t}\mathrm{d}t.$$

This gives:

$$\frac{K^\alpha}{\alpha} = c\frac{e^{\alpha\lambda t}}{\alpha\lambda} + C.$$

We find the constant C by plugging in the initial condition, so we get $C := \frac{K_0^\alpha}{\alpha} - \frac{sAL_0^\alpha}{\alpha\lambda}$ and in turn

$$K = \left[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(e^{\alpha\lambda t} - 1)\right]^{1/\alpha}.$$

Next we study the asymptotic behaviour of the ratio $\frac{K}{L}$ (also called the capital-labor ratio) as $t \to +\infty$:

$$\frac{K}{L} = \frac{1}{L_0 e^{\lambda t}}\left[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(e^{\alpha\lambda t} - 1)\right]^{1/\alpha}.$$

For simplicity we will first compute the asymptotic behaviour of

$$\left(\frac{K}{L}\right)^\alpha = \frac{1}{L_0^\alpha e^{\lambda\alpha t}}\left[K_0^\alpha + \frac{sAL_0^\alpha}{\lambda}(e^{\alpha\lambda t} - 1)\right].$$

We note that the only surviving term as t tends to $+\infty$ is the following:

$$\left(\frac{K}{L}\right)^{\alpha} \approx \frac{1}{L_0^{\alpha} e^{\lambda\alpha t}} \frac{sAL_0^{\alpha}}{\lambda} e^{\alpha\lambda t} = \frac{sA}{\lambda}.$$

Therefore, the capital-labor ratio converges to

$$\lim_{t\to\infty} \frac{K}{L} = \left(\frac{sA}{\lambda}\right)^{1/\alpha}.$$

This means that in the long term the national product per worker will be approximately constant

$$\frac{X}{L} = \frac{AK^{1-\alpha}L^{\alpha}}{L} = A\left(\frac{K}{L}\right)^{1-\alpha} \approx A\left(\frac{sA}{\lambda}\right)^{\frac{1-\alpha}{\alpha}} = A\left(\frac{sA}{\lambda}\right)^{\frac{1}{\alpha}-1}.$$

1.2.2 Method 2: Linear and Integrating factor

An equation is called linear if it is of the form:

$$y' + p(t)y = g(t)$$

for continuous functions p, g where p is assumed to not change signs.

Method formal steps

1. Starting from $y' + p(t)y = g(t)$, we multiply both sides by a function $\mu(t)$ that we will determine later:

$$\mu(t) \cdot y' + \mu(t)p(t) \cdot y = \mu(t)g(t).$$

2. If we had $\mu' = \mu(t)p(t)$ then observe that we can use the product rule:

$$\mu \cdot y' + \mu' \cdot y = \mu g(t) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = \mu g(t).$$

3. Therefore,

$$y(t) = \frac{1}{\mu(t)} \int_0^t \mu(s)g(s)\mathrm{d}s + c.$$

4. To find the desired function $\mu(t)$ we use the imposed condition, $\mu'(t) = \mu(t)p(t)$, to get:

$$\mu(t) = exp\left\{\int_0^t p(s)\mathrm{d}s\right\}.$$

Note that we ignore the constant of integration in step 4 since we just want to find some function satisfying $\mu'(t) = \mu(t)p(t)$ and not all such functions.

For linear equations we can also draw a direction field to obtain qualitative behaviour when an exact solution is not possible. The idea is to draw the curves along which $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and then study the regions where $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$.

1. As explained above we first draw the curves along which
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) = 0 \;\Rightarrow\; p(t)y = g(t) \;\Rightarrow\; y = \frac{g(t)}{p(t)}$$
provided that $p(t)$ is not 0.

2. Then we identify the regions where
$$p(t)y - g(t) > 0, p(t)y - g(t) < 0 \quad Rightarrow \;\; y > \frac{g(t)}{p(t)}, \;\; y < \frac{g(t)}{p(t)}$$
respectively, provided that $p(t) > 0$ (the inequalities are reversed otherwise).

Example-presenting the method

A rock contains two radioactive isotopes R_1, R_2 with R_1 decaying into R_2 with rate $5e^{-10t}$ kg/sec. So if $y(t)$ is the total mass of R_2, we obtain:
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \text{rate of creation of } R_2 \text{ - rate of decay of } R_2\\ &= 5e^{-10t} - ky(t),\end{aligned}$$
where $k > 0$ is the decay constant for R_2. Also assume that $y(0) = 40\,\mathrm{kg}$. Lets start by drawing the direction field to guess the solution:

1. The tangent is zero along the curve (equilibrium solution) so setting $\frac{dy}{dt} = 0$ and solving for y gives:
$$y(t) = \frac{5e^{-10t}}{k}$$

2. From this we see that $5e^{-10t} - ky(t) = \frac{dy}{dt} > 0$ when $\frac{5}{k}e^{-10t} > y$. Similarly we see that $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$ when $\frac{5}{k}e^{-10t} < y$.

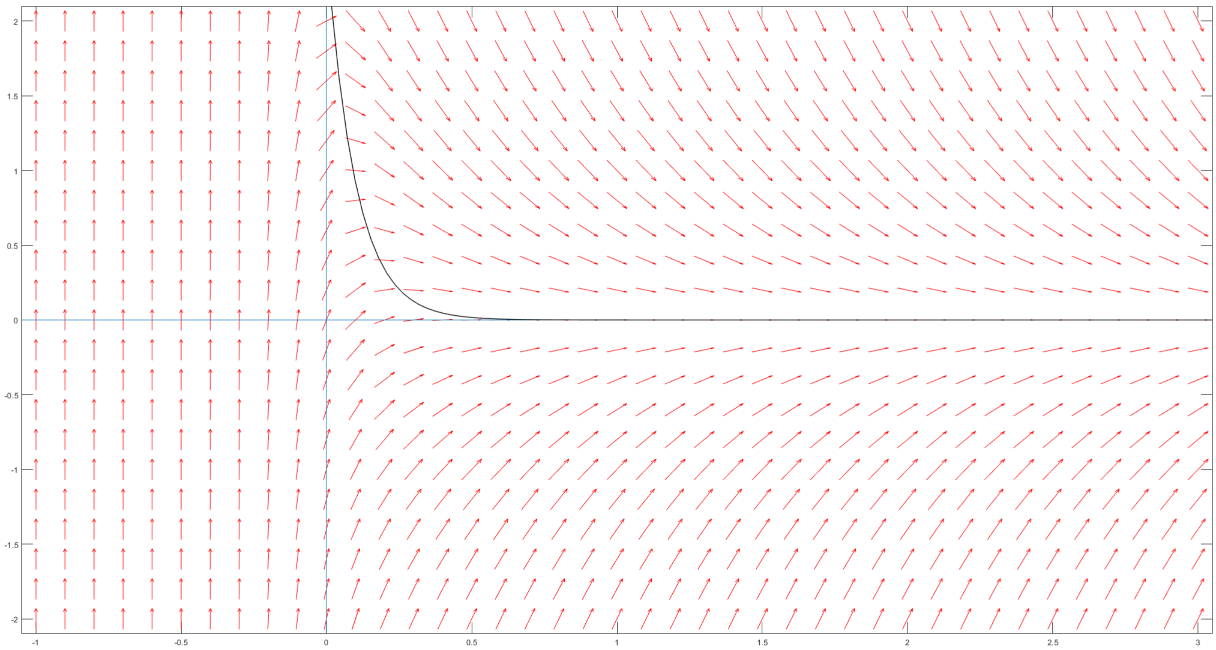


Figure 1.2.2: The black line is the equilibrium solution

3. So we observe that if $t > 0$, the solution $y \to 0$ as $t \to +\infty$.

Next we find the explicit solution:

1. We first multiply by the function $\mu(t)$, which we will determine specifically in the next step,
$$\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \mu(t)ky(t) = \mu(t)5e^{-10t}.$$

2. We require $\mu'(t) = \mu(t)k$, which can be easily solved to give:
$$\mu(t) = e^{kt}.$$

3. Then by product rule we have
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{kt}y(t)) = 5e^{(k-10)t} \Rightarrow y(t) = 5e^{-10t} + e^{-kt}c = 5e^{-10t} + 35e^{-kt}$$
where we used the initial condition $y(0) = 40\,\mathrm{kg}$ to determine that $c = 35$.

4. Therefore, we indeed obtain that $y \to 0$ as $t \to +\infty$.

General result

We can obtain a solution by a clever use of product-rule (§2.1).

Integrating factor

The $y' + p(t)y = g(t)$ have solutions of the form

$$y(t) = \frac{1}{\mu(t)} \int_0^t \mu(s)g(s)ds + c,$$

where $\mu(t) = exp\Big\{\int_0^t p(s)ds\Big\}$, provided that g and p are continuous.

Proof.

$$\mu(t) \cdot (y' + p(t)y) = \mu(t) \cdot g(t) \Rightarrow \mu(t) \cdot y' + \mu(t)p(t) \cdot y = \mu(t)g(t).$$

We note that if we pick $\mu(t)$ so that $\mu'(t) = \mu(t)p(t)$, we can then rewrite

$$\mu \cdot y' + \mu' \cdot y = \mu(t)g(t) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = \mu(t)g(t).$$

Therefore,

$$y(t) = \frac{1}{\mu(t)} \int_0^t \mu(s)g(s)\mathrm{d}s + c.$$

To determine $\mu(t)$ we use $\mu'(t) = \mu(t)p(t)$

$$\begin{aligned} \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} &= \mu(t)p(t) \\ \Rightarrow \frac{1}{\mu(t)} \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} &= p(t) \end{aligned}$$

we integrate both sides

$$\begin{aligned} \int_0^t \frac{1}{\mu(s)} \mathrm{d}\mu(s) &= \int_0^t p(s)\mathrm{d}s \\ \Rightarrow \ln|\mu(t)| - \ln|\mu(0)| &= \int_0^t p(s)\mathrm{d}s + c' \\ \Rightarrow \ln\left(\frac{|\mu(t)|}{|\mu(0)|}\right) &= \int_0^t p(s)\mathrm{d}s + c' \\ \Rightarrow \frac{|\mu(t)|}{|\mu(0)|} &= exp\Big\{\int_0^t p(s)\mathrm{d}s + c'\Big\} \\ \Rightarrow |\mu(t)| &= |\mu(0)| exp\Big\{\int_0^t p(s)\mathrm{d}s + c'\Big\} \end{aligned}$$

where, for simplicity, we set $\mu(0) = 1$ as well as $c' = 0$. Observe that since the exponential function is non-negative then the sign of $\mu(t)$ is determined by $\mu(0)$ (by continuity). Since we chose $\mu(0) > 0$ then we obtain

$$\mu(t) = exp\Big\{\int_0^t p(s)\mathrm{d}s\Big\}$$

$\square$

Examples

- Consider the equation
$$y' - 2y = t^2 e^{2t}.$$
First we look at the equilibrium solution:
  1. We have equilibrium solution $y' = 0 \Rightarrow y = -\frac{1}{2}t^2 e^{2t}$.
  2. We have positive/growing region $y' > 0 \Rightarrow t^2 e^{2t} > -2y$

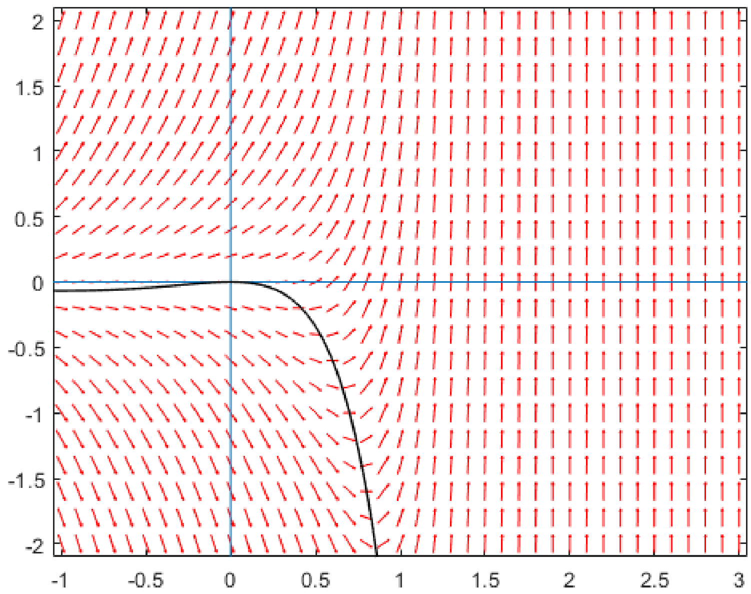


Figure 1.2.3: The black line is the equilibrium solution

Next we obtain the solutions (notice that there is no initial condition so we anticipate that we will have many solutions): 1$^{\text{st}}$: we multiply by $\mu(t)$
$$\mu y' - 2\mu y = \mu t^2 e^{2t}.$$
2$^{\text{nd}}$: We solve (and choose the constant of integration to be 0)
$$\mu'(t) = -2\mu(t) \Rightarrow \mu = exp\{-2t\}.$$
3$^{\text{rd}}$: We obtain
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-2t}y) = e^{-2t}t^2 e^{2t} = t^2$$
we integrate both sides to get
$$y(t) = e^{2t}\left(\frac{t^3}{3} + c\right).$$

- Consider equation (§2.1)
$$y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}.$$
We start with the 1$^{\text{st}}$ step of multiplying by $\mu(t)$ of our choice
$$\mu(t)y' + \mu(t)\frac{1}{2}y = \mu(t)\frac{1}{2}e^{t/3}.$$

2nd step: We observe that to make use of product rule we need

$$\begin{aligned}
&\mu'(t) = \frac{1}{2}\mu(t)\\
\Rightarrow &\ln|\mu(t)| = \frac{1}{2}t + c'\\
\Rightarrow &\mu(t) = e^{t/2}
\end{aligned}$$

by setting $c' = 0$ and assuming $\mu(t) \geq 0$ to simplify.
3rd step: by product rule we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)\cdot y) = \mu(t)\frac{1}{2}e^{t/3} = \frac{1}{2}e^{\frac{5t}{6}}\\
\Rightarrow &y(t) = e^{-t/2}(\frac{3}{5}e^{\frac{5t}{6}} + c)\\
\Rightarrow &y(t) = \frac{3}{5}e^{\frac{t}{3}} + ce^{-\frac{t}{2}}.
\end{aligned}$$

Applied example

- Returning to the compounded interest example, suppose that we also have deposits and withdrawals with rates $d(t), w(t)$ respectively. Then the equation will be

$$P' = r(t)P + d(t) - w(t).$$

To see that this is the expected differential equation observe that at time t, for a small amount of time Δt, we expect

$$P(t+\Delta t) \approx \underbrace{P(t)}_{\substack{\text{amount at}\\ \text{time } t}} + \underbrace{r(t)(\Delta t)P(t)}_{\substack{\text{interest earned}\\ \text{at time } t}} + \underbrace{d(t)\Delta t}_{\substack{\text{deposit}\\ \text{amount}}} - \underbrace{w(t)\Delta t}_{\substack{\text{withdrawal}\\ \text{amount}}}$$

where we have assumed that the interest rate, the deposit rate, and withdrawal rate do not change too much in the interval $[t, t+\Delta t]$. Notice that the presence of Δt in $d(t)\Delta t$ and $w(t)\Delta t$ is because $d(t)$ and $w(t)$ represent the *rates* for which we are depositing and withdrawing at time t respectively. Subtracting $P(t)$ and dividing by Δt gives

$$\frac{\Delta P}{\Delta t} \approx r(t)P(t) + d(t) - w(t)$$

and so letting Δt tend to 0 leads to the differential equation.

1. We first multiply by unknown factor μ

$$\mu P' - \mu r(t)P = \mu(d(t) - w(t)).$$

2. Then we obtain μ

$$\mu' = -\mu r(t) \Rightarrow \mu = exp\left\{-\int_0^t r(s)\mathrm{d}s\right\}.$$

3. Therefore, by product rule we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(P\mu) = exp\Big\{-\int_0^t r(s)\mathrm{d}s\Big\}(d(t) - w(t))$$
$$\Rightarrow P(t)exp\Big\{-\int_0^t r(s)\mathrm{d}s\Big\} = P(0) + \int_0^t exp\Big\{-\int_0^x r(s)\mathrm{d}s\Big\}(d(x) - w(x))\mathrm{d}x$$
$$\Rightarrow P(t) = exp\Big\{\int_0^t r(s)\mathrm{d}s\Big\}\Big(P(0) + \int_0^t exp\Big\{-\int_0^x r(s)\mathrm{d}s\Big\}(d(x) - w(x))\mathrm{d}x\Big)$$

4. This equation states that the present discounted value on the left, is the sum of the initial assets $P(0)$ plus the present discounted value of deposits minus withdrawals.

- When the price of a good is p, the total demand is $D(p) = a - bp$ and the total supply is $S(p) = \alpha + \beta p$, where a, b, α, and β are positive constants. Observe that the slope of the linear function respresenting D is negative while the slope of the linear function representing S is positive. This matches the expectation that when price is increased consumers are less interested in goods (making goods expensive prevents some from purchasing the item) and that suppliers are more inclined to produce that product (it is advantageous to sell items that generate more profit). When demand exceeds supply, price rises (people are willing to pay more for scarce items), and when supply exceeds demand it falls (common goods are easy to obtain, so consumers will simply look for lower prices). We assume that the speed at which the price changes is proportional to the difference between supply and demand. Specifically

$$p' = \lambda(D(p) - S(p))$$

for $\lambda > 0$.

1. We multiple by μ to get

$$\mu p' + \mu\lambda(b + \beta)p = \lambda\mu(a - \alpha).$$

2. We obtain μ

$$\mu' = \lambda(b + \beta)\mu \Rightarrow \mu = exp\{\lambda(b + \beta)t\}.$$

3. Therefore, by product rule we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu p) = \lambda(a - \alpha)exp\{\lambda(b + \beta)t\} \Rightarrow p(t) = c \cdot exp\{-\lambda(b + \beta)t\} + \frac{(a - \alpha)}{(b + \beta)}.$$

4. So as $t \to +\infty$ the price of this good converges to $\frac{(a-\alpha)}{(b+\beta)}$ (This is the equilibrium position for which demand is equal to supply. That is, the point for which the lines intersect.).

1.2.3 Method 3: Exact equations

Suppose the function $F(x, y)$ represents some physical quantity, such as temperature, in a region of the xy-plane. Then the level curves of F, where $F(x, y) =$ constant, could be interpreted as isotherms on a weather map (i.e curves on a weather map representing constant temperatures).

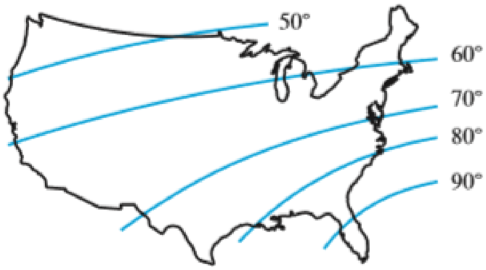


Figure 1.2.4: Level sets of constant temperature across the US.

Along one of these curves, $\gamma(x) = (x, y(x))$, of constant temperature we have, by Chain rule and the fact that the temperature, F, is constant on these curves:

$$0 = \frac{\mathrm{d}F(\gamma(x))}{\mathrm{d}x} = F_x + F_y \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Multiplying through by $\mathrm{d}x$ we obtain

$$0 = F_x \mathrm{d}x + F_y \mathrm{d}y.$$

Therefore, if we were not given the original function F but only an equation of the form:

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0,$$

we could set $F_x := M(x, y), F_y := N(x, y)$ and then by integrating figure out the original F.

1. First ensure that there is such an F, by checking the *exactness-condition*:

   $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

   This is because if there was such an F, then

   $$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

   where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ simply denote the partial derivatives with respect to the variables x and y respectively (where we hold the other variable constant while taking the derivative).

2. Second, integrate M, N with respect to x, y respectively:

   $$\int M(x, y)dx = \int F_x(x, y)dx = F(x, y) + a(y)$$
   $$\int N(x, y)dy = \int F_y(x, y)dy = F(x, y) + b(x)$$

   for some unknown functions a, b (these play the role of constant of integration when you integrate with respect to a single variable). So to obtain F it remains to determine either a or b.

3. Equate the above two formulas for $F(x, y)$:

$$\int M(x,y)\mathrm{d}x + a(y) = F(x,y) = \int N(x,y)\mathrm{d}y + b(x).$$

4. Since to find F it suffices to determine a or b, pick the integral that is easier to evaluate. Suppose that $\int M(x,y)\mathrm{d}x$ is easier to evaluate. To obtain $a(y)$ we differentiate both expression for F in y (for fixed x):

$$a'(y) = -\int M_y(x,y)\mathrm{d}x + N(x,y)$$

and then integrate in y:

$$a(y) = \int \left[-\int M_y(x,y)dx + N(x,y) \right] \mathrm{d}y + c.$$

Observe that a is only a function of y since if we differentiate the expression we found for a and use step 1 we find that

$$\begin{aligned} \frac{\partial a}{\partial x} &= \int \frac{\partial}{\partial x} \left[-\int M_y(x,y)\mathrm{d}x + N(x,y) \right] \mathrm{d}y \\ &= \int [-M_y(x,y) + N_x(x,y)]\mathrm{d}x \\ &= \int 0\mathrm{d}x \\ &= 0 \end{aligned}$$

Example-presenting the method

We measured the velocity field (u,v) of a two-dimensional incompressible flow and the curves are given by $\psi(x,y) = c$ for some yet unknown potential function ψ.

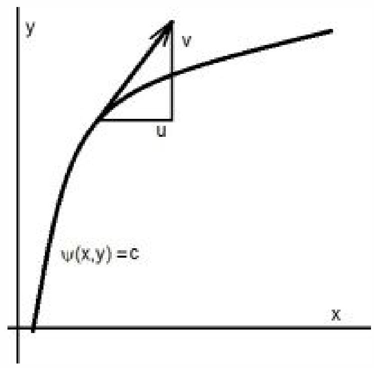


Figure 1.2.5: Velocity field of two dimensional flow.

For example suppose that $v(x,y) := -\frac{y}{x^2+y^2}$, $u(x,y) := \frac{x}{x^2+y^2}$ then by incompressibility our equation is:

$$udx - vdy = 0.$$

1. First we check the exactness condition:

$$-v_x = -\left(-\frac{y}{x^2+y^2} \right)_x = -\frac{2xy}{x^2+y^2}$$

$$u_y = -\frac{2xy}{x^2+y^2}.$$

2. Next we integrate u, v in y, x respectively:

$$\psi(x,y) = \int u(x,y)\mathrm{d}x + a(y) = \int \frac{x}{x^2+y^2}dx + a(y) = \frac{1}{2}\ln(x^2+y^2) + a(y)$$
$$\psi(x,y) = \int -v(x,y)dy + b(x) = \frac{1}{2}\ln(x^2+y^2) + b(x)$$

3. Next to obtain a(y) we equate the two ψ-formulas to obtain

$$\frac{1}{2}\ln(x^2+y^2) + a(y) = \frac{1}{2}\ln(x^2+y^2) + b(x)$$
$$\Rightarrow a(y) = b(x)$$
$$\Rightarrow a'(y) = 0$$

   and so $a(y) \equiv$ constant. Since a is constant then b is constant as well.

4. Therefore, the potential field is

$$\psi(x,y) = \frac{1}{2}\ln(x^2+y^2) + c.$$

5. Finally, we obtain the solution $(x, y(x))$ along which $\psi(x, y(x)) =$ constant $= C$:

$$\frac{1}{2}\ln(x^2+y(x)^2) = C - c \Rightarrow y(x) = \pm\sqrt{exp\{2(C-c)\} - x^2}.$$

6. For example, if we knew that $\psi(0,1) = 0$ then we would obtain $c = 0$. From this we see that the solution of the level set, say for $\psi = 60 = C$, is

$$x^2 + y^2 = exp\{2 \cdot 60\} \Leftrightarrow y(x) = \pm\sqrt{exp\{2 \cdot 60\} - x^2}.$$

   So the level curves are simply concentric circles.

General result:

Exact equation

If the equations $M(x,y) + N(x,y)y' = 0$ satisfy

1. M, N, M_y, N_x are continuous
2. the exactness condition:

$$M_y(x,y) = N_x(x,y)$$

then there exists a function, ψ, such that

$$\psi_x = M \text{ and } \psi_y = N.$$

Thus, along the solutions $\gamma(x) := (x, y(x))$ of the above equation we have:

$$\frac{d\psi(x,y)}{dx} = M(x,y)dx + N(x,y)dy = 0$$

and in turn we obtain the implicit solution $(x, y_c(x))$ for the following level set:

$$\psi(x, y_c(x)) = c.$$

Examples

- Consider the equation

$$(1 + e^x y + xe^x y)\mathrm{d}x + (xe^x + 2)\mathrm{d}y = 0.$$

First we check the exactness

$$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x}.$$

Second we integrate N(x,y)

$$F(x, y) = \int N(x, y)\mathrm{d}y + b(x) = (xe^x + 2)y + b(x).$$

Third, it remains to obtain b(x). We differentiate F in x

$$\begin{aligned} 1 + e^x y + xe^x y = M = F_x = (xe^x + e^x)y + b'(x) \\ \Rightarrow b(x) = \int [(1 + e^x y + xe^x y) - (xe^x + e^x)y]\mathrm{d}x + c = x + c. \end{aligned}$$

Therefore,

$$F(x, y) = (xe^x + 2)y + x + c.$$

So for a fixed level set $F \equiv C + c$ we obtain the solution:

$$y(x) = \frac{C - x}{xe^x + 2}.$$

- Consider the equation

$$(6xy - y^3)\mathrm{d}x + (4y + 3x^2 - 3xy^2)\mathrm{d}y = 0$$

First we check exactness

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}.$$

Second, we integrate $M(x, y)$

$$F(x, y) = \int M(x, y)\mathrm{d}x + a(y) = 3x^2 y - y^3 x + a(y).$$

Third we obtain $a(y)$:

$$\begin{aligned} a'(y) &= F_y - (3x^2 y - y^3 x)_y \\ &= 4y + 3x^2 - 3xy^2 - 3x^2 + 3y^2 x \\ &= 4y \\ \Rightarrow a(y) &= 2y^2. \end{aligned}$$

Therefore,

$$F(x,y) = 3x^2y - y^3x + 2y^2 + c.$$

So for a fixed level set $F \equiv C + c$ we obtain the implicit solution:

$$C = 3x^2y - y^3x + 2y^2.$$

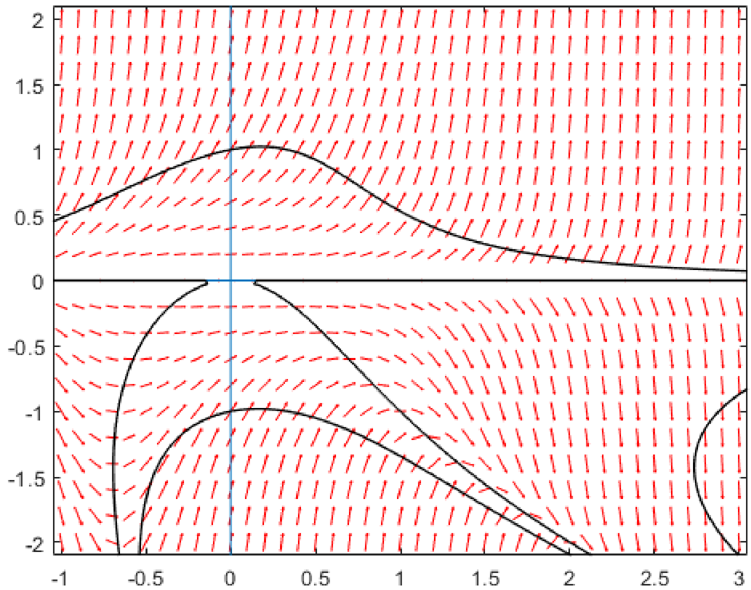


Figure 1.2.6: Direction field for $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$.

Applied examples

A geometric problem occurring often in engineering is that of finding a family of curves (orthogonal trajectories) that intersects a given family of curves orthogonally at each point. For example, we may be given the lines of force of an electric field and want to find the equation for the equipotential curves. Consider the level sets $F(x,y) = k$ then their slope is given by

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = -\frac{F_x}{F_y}.$$

Recall that two curves y_1, y_2 are perpendicular if their derivatives multiply to -1 (This definition comes from the fact that two lines are perpendicular if their slopes multiply to be -1. Hence, if we multiply the tangent vectors (slopes of tangent lines) then perpendicularity should be that the derivatives vectors multiply to be -1.) and so:

$$\frac{dy_2}{dx} = \frac{-1}{\frac{\mathrm{d}y_1}{\mathrm{d}x}} = \frac{F_y}{F_x} \Rightarrow F_y dx - F_x dy = 0$$

. Thus, by solving this equation we will obtain the implicit solution for the perpendicular curve. Consider $F(x,y) = xy$, then $xy = k$ are hyperbolas. We will show that $x^2 - y^2 = k$ are the curves perpendicular to them. First we check exactness for the equation

$$0 = F_y \mathrm{d}x - F_x \mathrm{d}y = x\mathrm{d}x - y\mathrm{d}y$$

we indeed have $M_y = 0 = N_x$. Second we evaluate F by integrating M

$$F(x,y) = \int M \mathrm{d}x + a(y) = \frac{x^2}{2} + a(y).$$

Third we differentiate $a(y)$

$$a'(y) = -y \Rightarrow a(y) = -\frac{y^2}{2} + c.$$

Therefore, for $F(x,y) = k$ we obtain

$$k = \frac{x^2}{2} - \frac{y^2}{2}.$$

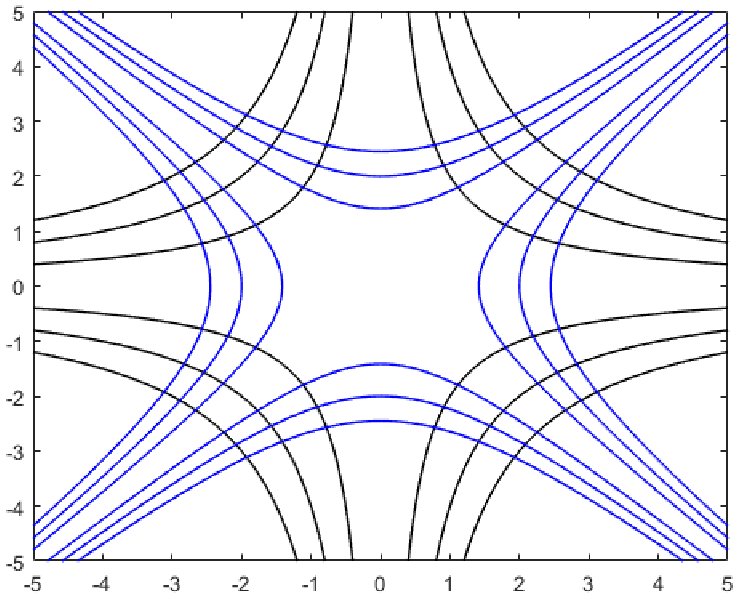


Figure 1.2.7: Hyperbolas perpendicular to each other

1.2.4 Method 4: Integrating factor and Exact

Some equations are close to being exact (they become exact after multiplying by an integrating factor). For example, the equation

$$(2y - 6x)\mathrm{d}x + (3x - 4x^2y^{-1})\mathrm{d}y = 0$$

is not exact but if we multiply both sides by $\mu(x,y) = xy^2$ we obtain

$$xy^2(2y - 6x)\mathrm{d}x + xy^2(3x - 4x^2y^{-1})\mathrm{d}y = 0$$

an exact equation. You can check that $F(x,y) = x^2y^3 - 2x^3y^2 + c$ solves the modified problem.

Method formal steps

1. If the equation is not exact, check whether

$$\frac{M_y - N_x}{N}$$

is a function of only x or

$$\frac{N_x - M_x}{M}$$

is a function of only y.

2. Then in the first case the correct integrating factor is

$$\mu(x) = exp\Big\{\int \frac{M_y - N_x}{N}\mathrm{d}x\Big\}$$

and in the second case

$$\mu(y) = exp\Big\{\int \frac{N_x - M_y}{M}\mathrm{d}y\Big\}$$

Example-presenting the method

In the context of the above example on perpendicular trajectories, suppose that we want to find the curves perpendicular to concentric circles. Then as explained above they satisfy the equation:

$$2y\mathrm{d}x - 2x\mathrm{d}y = 0.$$

This equation is not exact because $M_y = 2 \neq -2 = N_x$.

1. First we check the ratio
$$\frac{M_y - N_x}{N} = \frac{2+2}{-2x} = -\frac{2}{x}$$
and so indeed this ratio only depends on x and it is continuous away from the origin.

2. We integrate to obtain μ
$$\mu(x) = exp\Big\{\int \frac{-2}{x}\mathrm{d}x\Big\} = exp\big\{\ln\left(x^{-2}\right)\big\} = x^{-2}.$$

3. Therefore, by multiplying by μ we obtain an exact equation:
$$2yx^{-2}\mathrm{d}x - 2x^{-1}\mathrm{d}y = 0.$$

4. Next we carry out the exact equation steps. First we obtain F
$$F(x,y) = \int M\mathrm{d}x + a(y) = -2yx^{-1} + a(y).$$

5. Second we obtain the function a(y) by differentiating in y
$$-2x^{-1} + a'(y) = F_y = N(x,y) = -2x^{-1} \Rightarrow a(y) = c.$$

6. Therefore, we obtain F
$$F(x,y) = -2yx^{-1} + c$$
and note that for its level sets $F(x,y) = k$ we have
$$y = \frac{(c-k)}{2}x = m \cdot x.$$

In other words, lines going through the origin are perpendicular to concentric circles (as depicted in Figure 1.2.8).

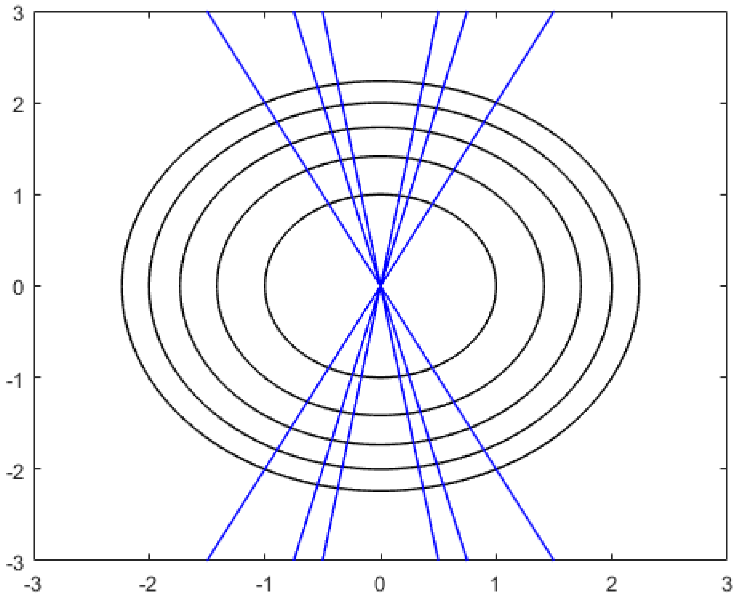


Figure 1.2.8: Lines perpendicular to circles

General result:

Integrating factor ratios

The equations $M(x,y) + N(x,y)y' = 0$ can be made exact if either

$$\frac{M_y - N_x}{N}$$

is continuous and depends only on x or

$$\frac{N_x - M_x}{M}$$

is continuous and depends only on y.

Proof. We are searching for a factor $\mu(x,y)$ that will satisfy the exact condition

$$\frac{\partial}{\partial y}(\mu(x,y)M(x,y)) = \frac{\partial}{\partial x}(\mu(x,y)N(x,y))$$
$$\Rightarrow M\mu_y - N\mu_x = (N_x - M_y)\mu.$$

This equation can be simplified if we assume that μ and $\frac{M_y - N_x}{N}$ are functions of only x:

$$0 - \mu_x(x) = \frac{N_x - M_y}{N}\mu(x).$$

Therefore, we obtain

$$\mu(x) = exp\left\{\int \frac{M_y - N_x}{N} \mathrm{d}x\right\}.$$

□

Generalized integrating factor

Consider equation
$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0.$$
If there are functions $\omega(x,y)$ and $R(x,y) = R(\omega(x,y))$ s.t.
$$\frac{M_y - N_x}{\omega_x N - \omega_y M} = R(\omega),$$
then we can use the integrating factor:
$$\mu(\omega(x,y)) := exp\left\{\int R(\omega)\mathrm{d}\omega\right\},$$
where we treat ω as a variable.

Proof.

By requiring exactness we obtain
$$\begin{aligned}
(\mu(\omega)M)_y &= (\mu(\omega)N)_x \Rightarrow \\
\mu'(\omega)\omega_y M + \mu(\omega)M_y &= \mu'(\omega)\omega_x N + \mu(\omega)N_x \Rightarrow \\
\frac{\mathrm{d}\mu}{\mu} &= \frac{M_y - N_x}{\omega_x N - \omega_y M}\mathrm{d}\omega \\
&= R(\omega)d\omega.
\end{aligned}$$
As a result, we can solve
$$\mu(\omega) = exp\left\{\int R(\omega)d\omega\right\},$$
where we treat ω as a variable. Therefore, we obtain the desired exactness condition.

□

Examples

- Consider the equation
$$(2x^2 + y)\mathrm{d}x + (x^2y - x)\mathrm{d}y = 0.$$
We first see that exactness is not satisfied
$$M_y = 1 \neq 2xy - 1 = N_x.$$
First we check if the first ratio only depends on x:
$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2y - x} = \frac{-2}{x}.$$
Indeed it does, and so we can multiply by
$$\mu(x) = exp\left\{\int \frac{-2}{x}dx\right\} = x^{-2}.$$

This gives us the exact equation

$$(2 + yx^{-2})\mathrm{d}x + (y - x^{-1})\mathrm{d}y = 0.$$

Then we repeat the steps for exact equations. First we obtain F

$$F(x, y) = \int M dx + a(y) = 2x - yx^{-1} + a(y).$$

Then we need to obtain $a(y)$:

$$-x^{-1} + a'(y) = F_y = N = y - x^{-1} \Rightarrow a(y) = \frac{y^2}{2} + c$$

and so F is

$$F(x, y) = 2x - yx^{-1} + \frac{y^2}{2} + c.$$

Therefore, for $F \equiv C$ we obtain the implicit solution:

$$2x - yx^{-1} + \frac{y^2}{2} = C - c.$$

- Consider the equation

$$2xy^3 + y^4 + (xy^3 - 2y)y' = 0.$$

1. You can check that this equation is not exact.
2. We will assume that the integrating factor μ is of the form $\mu = \mu(\omega(x, y))$.
3. Then we obtain the following equation:

   to have exactness we require

$$\begin{aligned} (\mu(\omega)M)_y &= (\mu(\omega)N)_x \Rightarrow \\ \mu'(\omega)\omega_y M + \mu(\omega)M_y &= \mu'(\omega)\omega_x N + \mu(\omega)N_x \Rightarrow \\ \frac{d\mu}{\mu} &= \frac{M_y - N_x}{\omega_x N - \omega_y M} d\omega. \end{aligned}$$

   In our case the last equation becomes

$$\frac{d\mu}{\mu} = \frac{6xy^2 + 3y^3}{\omega_x(xy^3 - 2y) - \omega_y(2xy^3 + y^4)} d\omega.$$

4. It is easy to notice identity $y(6xy^2 + 3y^3) = 2(2xy^3 + y^4)$.Because of that, we will take $\omega_y = \frac{-3}{y}$, or $\omega = \omega(y) = -3\ln y$.
5. By substituting this result into equation above, we finally get

$$\frac{d\mu}{\mu} = \frac{-3}{y} dy \Rightarrow \mu = \frac{1}{y^3}.$$

6. It is easy to check that $x^2 + xy + \frac{2}{y} = C$ is a solution.

- Consider the equation

$$dy + M dx = 0,$$

with $M := \frac{y}{2x} - \frac{\sqrt{\pi}e^{-(xy)^2}}{4x^2} erfi(xy)$, where $erfi(w) := \frac{2}{\sqrt{\pi}} \int_0^w e^{t^2} dt$ is the imaginary error function.

1. Since $M_y \neq 0$ the equation is not exact.

2. This equation satisfies
$$\frac{M_y}{y-xM} = R(xy) = xy.$$
3. Therefore, let the integrating factor be
$$\mu(xy) = exp\left\{\int R(\omega)d\omega\right\} = exp\{(xy)^2\}.$$

1.2.5 Method 5: Substitution methods

If an equation is of the form $y' = f(t, y)$ with f being nonlinear, then one can try to figure out a change of variables $(t, y) \mapsto z(t)$ that gives $z' = g(t, z)$ where now g is linear and thus the above methods apply.

Homogeneous equations

A function $f(t, y)$ is homogeneous of order α (or $\alpha-$homogeneous) if it satisfies:
$$f(\lambda t, \lambda y) = \lambda^\alpha f(t, y)$$
for all $\lambda \in \mathbb{R}$. Equivalently, such functions are of the form:
$$f(t,y) = f\left(t\cdot 1, t\cdot\frac{y}{t}\right) = t^\alpha \underbrace{f\left(1,\frac{y}{t}\right)}_{h\left(\frac{y}{t}\right)} = t^\alpha h\left(\frac{y}{t}\right) \text{ or}$$
$$f(t,y) = y^\alpha f\left(\frac{t}{y},1\right) = y^\alpha \underbrace{f\left(\frac{1}{\left(\frac{y}{t}\right)},1\right)}_{g\left(\frac{y}{t}\right)} = y^\alpha g\left(\frac{y}{t}\right).$$

For example, $f(t, y) = t^\alpha e^{\frac{y}{t}}$ is homogeneous of order α. If an equation is of the form $M(t, y) + N(t, y)y' = 0$, for continuous functions M, N that are both $\alpha-$homogeneous, we can obtain a solution by a clever use of chain-rule (§2.1).

Homogeneous equations

The equations $M(t, y) + N(t, y)y' = 0$,for $\alpha-$homogeneous M, N, have implicit solutions of the form
$$\int_{z(0)}^{z} \frac{1}{h(x)-x}dx = \ln|t| + c,$$
where $z(t) := \frac{y}{t}$ and $h\left(\frac{y}{t}\right) := -\frac{M(t,y)}{N(t,y)}$.

Proof. 1. We rewrite the equation as
$$y' = -\frac{M(t,y)}{N(t,y)} =: f(t,y)$$
where we note that $f(t, y))$ is $0-$homogeneous due to the ratio i.e. $-\frac{M(t,y)}{N(t,y)} =: h\left(\frac{y}{t}\right)$.

2. Therefore we can rewrite the equation as

$$y' = h\Big(\frac{y}{t}\Big).$$

3. By change of variables $z(t) := \frac{y(t)}{t}$ we can rewrite the equation as

$$t\frac{dz}{dt} + z = y' = h\Big(\frac{y}{t}\Big) = h(z).$$

By rearranging it we get

$$\begin{aligned}\frac{dz}{dt} &= \frac{h(z)-z}{t}\\ \Rightarrow \int_{z_0}^{z}\frac{1}{h(x)-x}\mathrm{d}x &= \int_{t_0}^{t}\frac{1}{t}\mathrm{d}t + c\end{aligned}$$

where $z(t_0) = z_0$. Let $\Phi(z) := LHS = \int_{z_0}^{z}\frac{1}{h(x)-x}\mathrm{d}x$ then we obtain the implicit solution:

$$\Phi(z) = \ln|t| + c. \qquad \square$$

Bernoulli equation

The non-linear equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

are called **Bernoulli equations**. By substituting $v = y^{1-n}$ we obtain the linear equation:

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)P(x)v = (1-n)Q(x)$$

which we can solve by means of an integrating factor.

Riccati Equation

The non-linear equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A(x)y^2 + B(x)y + C(x)$$

are called **Riccati equations**. By substituting $y = f + \frac{1}{v}$, where f is a particular solution of the above, we obtain the linear equation:

$$\frac{dv}{dx} + (B + 2fA)v = -A.$$

1.2.6 Method 6: Linearize

Example-presenting the method

- (Energy balance equation (EBM)) The global mean temperature T can be modeled by the [Wal15]

$$R\frac{\mathrm{d}T}{\mathrm{dt}} = Q(1-\alpha) - \sigma T^4,$$

where

- T (K,kelvins)is the average temperature in the Earth's photosphere (upper atmosphere,where the energy balance occurs in this model) (1kelvin= 1C);
- t(years)is time;
- R ($W - yr/m^2K$) is the averaged heat capacity of the Earth/atmosphere system(heat capacity is the amount of heat required to raise the temperature of an object or substance 1 kelvin(=1 C));
- Q (W/m^2) is the annual global mean incoming solar radiation (or insulation) per square meter of the Earth's surface;
- α (dimensionless) is planetary albedo(reflectivity),and
- $\sigma(W/m^2K^4)$ is a constant of proportionality, the Stefan-Boltzmann constant.

- Let T_S be the equilibrium surface temperature

$$T_S := (\frac{(1-\alpha)Q}{\sigma})^{1/4} = 288K.$$

  We will make the assumption that T is close to T_S over time.

- By Taylor expanding the non-linear term we obtain

$$T^4 = T_S^4 + 4T_S^3(T - T_S) + O(|(T - T_S)|).$$

- Substituting in the equation we obtain

$$R\frac{\mathrm{d}Y}{\mathrm{dt}} = Q(1-\alpha) - \sigma T_S^4 - (\sigma \cdot 4T_S^3)Y,$$

  where $Y := T - T_S$. This a linear ode that can solved by integrating factor.

1.2.7 Summary

Given a specific first-order differential equation to be solved, we can attack it by means of the following steps:

1. Is it separable? If so, separate the variables and integrate.
2. Is it linear? That is, can it be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = Q(t)?$$

   If so, multiply by the integrating factor $\mu(t) = exp\{\int P(s)\mathrm{d}s\}$.
3. Is it exact? That is,when the equation is written in the form:

$$M(t,y) + N(t,y)\mathrm{d}y = 0$$

   with $M_y = N_t$.
4. If the equation is not exact, do we have either $\frac{M_y - N_t}{N}$ or $\frac{M_y - N_t}{M}$ being a function of only t, y respectively? If so then it can be made exact.
5. If the equation as it stands is not separable, linear, or exact, is there a plausible substitution that will make it so? For instance, is it homogeneous?

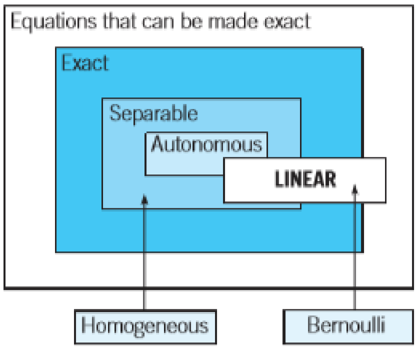


Figure 1.2.9: Summarizing diagram of all the methods

1.3 Existence and Uniqueness

If the first order equation $y' = f(t, y)$ satisfies that f, f_y are continuous as functions of (t, y) for $|t| < a, |y| < b$, then there is some interval $t \in [0, h]$ in which there exists a unique solution $y = \varphi(t)$.

1.3.1 Method Formal steps: Picard's method

Observe that by integrating both sides we obtain by fundamental theorem of calculus:

$$y(t) - y_0 = \int_{t_0}^{t} f(s, y(s))\mathrm{d}s,$$

where $y_0 = y(t_0)$. We will approximate the solution y as follows

1. Let $\varphi_0(t)$ be an initial guess to the solution eg. $\varphi_0(t) \equiv y_0$, then we define the next step as:
$$\varphi_1(t) := y_0 + \int_{t_0}^{t} f(s, \varphi_0(s))\mathrm{d}s.$$

2. Similarly, given φ_n we define
$$\varphi_{n+1}(t) := y_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\mathrm{d}s.$$

Example-Presenting the method

Consider the initial value problem:
$$y' = y, y(0) = 1$$

1. Let $\varphi_0 \equiv 1$ then
$$\varphi_1(t) := 1 + \int_{0}^{t} f(s, \varphi_0(s))\mathrm{d}s = 1 + \int_{0}^{t} 1\mathrm{d}s = 1 + t.$$

2. We repeat

$$\varphi_2(t) := 1 + \int_0^t f(s, \varphi_1(s))\mathrm{d}s = 1 + \int_0^t (1+s)\mathrm{d}s = 1 + t + \frac{t^2}{2}.$$

3. Assume that $\varphi_n(t) := 1 + t + ... + \frac{t^n}{n!}$ then

$$\varphi_{n+1}(t) := 1 + \int_0^t f(s, \varphi_n(s))\mathrm{d}s = 1 + \int_0^t \sum_{k=0}^{n} \frac{t^k}{k!}\mathrm{d}s = 1 + t + ... + \frac{t^{n+1}}{(n+1)!}.$$

4. Therefore, as expected we obtained the approximation to $y(t) = e^t$, which is indeed the solution of the above initial value problem.

1.3.2 Picard-Lindelöf Theorem

In this section we will prove the Picard-Lindel**ö**f Theorem. To do this we will first reformulate the differential equation into an "integral equation". The reason we do this is because integral are more lenient with respect to regularity (think of how shabby a Riemann integrable function can look) while derivatives are not. This reformulation is the goal of the following lemma:

Integral equation reformulation

Suppose $f(x,t)$ is a continuous function on a domain $(a,b)\times(c,d)$. Then if x is a solution to

$$x'(t) = f(x(t),t), \quad \text{for } t \in (c,d) \quad \text{such that } x(t_0) = x_0$$

whose derivative is continuous and for $t_0 \in (c,d)$, then x satisfies

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)\mathrm{d}s$$

for all $t \in (c,d)$. Conversely, suppose x is a continuous function that satisfies

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)\mathrm{d}s$$

for all $t \in (c,d)$ then x is differentiable withcontinuous derivative and x solves

$$x'(t) = f(x(t),t), \quad \text{for } t \in (c,d) \quad \text{such that } x(t_0) = x_0.$$

Proof. First suppose that x solves

$$x'(t) = f(x(t),t), \quad \text{for } t \in (c,d) \quad \text{such that } x(t_0) = x_0.$$

and has a continuos derivative. Since x is continuous then $f(x(t),t)$ is a continuous function of t. Integrating both sides from t_0 to any $t \in (c,d)$ gives, by the Fundamental Theorem of Calculus,

$$x(t) - x_0 = x(t) - x(t_0) = \int_{t_0}^t x(s)\mathrm{d}s = \int_{t_0}^t f(x(s), s)\mathrm{d}s.$$

Since t was arbitrary we conclude that for all $t \in (c,d)$ we have

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)\mathrm{d}s.$$

Conversely, suppose x is a continuous function that satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s.$$

Observe that since x is continuous then $f(x(s), s)$ is continuous. By the Fundamental Theorem of Calculus we conclude that the map

$$t \mapsto \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$$

is differentiable for all t. Since constant functions are differentiable we conclude that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$$

is differentiable for all t. By the Fundamental Theorem of Calculus we have

$$x'(t) = f(x(t), t)$$

for all $t \in (c, d)$. Since $f(x(t, t))$ is continuous then $x'(t)$ is continuous. So x solves the ODE and has continuous derivative. In particular, notice that $x(t_0) = x_0$. $\square$

Now we have exchanged the goal of demonstrating existence and uniqueness of an IVP for demonstrating existence and uniqueness of an integral equation. Notice that this reformulation has x appear on both sides of the desired equation. This allows us to use "fixed point methods" to achieve our goal. We recall, for convenience, the Banach Fixed Point Theorem which is proven in the appendix of analysis results:

Banach Fixed Point Theorem

Suppose X is a non-empty and complete metric space. Suppose also, that $f : X \to X$ satisfies

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ and where $0 \leq k < 1$. Then there exists a unique point $z \in X$ such that $f(z) = z$.

We are now ready to prove the existence and uniqueness theorem.

Picard-Lindelöf Theorem

Suppose $f : [a, b] \times [c, d] \to \mathbb{R}$ satisfies, for each fixed $t \in [c, d]$ and for any $x_1, x_2 \in [a, b]$

$$|f(x_1, t) - f(x_2, t)| \leq M|x_1 - x_2|$$

where $M \geq 0$ and does not depend on t. Then, if $t_0 \in (c, d)$ and $x_0 \in (a, b)$, there exists $\epsilon > 0$ and a continuous function x such that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s$$

for $t \in [t_0 - \epsilon, t_0 + \epsilon]$.

Proof. Observe that for any $x, y \in (a, b)$ we have, for $t \geq t_0$

$$\begin{aligned}\left|\left(x_0+\int_{t_0}^t f(x,s)\mathrm{d}s\right)-\left(x_0+\int_{t_0}^t f(y,s)\mathrm{d}s\right)\right| &= \left|\int_{t_0}^t \big(f(x,s)-f(y,s)\big)\mathrm{d}s\right| \\ &\leq \int_{t_0}^t |f(x,s)-f(y,s)|\mathrm{d}s \\ &\leq M(t-t_0)|x-y| \\ &= M|t-t_0||x-y|\end{aligned}$$

A similar computation holds for $t < t_0$ with the same conclusion. Thus, for all t we have

$$\left|\left(x_0+\int_{t_0}^t f(x,s)\mathrm{d}s\right)-\left(x_0+\int_{t_0}^t f(y,s)\mathrm{d}s\right)\right| \leq M|t-t_0||x-y|$$

and in particular if we restrict the consideration of the above inequality to $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}]$ then we can further conclude that

$$\left|\left(x_0+\int_{t_0}^t f(x,s)\mathrm{d}s\right)-\left(x_0+\int_{t_0}^t f(y,s)\mathrm{d}s\right)\right| \leq \frac{1}{2}|x-y|$$

for $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}]$. This constraint shows that the dependence of $x_0 + \int_{t_0}^t f(x,s)\mathrm{d}s$ on x is that of a contraction. Next observe that for $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}]$ we have

$$\left|\left(x_0+\int_{t_0}^t f(x,s)\mathrm{d}s\right)-x_0\right| = \left|\int_{t_0}^t f(x,s)\mathrm{d}s\right| \leq \|f\|_{C([a,b]\times[c,d])}|t-t_0|.$$

Thus, if we choose $t \in \left[t_0 - \frac{\min\{x_0-a,b-x_0\}}{\|f\|_{C([a,b]\times[c,d])}}, t_0 + \frac{\min\{x_0-a,b-x_0\}}{\|f\|_{C([a,b]\times[c,d])}}\right]$ then we may conclude that

$$a \leq x_0 + \int_{t_0}^t f(x,s)\mathrm{d}s \leq b.$$

Thus, in this constraint the output of $x_0 + \int_{t_0}^t f(x,s)\mathrm{d}s$ is in between a and b. We restrict our attention to $t \in [t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}] \cap \left[t_0 - \frac{\min\{x_0-a,b-x_0\}}{\|f\|_{C([a,b]\times[c,d])}}, t_0 + \frac{\min\{x_0-a,b-x_0\}}{\|f\|_{C([a,b]\times[c,d])}}\right] = I$ so that both of the above described conditions are true. Now we define $\mathcal{F} : C(I;[a,b]) \to C(I;[a,b])$ by

$$\mathcal{F}(x) = x_0 + \int_{t_0}^t f(x(s),s)\mathrm{d}s.$$

We verify that $\mathcal{F}$ is well defined and that $\mathcal{F}$ is a contraction. Let $x \in C(I;[a,b])$. Then x is bounded and so

$$\mathcal{F}(x) = x_0 + \int_{t_0}^t f(x(s),s)\mathrm{d}s.$$

is a continuous function. In particular, since x is a function that maps to $[a,b]$ the above expression makes sense and is defined for all $t \in I$. By construction, since $t \in I$ and $x(I) \subset [a,b]$ then a previous computation shows that

$$a \leq x_0 + \int_{t_0}^t f(x(s),s)\mathrm{d}s \leq b.$$

for all $t \in I$. The previous computation still holds since $f(x(s),s) \leq \|f\|_{C([a,b]\times[c,d])}$ for all $s \in I$

remains true. Thus, $\mathcal{F}(x) \in C(I;[a,b])$. Next we show that $\mathcal{F}$ is a contraction. By mimicking a previous computation we have, since $t \in I$ and $x(I), y(I) \subset [a,b]$ for $x, y \in C(I;[a,b])$, that for all $t \in I$

$$\begin{aligned}\left|\left(x_0 + \int_{t_0}^{t} f(x(s),s)\mathrm{d}s\right) - \left(x_0 + \int_{t_0}^{t} f(y(s),s)\mathrm{d}s\right)\right| &\leq M \int_{t_0}^{t} |x(s) - y(s)|\mathrm{d}s \\ &\leq M|t - t_0|\|x - y\|_{C(I;[a,b])} \\ &\leq \frac{1}{2}\|x - y\|_{C(I;[a,b])}\end{aligned}$$

where the last inequality follows from the choice $t \in I$. From this we conclude that

$$|(\mathcal{F}(x))(t) - (\mathcal{F}(y))(t)| \leq \frac{1}{2}\|x - y\|_{C(I;[a,b])}$$

for all $t \in I$. Hence,

$$\|\mathcal{F}(x) - \mathcal{F}(y)\|_{C(I;[a,b])} \leq \frac{1}{2}\|x - y\|_{C(I;[a,b])}$$

which shows that $\mathcal{F}$ is a contraction on $C(I;[a,b])$. By the Banach Fixed Point Theorem we conclude that there is a fixed point $x^* \in C(I;[a,b])$ satisfying

$$x^*(t) = x_0 + \int_{t_0}^{t} f(x^*(s),s)\mathrm{d}s$$

for all $t \in I$. □

1.3.3 Generalized Existence and Uniqueness*

We prove a general form of existence and uniqueness known as the *Osgood's existence and uniqueness theorem*:

Osgood's existence and uniqueness

Suppose $f(x,t)$ is continuous in both variables and satisfies, for each $x \in [a,b]$,

$$|f(x,t_1) - f(x,t_2)| \leq \omega(|t_1 - t_2|)$$

where $\omega : [0,\infty) \to [0,\infty)$ satisfies $\omega^{-1}(\{0\}) = \{0\}$, $\frac{1}{\omega}$ is Riemann integrable on $[\delta, 1]$ for all $\delta > 0$, and

$$\lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{1}{\omega(s)}\mathrm{d}s = +\infty.$$

Then there exists a unique y that satisfies

$$y'(x) = f(x, y(x))$$

for $x \in [a, a+\epsilon)$ as well as $y(a) = y_0$.

Proof. Since f is continuous then by the *Peano's existence theorem* we have that a solution, y, exists for $x \in [a, a+\epsilon)$ where $\epsilon > 0$. We now demonstrate uniqueness. Suppose not. Then there exist two solutions y_1, y_2 to this initial value problem. Since the solutions are distinct then there is a point x^* such that $y_1(x^*) \neq y_2(x^*)$. In particular, we see that

$$A = \left\{x \in [a, a+\epsilon) \;\middle|\; y_1(x) \neq y_2(x)\right\} \neq \phi.$$

Observe that by continuity that A is also open. Choose any point in A and consider the largest interval containing the chosen point. Suppose this interval has boundary points c and d. I claim that we must have $y_1(c) = y_2(c)$ and $y_1(d) = y_2(d)$. Suppose not. If this fails for c then notice that by continuity there will be a $\delta > 0$ such that $y_1(x) \neq y_2(x)$ for $x \in (c - \delta, c)$. This contradicts how we found the endpoint c. A similar proof works for d. Observe also that we either have $y_1(x) > y_2(x)$ for all $x \in (c, d)$ or $y_2(x) > y_1(x)$ for all $x \in (c, d)$. This is because $(c, d) \subset A$ and if both situations occurred in (c, d) then by the Intermediate Value Theorem there would be a point in $(c, d) \subset A$ where y_1 and y_2 agree. Assume, without loss of generality, that $y_1(x) > y_2(x)$ for all $x \in (c, d)$. Then, for $x \in (c, d)$ we have

$$\begin{aligned} y_1'(x) - y_2'(x) &= f(x, y_1(x)) - f(x, y_2(x)) \leq |f_1(x, y_1(x)) - f(x, y_2(x))| \\ &\leq \omega(|y_1(x) - y_2(x)|) = \omega(y_1(x) - y_2(x)). \end{aligned}$$

Hence, for $x \in (c, d)$

$$\frac{(y_1(x) - y_2(x))'}{\omega(y_1(x) - y_2(x))} \leq 1.$$

By Preiss and Uher's version of the change of variables theorem we have for each $t > c$ and for fixed $t^* > t$ that

$$\int_t^{t^*} \frac{(y_1(x) - y_2(x))'}{\omega(y_1(x) - y_2(x))} \mathrm{d}x = \int_{(y_1 - y_2)(t)}^{(y_1 - y_2)(t^*)} \frac{1}{\omega(s)} \mathrm{d}s$$

where we choose t^* so that $(y_1 - y_2)(t^*) < 1$ (this is possible since $y_1 - y_2$ is continuous and is 0 at $t = c$. Hence,

$$\int_{(y_1 - y_2)(t)}^{(y_1 - y_2)(t^*)} \frac{1}{\omega(s)} \mathrm{d}s \leq t^* - t.$$

Letting t tend to c gives

$$+\infty = \lim_{t \to c^+} \int_{(y_1 - y_2)(t)}^{(y_1 - y_2)(t^*)} \frac{1}{\omega(s)} \mathrm{d}s \leq t^* - c < +\infty$$

which is a contradiction (note that we have used that $(y_1 - y_2)(t)$ tends to 0 as t tends to c). We conclude that the solution is unique. □

Remarks

We remark that the Osgood condition can be weakened to requiring

$$|f(x, t_1) - f(x, t_2)| \leq \omega(|t_1 - t_2|)\varphi(x)$$

for each $x \in [a, b]$ and any t_1, t_2 where ω satisfies the conditions stated in Osgood's existence and uniqueness theorem and $\varphi \geq 0$ is Riemann integrable on $[0, 1]$. This strengthening of Osgood's theorem is referred to as the Montel-Tonelli uniqueness theorem.

1.4 Autonomous dynamics and Logistic growth

The equations of the form

$$\frac{dy}{dt} = f(y)$$

are called **autonomous**. Such equations might not have explicit solutions, but it is possible to draw qualitative solutions for them.

Method formal steps

1. First we draw the curves $\varphi_i(t) = (t, y(t))$ where $f(y) = 0$ (called the **equilibrium solutions** or **critical points**).
2. These will separate the regions into $y' = f(y) > 0$ and $y' = f(y) < 0$.
3. We classify each φ_i as **asymptotically stable** if for $y(t)$ starting close to φ_i (i.e. $|y_0 - \varphi_i(0)| < \varepsilon$)
$$\lim_{t\to\infty} y(t) = \varphi_i$$
irrespective of whether $y_0 < \varphi_i(0), y_0 > \varphi_i(0)$ and **asymptotically unstable** if solutions that start close to the $\varphi_i(t)$ curve, move away from it.

Example-Presenting the method

En route to studying the competing species we will need the logistic equation (2.5): Let $y(t)$ be the population of a given species at time t then

$$\frac{dy}{dt} = r\Big(1 - \frac{y}{K}\Big)y,$$

where $r > 0$ is called the intrinsic growth rate and K is the saturation level. Since y is a physical quantity, the $y < 0$ is ignored.

1. First we find the equilibrium solutions: $r\big(1 - \frac{y}{K}\big)y = 0 \Rightarrow y = K$ or $y = 0$. So the equilibrium solutions are $\varphi_1(t) \equiv 0, \varphi_2(t) \equiv K$.
2. We have $y' = r(1 - \frac{y}{K})y > 0$ when $K > y$ and $y > 0$ ($y < 0$ is ignored). Therefore, the solutions started from below K will be growing upwards to $y = K$.
3. On the other hand, $y' = r(1 - \frac{y}{K})y < 0$ when $K < y$ and $y > 0$. Therefore, the solutions started from above K will be decaying downwards to $y = K$.

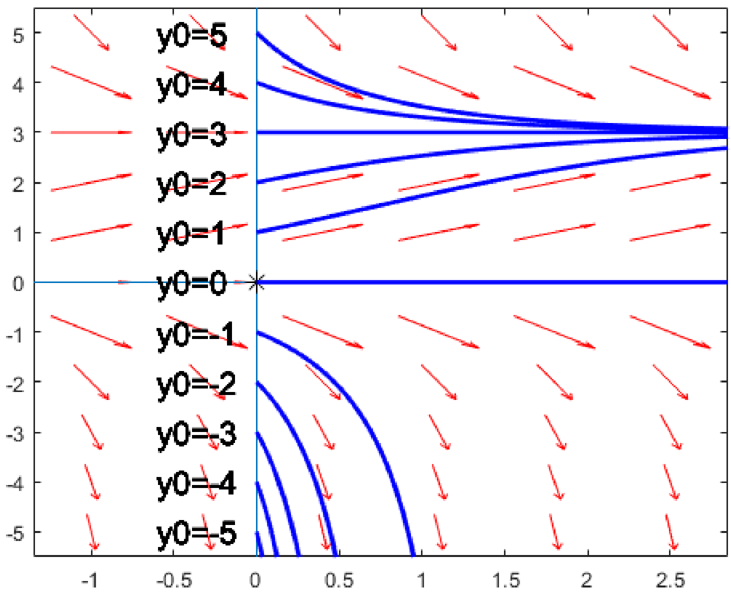


Figure 1.4.1: Direction field for logistic with $K = 3$.

4. So we observe that irrespective of the initial value the solution converges to the saturation level: $\lim_{t\to\infty} y = K$. Therefore, $\varphi_2(t) \equiv K$ is the asymptotically stable solution.
5. On the other hand, we observe that if y is really small (i.e. close to $\varphi_1 = 0$) but still positive, the solutions still move away from φ_1 and go towards φ_2. Therefore, φ_1 is the asymptotically unstable solution.
6. Physically that the population dynamics will return to the saturation/capacity level K; the most the ecosystem can withhold.

Examples

1. Consider the equation
$$y' = (y-1)(y-2)(y-3).$$
   (a) First we identify the equilibrium solutions:
$$y' = 0 \Rightarrow \varphi_1(t) \equiv 1,\ \varphi_2(t) \equiv 2,\ \text{and } \varphi_3(t) \equiv 3.$$
   (b) Second we identify the sign of y' in each region:
   - Above $y > 3$ we have $y' > 0$ and so the solution will growth to infinity
   - In $y \in [2,3]$ we have $y' < 0$ and so the solution will decay downwards to φ_2.
   - In $y \in [1,2]$ we have $y' > 0$ and so the solution will grow upwards to 2. Therefore, in either case
$$y(t) \to \varphi_2$$
     and so φ_2 is asymptotically stable.
   - In $y \in [0,1]$ we have $y' < 0$ and so the solution will decay downwards to minus infinity.
   - Therefore, φ_1, φ_3 will be asymptotically unstable.

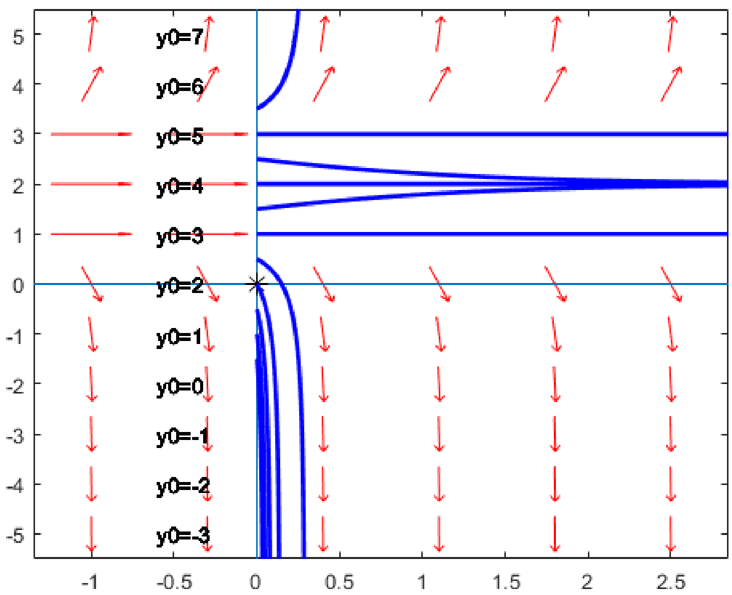


Figure 1.4.2: Direction field for $y' = (y-1)(y-2)(y-3)$.

2. Consider the equation
$$y' = e^{-y} - 1.$$

(a) First we identify the equilibrium solution/-s:

$$y' = 0 \Rightarrow \varphi(t) = 0.$$

(b) Second we identify the sign of y' in each region:

- Above $y > 0$ we have $y' = e^{-y} - 1 < 0$ and so it will decay downwards to 0.
- Below $y < 0$ we have $y' = e^{-y} - 1 > 0$ and so it will grow upwards to 0. Therefore, φ is asymptotically stable.

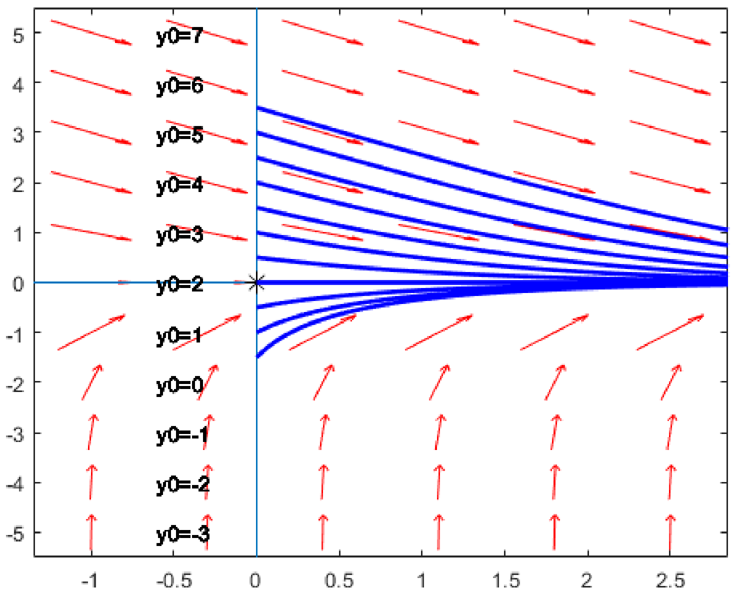


Figure 1.4.3: Direction field for $y' = e^{-y} - 1$.

Applied examples

- We return to the price of good example from the integrating factor section. When the price of a good is p, the total demand is D(p) = a - bp and the total supply is $S(p) = \alpha + \beta p$, where a, b, α, and β are positive constants. When demand exceeds supply, price rises, and when supply exceeds demand it falls. The speed at which the price changes is proportional to the difference between supply and demand. Specifically

$$p' = \lambda(D(p) - S(p))$$

for $\lambda > 0$.

1. We obtained
$$p(t) = c \cdot exp\{-\lambda(b+\beta)t\} + \frac{(a-\alpha)}{(b+\beta)}.$$

2. So as $t \to +\infty$ the price of this good converges to $\frac{(a-\alpha)}{(b+\beta)}$. Now we will show that $\varphi = \frac{(a-\alpha)}{(b+\beta)}$ in fact an asymptotically stable solution.

3. We first obtain the equilibrium solutions

$$p' = 0 \implies D(p) = S(p) \implies a - bp = \alpha + \beta p \implies p = \frac{(a-\alpha)}{(b+\beta)}.$$

4. Since $D(p) = S(p)$, this is the price where supply will match demand. We have $p' > 0$ when $p < \frac{(a-\alpha)}{(b+\beta)}$ and so the price will grow upwards to the stable price.

5. We have $p' < 0$ when $p > \frac{(a-\alpha)}{(b+\beta)}$ and so the price will decay downwards to the stable price.

1.5 Problems

The questions labeled by (*) are trickier.

- **Linear integrating factor (2.2)**
  - Find the general solution and use it to determine how solutions behave as $t \to +\infty$
    1. $y' + 3y = t + e^{-2t}$
    2. $y' + y = te^{-t} + 1,$
    3. (*)[1] $y' + \frac{1}{t}y = 3\cos(2t), t > 0$
  - Find the general solution and use it to determine the asymptotic behavior for different values of a
    1. $y' - \frac{1}{2}y = 2\cos(t), y(0) = a$ as $t \to +\infty,$
    2. (*) $ty' + (t+1)y = 2te^{-t}, y(1) = a, 0 < t$ as $t \to 0.$
    3. (*) A rock contains two radioactive isotopes R_1, R_2 with R_1 decaying into R_2 with rate $5e^{-20t}$kg/sec. So if $y(t)$ is the total mass of R_2, we obtain:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \text{rate of creation of } R_2 \text{ - rate of decay of } R_2\\ &= 5e^{-20t} - ky(t),\end{aligned}$$

       where $k > 0$ is the decay constant for R_2. Also assume that $y(0) = 40\,\text{kg}$.
  - Suppose $x : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), and satisfies $x'(t) \leq c(t)x(t) + b(t)$ on (a,b) where $c, b : [a,b] \to \mathbb{R}$ are continuous.
    1. By using an integrating factor, μ, show that we can rewrite the inequality $x'(t) \leq c(t)x(t) + b(t)$ as

$$\big(\mu(t)x(t)\big)' \leq \mu(t)b(t).$$

    2. By further rewriting the inequality conclude that the function

$$F(t) = \mu(t)x(t) - \int_a^t \mu(s)b(s)\mathrm{d}s$$

       is non-increasing on (a,b) and continuous on $[a,b]$. Conclude that F is non-increasing on $[a,b]$ and hence

$$F(t) \leq F(a).$$

       Conclude that

$$x(t) \leq x(a)e^{\int_a^t c(s)\mathrm{d}s} + e^{\int_a^t c(s)\mathrm{d}s}\int_a^t e^{\left[-\int_a^s c(s)\mathrm{d}s\right]}b(s)\mathrm{d}s$$

       for $t \in [a,b]$.
- **Separable (2.1):**
  - For each of the questions (1) – (5) you should:
    (a) find the solution of the given initial value problem
    (b) determine the interval in which the solution is defined:

[1] We follow the standard book practice of using a (*) to indicate a question is trickier.

1. $y' = (1-2x)y^2, \;\; y(0) = -1/6,$
2. $y' = (1-2x)/y, \;\; y(1) = -2,$
3. (*) $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r^2}{\theta}, \;\; r(1) = 2,$
4. (*) $y' = 2x/(1+2y), \;\; y(2) = 0,$
5. (*) $y' = y^2 + 1, \;\; y(0) = 0$ (only the interval containing 0).

– For each of the following questions you should:
  (a) find the solution of the given initial value problem
  (b) determine the behaviour as $t \to +\infty$:
   1. (*) $y' = \cos^2(y), \;\; y(0) = 2,$
   2. (*) $y' = t^{\frac{y(4-y)}{3}}, \;\; y(0) = 1.$

– (*) Homogeneous equations problem 2.2-(25).

- **Homogeneous equations problem**

  1. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+xy+y^2}{x^2}$

- **Autonomous equations**

  1. Draw the phase lines and identify which solutions are asymptotically stable/unstable.
     (a) $\frac{\mathrm{d}y}{\mathrm{d}t} = ay + by^2, \;\; a > 0, \;\; b > 0, \;\; -\infty < y_0 < \infty$,
     (b) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2), \;\; y_0 \geq 0,$
     (c) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^y - 1, \;\; -\infty < y_0 < \infty$.
  2. (**) Suppose $f : [c,d] \to \mathbb{R}$ and suppose $x : [a,b] \to \mathbb{R}$ is a function which is continuous on $[a,b]$, differentiable on (a,b), $x([a,b]) \subset [c,d]$, and x satisfies $x'(t) = f(x(t))$ for $t \in [a,b]$. Show that x is monotone on $[a,b]$.

     Hints:

     (a) Suppose not. Then, without loss of generality, we can find points $a \leq t_1 < t_2 < t_3 \leq b$ such that $x(t_1) < x(t_2)$ but $x(t_3) < x(t_2)$. Use this to argue that there is a point $t_4 \in (t_1, t_3)$ such that x attains its maximum on $[t_1, t_3]$ at t_4. Then construct $t_5 \in (t_1, t_4)$ and $t_6 \in (t_4, t_3)$ such that $x'(t_5) > 0$, $x'(t_6) < 0$, $x(t_5) < x(t_4)$ and $x(t_6) < x(t_4)$.
     (b) If $x(t_5) = x(t_6)$ use the differential equation satisfied by x to conclude we are done. Otherwise, without loss of generality, we may assume $x(t_5) < x(t_6)$. Show that the collection of $t \in (t_5, t_4)$ such that $x(t) = x(t_6)$ is non-empty.
     (c) Now suppose, for $t \in (t_5, t_4)$, $x'(t) < 0$ whenever $x(t) = x(t_6)$ and consider the set
     $$A = \{t \in (t_5, t_4) \mid x(t) \neq x(t_6)\}.$$
     Show that if $t_* \in A$ then there exists $\epsilon > 0$ such that $t_* \pm \epsilon \in A$. Finally show that the set $x^{-1}(A)$ has the property that if $s \in x^{-1}(A)$ then $s \pm \delta \in x^{-1}(A)$ for $\delta > 0$ small enough (you will need to use that x is continuous here).
     (d) Conclude that $A = \bigcup_{i=1}^{\infty}(a_i, b_i)$. To see this for note that if a point is in A then nearby points are in A. Thus, we may consider the maximal neighbourhood of a point $x \in A$. Show that if A_x, A_y are maximal neighbourhoods of x and y then either $A_x = A_y$ or $A_x \cap A_y = \phi$. Since rationals are countable conclude that the amount of intervals A_x is countable.
     (e) Pick any maximal interval (a_i, b_i) from A. Observe that $a_i, b_i \notin A$. Conclude, by continuity, that there is $\delta > 0$ such that $x(a_i) > x(t)$ if $0 < t - a_i < \delta$ and $x(t) > x(b_i)$ if $0 < b_i - t < \delta$. Conclude that there is a point $c \in (a_i, b_i)$ such that $x(a_i) = x(c) = x(b_i)$ (they all equal $x(t_6)$). This is a contradiction. Thus, the assumption $x'(t) < 0$ whenever $x(t) = x(t_6)$ is wrong. Conclude that there exists $t_7 \in (t_5, t_4)$ such that $x(t_7) = x(t_6)$ and $x'(t_7) \geq 0$ and use the differential equation to argue that we have a contradiction as in 2b.

3. (**) In this exercise we construct an initial value problem for an autonomous differential equation which has no solution but the function f is continuous at the initial condition.
   (a) Suppose $f : [c, d] \to \mathbb{R}$ is nowhere 0. Show that the differential equation $x'(t) = f(x(t))$ has no constant solutions. That is, if $C \in [c, d]$ is any constant, then $x \equiv C$ is not a solution to $x'(t) = f(x(t))$.[2]
   (b) Suppose $f : [c, d] \to \mathbb{R}$ is continuous on $[c, d]$ and differentiable on (c, d). Then:
      i. if $t, s \in (c, d)$ with $t < s$ and $f'(t) < 0$ and $f'(s) > 0$ then there exists $r \in (t, s)$ such that $f'(r) = 0$.
      ii. if $t, s \in (c, d)$ with $t < s$ and p satisfies $f'(t) < p < f'(s)$ then there exists $r \in (t, s)$ such that $f'(r) = p$.

      Hint: Apply the previous problem to $g(x) = f(x) - xp$.
   (c) Suppose $f : [c, d] \to \mathbb{R}$ and $x : [a, b] \to \mathbb{R}$ satisfies $x([a, b]) \subset [c, d]$, x is continuous on $[a, b]$ but differentiable on (a, b), x satisfies $x'(t) = f(x(t))$ for $t \in (a, b)$. Then if $t, s \in (\min\{x(a), x(b)\}, \max\{x(a), x(b)\})$ with $t < s$, and p is such that $f(t) < p < f(s)$ then there exists $r \in (t, s)$ such that $f(r) = p$.

      Note: You will need to use 2 to conclude.
   (d) Consider $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \neq \pm\frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 + \frac{1}{n} & x = \pm\frac{1}{n} \end{cases}.$$

      Show that the problem $x'(t) = f(x(t))$ with $x(0) = 0$ has no solutions using the previous questions. Also, verify by directly integrating that this initial value problem has no solutions.

[2] This is referred to as Darboux's Theorem.

Chapter 2

Second order equations

The general form of 2^{nd} order equation is

$$y'' = f(t, y, y').$$

We call them **linear non-homogeneous** if the equation can be written in the form

$$y'' + p(t)y' + q(t)y = g(t)$$

and **linear homogeneous** if, in addition to being linear non-homogeneous, $g(t) = 0$

$$y'' + p(t)y' + q(t)y = 0.$$

The method of characteristic equations is for homogeneous equations and the methods of undetermined coefficients and of variation of parameters for homogeneous equations.

2.1 Method 1: Characteristic equation

If the equation is linear homogeneous and further $p(t), q(t)$ are constant, then the equation is referred to as a **constant-coefficients** equation:

$$ay'' + by' + cy = 0$$

and we can apply the method of characteristic equations to solve such an equation. Note that a is assumed to be non-zero since we are working with a *second* order equation.

Method formal steps

1. We assume that the solution is of the form $y(t) = e^{rt}$ (this is called making an ansatz). This gives

   $$(ar^2 + br + c)e^{rt} = 0 \Longrightarrow ar^2 + br + c = 0,$$

   which equation is called the **characteristic equation.**

2. So to solve the above ODE, it suffices to find the two roots r_1, r_2.

3. Then the general solution is of the form:

   $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Example-presenting the method

Consider a mass m hanging at rest on the end of a vertical spring of length l, spring constant k and damping constant γ (as depicted in Figure 2.1.1).

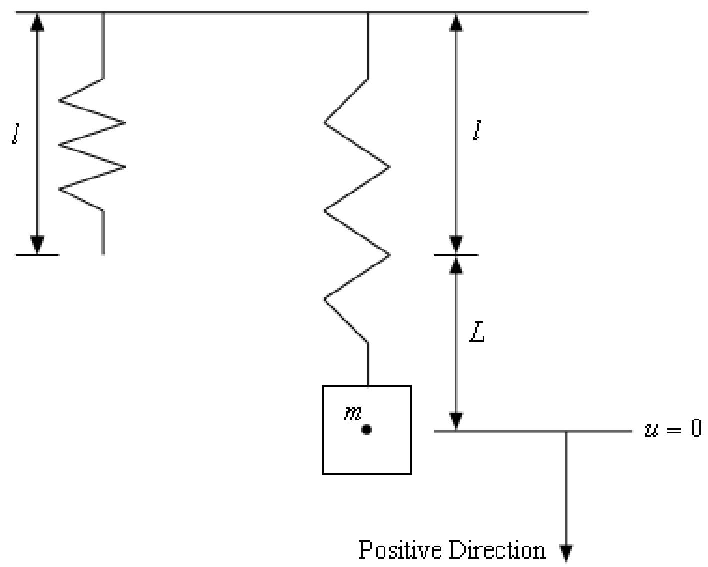


Figure 2.1.1: Spring mass

Let $u(t)$ denote the displacement, in units of feet, from the equilibrium position. Note that since $u(t)$ represents the amount of displacement from the spring's equilibrium position (the position obtained when the downward force of gravity is matched by the will of the spring to not allow the mass to stretch the spring further) then $u(t)$ should increase downward. Then by Newton's Third Law one can obtain the equation

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where $F(t)$ is any external force, which for simplicity we will assume to be zero.

1. First we obtain the characteristic equation:

$$mr^2 + \gamma r + k = 0.$$

2. Suppose that $m = 1\text{lb}, \gamma = 5\text{lb/ft/s}$ and $k = 6\text{lb/ft}$ then we obtain the roots $r_1 = -2$, $r_2 = -3$.

3. Therefore, the general solution will be

$$u(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

4. Further if $u(0) = 0$, $u'(0) = 1$ we obtain $c_1 = 1, c_2 = -1$:

$$u(t) = e^{-2t} - e^{-3t}.$$

Examples

- Consider the IVP

$$4y'' - y = 0,\ y(-2) = 1,\ y'(-2) = -1.$$

1. We obtain the characteristic equation $4r^2 - 1 = 0 \Rightarrow r = \pm\frac{1}{2}$ and so the general solution will be
$$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}}.$$
2. Using the initial conditions we obtain:
$$1 = c_1 e^{-1} + c_2 e \text{ and } -1 = \frac{1}{2}(c_1 e^{-1} - c_2 e).$$
3. Solving these two equations gives: $c_1 = \frac{-1}{2}e, c_2 = \frac{3}{2}e^{-1}$ and so the solution for our IVP is:
$$y(t) = -\frac{1}{2}e^{1+\frac{t}{2}} + \frac{3}{2}e^{-\frac{t}{2}-1}.$$
4. Therefore, as $t \to +\infty$ we obtain $y \to -\infty$.

- Consider the IVP
$$y'' + 5y' + 6y = 0,\ y(0) = 2,\ y'(0) = \beta$$

1. The characteristic equation is $r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3$ and so the general solution will be:
$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$
2. Using the initial conditions we obtain:
$$2 = c_1 + c_2 \text{ and } \beta = -2c_1 - 3c_2.$$
3. Solving these two equations gives: $c_1 = (6 + \beta)$, $c_2 = -(4 + \beta)$ and so the solution for our IVP is:
$$y(t) = (6 + \beta)e^{-2t} - (4 + \beta)e^{-3t}.$$
4. Therefore, as $t \to +\infty$ we obtain $y \to 0$.

2.1.1 Wronskian

Now we will show that the general solution of linear homogeneous ode is always of the form:
$$y(t) = c_1 y_1 + c_2 y_2,$$
where the y_i are solutions for the differential equation that satisfy a linear independence condition that is called the **Wronskian**. Then $\{y_1, y_2\}$ will be called the **fundamental solutions** because they can be used to generate all other solutions.

Method formal steps

Consider arbitrary initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

1. Assuming that $y = c_1 y_1 + c_2 y_2$ holds for some choice of c_1, c_2 then we certainly expect that the follow equations will hold:
$$\begin{aligned} y_0 &= y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \\ y_0' &= y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{aligned}$$
which can be rewritten in matrix form as:
$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

This leads us to studying the matrix

$$W_{matrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}.$$

2. Then we compute the determinant of this matrix, referred to as the *Wronskian*,

$$W = \det(W_{matrix}) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0).$$

3. If it is not zero, then the general solution will be of the form $y = c_1y_1 + c_2y_2$ (Although we only found coefficients that allow $c_1y_1 + c_2y_2$ to match y when $t = t_0$ it will turn out that both functions agree for all t).

4. If it is zero then these y_1, y_2 will not generate all solutions (In this case it is possible to choose y_0 and y_0' to make the system have no solutions at t_0. If we can't solve it at t_0 there is no hope for general t).

Example-presenting the method

Going back to the spring example, the characteristic equation is

$$mr^2 + \gamma r + k = 0.$$

Assume that it has two distinct real roots r_1, r_2 and so we can easily check that $y_1(t) = e^{r_1t}, y_2(t) = e^{r_2t}$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W\left(e^{r_1t}, e^{r_2t}, t\right) = e^{(r_1+r_2)t} \underbrace{(r_2 - r_1)}_{\text{distinct roots}} \neq 0.$$

Therefore, all solutions will be of the form: $y = c_1e^{r_1t} + c_2e^{r_2t}$ for some choice of c_1 and c_2.

General results:

Generalized solution

Suppose that y_1, y_2 are solutions of

$$y'' + p(t)y' + q(t)y = 0.$$

Then the family of solutions

$$y = c_1y_1 + c_2y_2$$

for arbitrary c_1, c_2, includes all possible solutions if and only if there is a t_* where the Wronskian of $y_1(t_*), y_2(t_*)$ is not zero.

Proof. Consider general solution $\varphi(t)$ of the above ODE. We will show that there are constants a, b s.t. $\varphi(t) = ay_1 + by_2$. Let t_* be the time for which $W(y_1, y_2, t_*) \neq 0$ and let $K_0 = \varphi(t_*), K_1 = \varphi'(t_*)$. Then

$$\begin{bmatrix} y_1(t_*) & y_2(t_*) \\ y_1'(t_*) & y_2'(t_*) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$$

has a solution $\binom{a}{b}$ because the matrix is invertible. So if $\zeta(t) := ay_1(t) + by_2(t)$ we have $\zeta(t_*) = K_0$, $\zeta'(t_*) = K_1$. Therefore, the existence and uniqueness theorem for 2nd order ODEs gives us $\varphi(t) = \zeta(t) = ay_1(t) + by_2(t)$ for all t. ◻

In fact if $W(y_1, y_2, t_*) \neq 0$ for one t_* , then the Wronskian is actually never zero for all t s.t. $W(y_1, y_2, t) \neq 0$. This is proved via Abel's identity:

Proposition 2.1.1 (Abel's identity)**.** *Let y_1, y_2 be solutions to*

$$y'' + p(t)y' + q(t)y = 0$$

then for any t_* we can write

$$W(y_1, y_2, t) = W(y_1, y_2, t_*) exp\left\{ - \int_{t_*}^{t} p(s) ds. \right\}$$

Proof. Next we prove Abel's identity which will imply $W(y_1, y_2, t_*) \neq 0 \Rightarrow W(y_1, y_2, t) \neq 0$. Differentiating the Wronskian we obtain

$$W' = y_1 y_2'' - y_1'' y_2.$$

Plugging in the ODE for y_1'' and y_2'' gives

$$\begin{aligned} W' &= y_1(-py_2' - q(t)y_2) - (-py_1' - q(t)y_1)y_2 \\ &= -p(y_1 y_2' - y_1' y_2) \\ &= -pW. \end{aligned}$$

Therefore, we obtain the first order ODE $W' = -p(t)W$ which is solved by

$$W(t) = W(t_*) exp\left\{ - \int_{t_*}^{t} p(s) ds \right\}.$$

□

One application of this is in disproving that two functions y_1, y_2 are the fundamental solutions for some second order linear non-homogeneous constant coefficient ODE. For example, let $y_1 = 1 - t$, $y_2 = t^3$ then their Wronskian is

$$W(y_1, y_2, t) = t^2(3 - 2t)$$

and so $W(y_1, y_2, 0) = 0$ and $W(y_1, y_2, 1) = 1$. Therefore, these y_1, y_2 cannot be solutions to any such ODE (If such an ODE existed then since the Wronskian is non-zero at $t = 1$ then by Abel's identity the Wronskian is nowhere zero. However, the Wronskian is 0 at $t = 0$).

Examples

- Consider the equation $y'' - 2y' + y = 0$ and functions $y_1 := e^t, y_2 := te^t$

  1. One can easily check that both y_1, y_2 solve the above ODE, so now we will check if they are fundamental solutions.
  2. The Wronskian is

  $$W\big(e^t, te^t, t\big) = e^t\big(e^t + te^t\big) - te^{2t} = e^{2t} \neq 0$$

  3. So indeed a general solution for the above ODE is $y = ae^t + bte^t$.

- Consider the ODE $y'' - y' - 2y = 0$ and functions $y_1 := e^{2t}$, $y_2 := -2e^{2t}$.

  1. One can easily check that both solve the ODE.

2. Their Wronskian is

$$W\left(e^{2t}, -2e^{2t}, t\right) = -2e^{4t} - \left(-4e^{2t}e^{2t}\right) = 0$$

and so they do not form a linearly independent set and in turn a fundamental solution.

2.1.2 Complex roots

In some cases the roots are complex (when $b^2 - 4ac < 0$). For example, suppose that there is no damping in the above spring example ($\gamma = 0$), then the equation will be:

$$mu'' + ku = 0.$$

Therefore, the roots will be $r = \pm\sqrt{-k/m} = \pm i\sqrt{k/m} =: \pm i\omega$, where we define $i := \sqrt{-1}$ called the imaginary unit as well as $\omega = \sqrt{\frac{k}{m}}$. The main result we will need is **Euler's formula**

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

Here we can easily check that $y_1(t) = \cos(\omega t)$ and $y_2(t) = \sin(\omega t)$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W(\cos(\omega t), \sin(\omega t), t) = \omega\cos^2(\omega t) + \omega\sin^2(\omega t) = \omega \neq 0.$$

This is to be expected since we don't imagine that the imaginary part of this solution (i.e $\sin(\omega t)$) interferes with the real part of this solution (i.e $\cos(\omega t)$ and so they should be independent. Therefore, all solutions will be of the form: $y = c_1\cos(\omega t) + c_2\sin(\omega t)$, where c_i could be complex constants. Physically this periodicity is expected because there is no external force or damping to remove energy from the spring and so it can keep oscillating forever.

Examples

- Consider the equation $y'' + y = 0,\ y(\pi/3) = 2,\ y'(\pi/3) = -4$

1. The roots are $r^2 + 1 = 0 \Rightarrow r = \pm i$ and so the general solution is (with $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$)

$$y(t) = c_1e^{it} + c_2e^{-it} = a_1\cos(t) + a_2\sin(t).$$

2. Using the initial conditions we obtain:

$$2 = a_1\frac{1}{2} + a_2\frac{\sqrt{3}}{2} \text{ and } -4 = -a_1\frac{\sqrt{3}}{2} + a_2\frac{1}{2}.$$

3. Solving these two equations gives: $a_1 = (1+2\sqrt{3}), a_2 = -(2-\sqrt{3})$ and so the solution for our IVP is:

$$y(t) = (1 + 2\sqrt{3})\cos(t) - (2 - \sqrt{3})\sin(t).$$

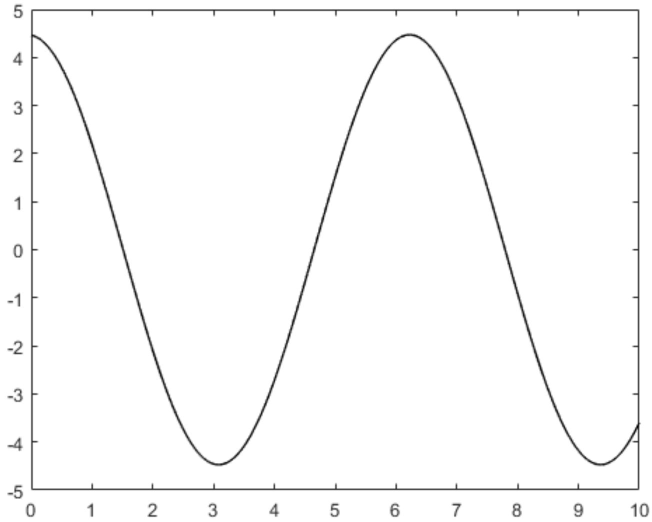


Figure 2.1.2: Spring mass

4. So as $t \to \infty$ the system simply keeps oscillating steadily (depicted in Figure 2.1.2). Physically this is because it is damping free $\gamma = 0$.

- Consider the equation $y'' - 2y' + 5y = 0,\ y(\pi/2) = 0,\ y'(\pi/2) = 2$

1. The roots are $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and so the general solution is (with $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$)
$$y(t) = c_1 e^{t(1+2i)} + c_2 e^{t(1-2i)} = e^t(a_1 \cos(2t) + a_2 \sin(2t)).$$
2. Using the initial conditions we obtain:
$$0 = e^{\frac{\pi}{2}}(a_1 \cdot (-1) + a_2 \cdot 0) \ \text{ and } \ 2 = e^{\frac{\pi}{2}}(a_1 \cdot (-1) + a_2 \cdot (-2)).$$
3. Solving these two equations gives: $a_1 = 0$, $a_2 = -e^{-\pi/2}$ and so the solution for our IVP is:
$$y(t) = -e^{t-\pi/2} \sin(2t).$$

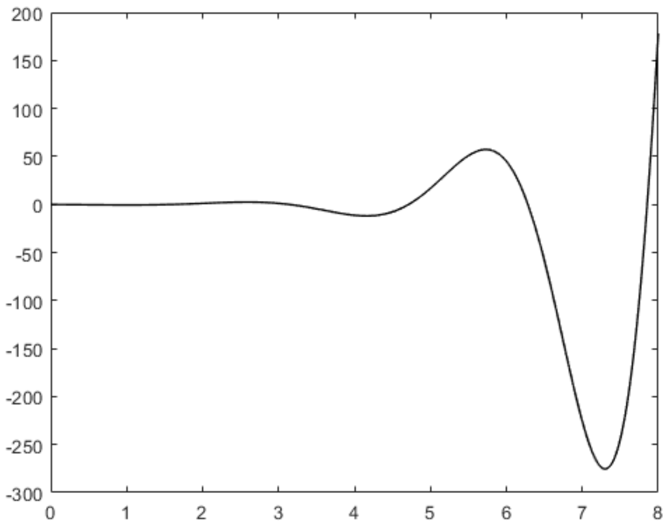


Figure 2.1.3: Spring mass

4. So as $t \to \infty$ the system simply keeps oscillating with increasing amplitude. Physically this is because the damping is negeative $\gamma = -2 < 0$ and so instead of removing energy, it adds.

2.1.3 Repeated roots

In some cases the roots are equal (when $b^2 - 4ac = 0$). For example, suppose that $\gamma^2 \approx 4km$ (called **critically damped**), then the roots will be

$$r_1 = r_2 = -\frac{\gamma}{2m} =: r.$$

This only gives one solution $y_1 = e^{rt}$, but to find the general one we require a second solution y_2 that is linearly independent: $W(y_1, y_2, t_*) \neq 0$ for some t_*. It turns out (proved below) that $y_2(t) := te^{rt}$ is such a function:

$$W(y_1, y_2, t) = e^{rt}\big(e^{rt} + rte^{rt}\big) - re^{rt}te^{rt} = e^{2rt} \neq 0.$$

Example

- Consider the IVP
$$y'' - 2y + 2 = 0, \; y(0) = 1, \; y'(0) = 2$$
  1. The root of the characteristic equation is $r = 1$ and so the two solutions are $y_1 = e^t$, $y_2 = te^t$. Thus, the general solution will be of the form
  $$y = ae^t + bte^t.$$
  2. The initial conditions give $1 = a$, $2 = a + b \Rightarrow a = 1$, $b = 1$ and so the solution satisfying these conditions is
  $$y = e^t + te^t.$$
  3. This solution goes to infinity as $t \to +\infty$.
- Consider the IVP
$$y'' - 6y' + 9y = 0, \; y(0) = 0, \; y'(0) = 2$$
  1. The root is $r = 3$ and so the independent solutions are $y_1 = e^{3t}, y_2 = te^{3t}$. Thus, the general solution will be
  $$y = ae^{3t} + bte^{3t}.$$
  2. The initial conditions give $0 = a$, $2 = 3a + b \Rightarrow a = 0$, $b = 2$ and so the solution satisfying these conditions is
  $$y = 2te^t.$$
  3. This solution goes to infinity as $t \to +\infty$.

General result

Repeated root

If the ODE $ay'' + by' + cy = 0$ has a characteristic equation with repeated root $r := \frac{-b}{2a}$, then its general solution is of the form:

$$y = c_1 e^{rt} + c_2 te^{rt}.$$

Proof. For $y_2 := g(t)y_1$ we will first find which ODE $g(t)$ must satisfy in order that y_2 is a solution of our ODE.

$$\begin{aligned}&a(g(t)y_1)'' + b(g(t)y_1)' + c = 0\\ \Rightarrow &a\big(g''(t)e^{rt} + 2g'(t)re^{rt}\big) + bg'e^{rt} = 0\end{aligned}$$

where we used that y_1 satisfies the ODE $ay'' + by" + cy = 0$

$$\begin{aligned}0 = a(g''(t) + g'(t)(2ar + b)) &= ag''(t) + g'(t)\left(2a\frac{-b}{2a} + b\right) = ag''\\ \Rightarrow &ag'' = 0\\ \Rightarrow &g = c_1 + c_2t.\end{aligned}$$

We conclude that

$$y_2(t) = (c_1 + c_2t)e^{rt} = c_1e^{rt} + c_2te^{rt}.$$

Since we are interested in finding an independent solution (so that we can find the general solution) we may as well take $c_1 = 0$ and $c_2 = 1$ since for any $a, b \in \mathbb{R}$ we have

$$ae^{rt} + b\big(c_1e^{rt} + c_2te^{rt}\big) = (a + bc_1)e^{rt} + bc_2te^{rt}.$$

That is, any linear combination of the solutions e^{rt} and $c_1e^{rt} + c_2te^{rt}$ can be generated by e^{rt} and te^{rt} by a different set of coefficients. The opposite is also true. We conclude that the candidates for fundamental solutions are e^{rt} and te^{rt}. As shown earlier, by means of a Wronskian computation, these solutions are independent. Thus, the general solution is of the form

$$y = d_1e^{rt} + d_2te^{rt}$$

□

2.1.4 Stability

Consider nonhomogeneous equation of the form

$$y'' + ay' + by = c,$$

where a, b, c are constants. If we have a solution y_h for the homogeneous problem, then we can construct a solution for the nonhomogeneous problem:

$$y = y_h + \frac{c}{b}.$$

The solution $y_s := \frac{c}{b}$ is called **globally stable** when for all solutions y we have $y \to y_s$ as $t \to +\infty$, which is equivalent to saying $y_h \to 0$ as $t \to +\infty$.

Method formal steps

1. If the characteristic equation has two real distinct roots r_1, r_2 then the general solution is

$$y_h = c_1e^{r_1} + c_2e^{r_2t}$$

and so y_s is stable iff $r_1, r_2 < 0$.

2. If the characteristic equation has two complex roots r_1, r_2 then $\Big($if we let $\beta = \frac{\sqrt{4b-a^2}}{2})\Big)$

$$y_h = e^{-\frac{at}{2}}(c_1\cos(\beta t) + c_2\sin(\beta t))$$

and so y_s is stable iff $a > 0$.

3. If the characteristic equation has a double root $r = r_1 = r_2$ then

$$y_h = e^{rt}(c_1 + c_2 t)$$

and so y_s is stable iff $r < 0$.

In fact, as we will prove below it suffices to check whether the coefficients a, b are positive.

Example-Presenting the method

- **[CW]** In the price adjustment example (from the section on linear integrating factor and autonomous dynamics), we assumed that the demand and supply are functions of the price alone:
$$D(P) = a - bP \text{ and } S(P) = \alpha + \beta P.$$
However, buyers may also base their behavior on whether the price is increasing or decreasing. For example, if the price of newer versions of a phone brand have been increasing steadily or in an accelerating manner, they may decide to switch to another brand. So the demand will also be a function of the derivative of the price P' (growth) and the second derivative of the price P'' (steady or accelerating growth).
To keep things simple we will consider the following updated models:
$$D(P) = a - bP + mP' + nP'' \text{ and } S(P) = \alpha + \beta P + uP' + wP'',$$
where $a, b, \alpha, \beta > 0$ and m, n, u, w can be any sign. For now we will study it from the buyers perspective and set $u = w = 0$. To obtain an ODE for it, we assume that the market is cleared and thus $D(P) = S(P)$.
$$a - bP + mP' + nP'' = \alpha + \beta P \Rightarrow P'' + \frac{m}{n}P' - \frac{b+\beta}{n}P = \frac{\alpha - a}{n}.$$

1. Once we obtain the solution y_h of the homogeneous problem:
$$P'' + \frac{m}{n}P' - \frac{b+\beta}{n}P = 0$$
then for the above unhomogeneous ode the solution will simply be
$$y := y_h + \left(-\frac{b+\beta}{n}\right)^{-1}(\frac{\alpha - a}{n}) = y_h + \frac{a - \alpha}{b+\beta}.$$

2. The characteristic equation is:
$$r^2 + \frac{m}{n}r - \frac{b+\beta}{n} = 0.$$
Its roots are:
$$r_{1,2} = -\frac{m}{2n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}}.$$
(a) If $\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n} > 0$ then we obtain two distinct real roots r_1, r_2 and the solution will be

$$P = c_1 exp\left\{\left(-\frac{m}{2n} - \frac{1}{2}\sqrt{\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}}\right)t\right\}$$
$$+ c_2 exp\left\{\left(-\frac{m}{2n} + \frac{1}{2}\sqrt{\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}}\right)t\right\} + \frac{a-\alpha}{b+\beta}.$$

Since $\frac{b+\beta}{n} < 0 \Leftrightarrow n < 0$, then if $n < 0$ and $m < 0$ then both roots are negative and thus as $t \to +\infty$ we obtain:

$$P = c_1 e^{r_1 t} + c_1 e^{r_2 t} + \frac{a-\alpha}{b+\beta} \xrightarrow{t\to+\infty} \frac{a-\alpha}{b+\beta}.$$

Intuitively this means that when the demand $D(P)$ depends negatively on P'' ($n < 0$), the buyer will be averse to accelerating prices and so demand will not rise but simply converge to the equilibrium price $\frac{a-\alpha}{b+\beta}$.
For example, suppose $a = 42, b = 1, \alpha = -6, \beta = 1, m = -4, n = -1$, then our ODE will be:

$$P'' + 4P' + 2P = 48$$

and the equilibrium will be $\frac{a-\alpha}{b+\beta} = 24$. Solving this ODE with $P(0) = 1,\ P'(0) = 0$ gives us:

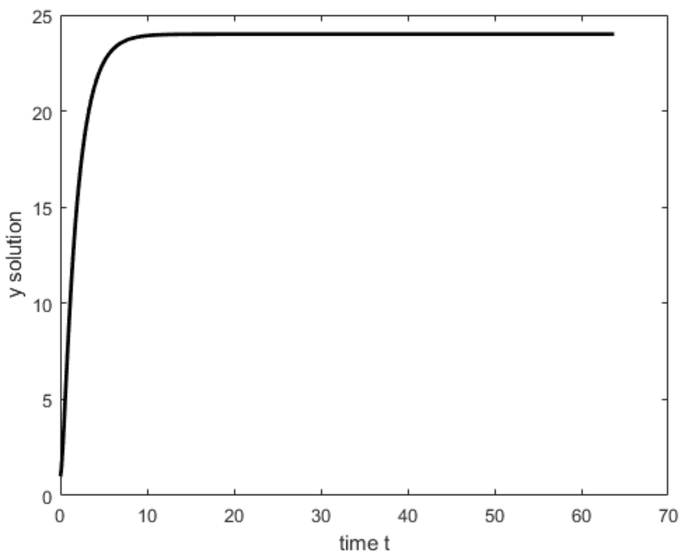


Figure 2.1.4: The solution stabilizes around the equilibrium price $\frac{\alpha-a}{b+\beta} = 24$

This agrees with the result below, namely the coefficients $\frac{m}{n}, -\frac{b+\beta}{n}$ are both positive and so the y_s is globally stable.

(b) If $\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n} < 0$ then we obtain two distinct complex roots r_1, r_2 and the solution will be (with $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$)

$$P = c_1 e^{-\frac{m}{2n}t} exp\left\{-\frac{i}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|})t\right\}$$
$$+ c_2 e^{-\frac{m}{2n}t} exp\left\{\frac{i}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|})t\right\} + \frac{\alpha-a}{b+\beta}$$
$$= e^{-\frac{m}{2n}t}\left[a_1 \cos\left(\frac{t}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|}\right) + a_2 \sin\left(\frac{t}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|}\right)\right] + \frac{a-\alpha}{b+\beta}.$$

So this solution will diverge or go to zero depending on whether $\frac{m}{n} > 0$ or $\frac{m}{n} < 0$ respectively. For example, suppose $a = 40, b = 2, m = -2, \alpha = -5, \beta = 3, n = -1$ then our ODE will be:

$$P'' + 2P' + 5P = 45$$

and for $m = 2$

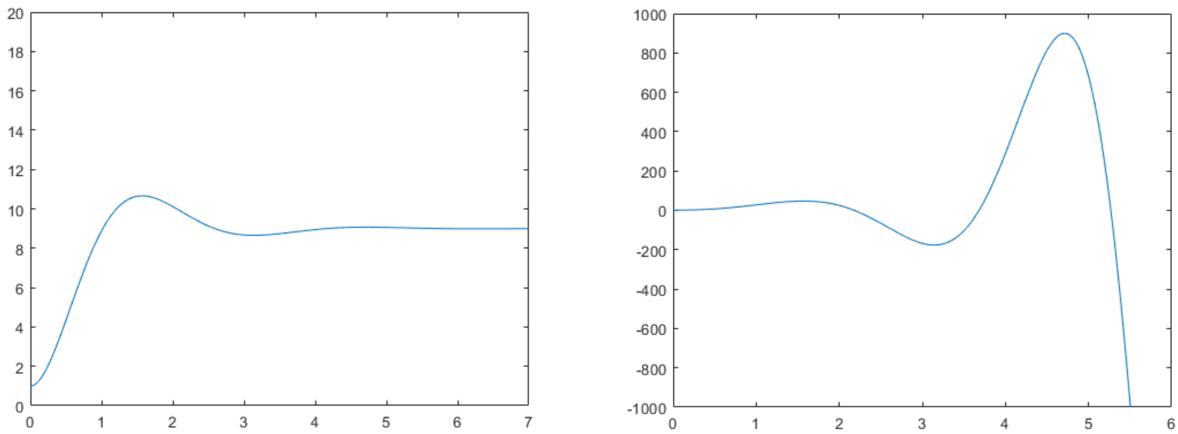
$$P'' - 2P' + 5P = 45.$$

The corresponding solutions will be

$$P(t) = e^{-t}[a_1 \cos(2t) + a_2 \sin(2t)] + 9$$

and

$$P(t) = e^{t}[a_1 \cos(2t) + a_2 \sin(2t)] + 9.$$



(a) The solution stabilizes around the equilibrium price $\frac{a-\alpha}{b+\beta} = 9$

(b) The solution oscillates with increasing amplitude.

As expected from the result below the first ODE has globally stable solution due to the positivity of the coefficients whereas the second ODE does not.

Intuitively, when $m = -2 < 0$, the demand $D(P)$ will depend negatively on growing price P' and so the price will have to drop to market equilibrium. When $m = 2 > 0$, the buyer will not stop even if the price is growing eg. for vital goods such as bread, and so the price is free to keep growing. But why is it oscillating? This is because the condition $(\frac{m}{n})^2 + 4\frac{b+\beta}{n} < 0$ forces that $n < 0$ and so the buyer will always be averse to accelerating growth in the price, which in turn causes the downturns in price.

(c) If $(\frac{m}{n})^2 + 4\frac{b+\beta}{n} = 0$ then we obtain one double root $r = r_1 = r_2$ and the solution will be

$$P = c_1 e^{-\frac{m}{2n}t} + c_2 t e^{-\frac{m}{2n}t} + \frac{a-\alpha}{b+\beta}.$$

Similarly, depending on the sign of m, the solutions will either diverge $(m > 0)$ or converge to market equilibrium price $(m < 0)$.

General solution for non-homogeneous and Stability for second order

Stability criterion for second order nonhomogeneous ODEs

Consider non-homogeneous equation

$$y'' + ay' + by = f(t).$$

Then the solution is of the form $y = y_h + y_s$, where y_h solves the homogeneous problem and y_s is any solution of the nonhomogeneous problem. We have that

$$\lim_{t\to\infty} y = y_s \text{ iff } a > 0, b > 0.$$

In other words, y_s is globally stable iff $a > 0, b > 0$ iff the real parts of the roots of the characteristic equation are both negative.

Proof. As with constant $f(t)$, we again obtain that the generalized solution is of the form

$$y = c_1 y_1 + c_2 y_2 + y_s =: y_h + y_s.$$

So by studying when $y_h \to 0$, we can identify when y_s is the globally stable solution.

1. If the characteristic equation has two real distinct roots r_1, r_2 then

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

and so y_s is stable iff $r_1, r_2 < 0$. The roots are

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

We have $r_1 < 0 \Leftrightarrow a > 0, b > 0$ and $r_2 < 0 \Leftrightarrow a > 0$. So to have both conditions we must require $a > 0$ and $b > 0$.

2. If the characteristic equation has two complex roots r_1, r_2 then $\left(\text{for } \beta = \frac{\sqrt{4b-a^2}}{2}\right)$

$$y_h = e^{-\frac{at}{2}}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

and so y_s is stable iff $a > 0$. The condition $b > 0$ follows from $a^2 < 4b$.

3. If the characteristic equation has a double root $r = r_1 = r_2$ then

$$y_h = e^{rt}(c_1 + c_2 t)$$

and so y_s is stable iff $r = \frac{-a}{2} < 0 \Rightarrow a > 0$. The condition $b > 0$ follows from $a^2 = 4b$. □

Examples

- Solve the IVP and determine long term behaviour

$$y'' + y = 9, \ y(\pi/3) = 2, \ y'(\pi/3) = -4$$

1. As showed in the complex roots section the solution to the homogeneous problem is:

$$y_h(t) = \left(1 + 2\sqrt{3}\right)\cos(t) - \left(2 - \sqrt{3}\right)\sin(t).$$

2. So the solution to our problem is:
$$y = y_h + 9.$$
3. However, y_h will keep oscillating steadily around the constant solution 9.

- Solve the IVP and determine long term behaviour
$$y'' + 5y' + 6y = 3, y(0) = 2, y'(0) = 1$$

1. The solution to the homogeneous problem is:
$$y_h(t) = 7e^{-2t} - 5e^{-3t}.$$
2. So the solution to our problem is:
$$y = y_h + \frac{3}{6} = y_h + \frac{1}{2}.$$
3. Therefore, y will converge to the constant solution $y_s \equiv \frac{1}{2}$.

2.2 Method 2: Undetermined coefficients

We will now consider non-homogeneous equations with constant coefficients of the form
$$ay'' + by' + cy = f(t).$$
By managing to find a particular solution y_{nh}, then we can generate every other one. Let v be any another solution, then
$$a(v - y_{nh})'' + b(v - y_{nh})' + c(v - y_{nh}) = f(t) - f(t) = 0.$$
Therefore, by finding the fundamental set of solutions y_1, y_2 for the homogeneous problem we have
$$v - y_{nh} = c_1y_1 + c_2y_2 \Rightarrow v = y_{nh} + c_1y_1 + c_2y_2.$$
So we managed to generate any solution starting from y_{nh}, y_1, y_2. Here we will find y_{nh} for $f(t)$ of the following possible forms:
$$f_1(t) := Ct^m e^{r_* t}, \ f_2(t) := Ct^m e^{\alpha} \cos(\beta t), \ f_3(t) := Ct^m e^{\alpha t} \sin(\beta t).$$
In fact once we obtain solutions $y_i, i = 1, 2, 3$ for them, we also obtain solutions for their sums. For example, consider the equation
$$ay'' + by' + cy = t^m e^{r_* t} + \sin(\beta t).$$
Observe that if y_1, y_2 solve
$$\begin{aligned} ay'' + by' + cy &= t^m e^{r_* t} \\ ay''by' + cy &= \sin(\beta t) \end{aligned}$$
respectively then their sum, $y_1 + y_2$, solves
$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = t^m e^{r_* t} + \sin(\beta t).$$

Method formal steps

1. If $f = Ct^m e^{r_* t}$ then we make the ansatz (assume the solution to be of the form)

$$y_{nh}(t) = t^s(a_0 + a_1 t + ... + a_m t^m)e^{r_* t}.$$

Now the way we pick the exponent s, depends on whether or not r_* is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof 2.2below.

   (a) If r_* is not a root, then we set $s := 0$.
   (b) If r_* is a simple root, then we set $s := 1$.
   (c) If r_* is a double root, then we set $s := 2$.

2. If $f = Ct^m e^{\alpha} \cos(\beta t)$ or $ct^m e^{\alpha t} \sin(t)$ then we make the ansatz

$$y_{nh}(t) = t^s e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m)\cos(\beta t) + (b_0 + a_1 t + ... + b_m t^m)\sin(\beta t)].$$

Now the way we pick the exponent s, depends on whether or not $\alpha + i\beta$ is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.

   (a) If $\alpha + i\beta$ is not a root, then we set $s := 0$.
   (b) If $\alpha + i\beta$ is a root, then we set $s := 1$.

Example-presenting the method

Resuming the spring example, let $u(t)$ denote the displacement from the equilibrium position. Then by Newton's Third Law one can obtain the equation

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where $F(t)$ is any external force. Above we assumed that $F(t) = 0$, and now we will take it to be any of the above mentioned functions. For example, consider the equation

$$y'' + 3y' + 2y = \sin(t).$$

Here we are shaking the spring system periodically in time.

1. First we are in the second case and so we make the ansatz

$$y_{nh}(t) = t^s e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m)\cos(\beta t) + (b_0 + a_1 t + ... + b_m t^m)\sin(\beta t)]$$

which simplifies because $m = 0, \alpha = 0$ and $\beta = 1$:

$$y_{nh}(t) = t^s(a_0 \cos(t) + b_0 \sin(t)).$$

2. Next we pick s, depending on whether $a + i\beta = i$ is a root of our ODE's characteristic equation:

$$r^2 + 3r + 2 = 0.$$

3. Its roots are $r_1 = -2$, $r_2 = -1$ and so we set $s = 0$ and have

$$y_{nh}(t) = a_0 \cos(t) + b_0 \sin(t).$$

4. Plugging into our ODE we obtain

$$y'' + 3y' + 2y = -(a_0 \cos(t) + b_0 \sin(t)) + 3(-a_0 \sin(t) + b_0 \cos(t)) + 2(a_0 \cos(t) + b_0 \sin(t))$$

$$= (a_0 + 3b_0)\cos(t) + (-3a_0 + b_0)\sin(t)$$

and so to have this be equal to $\sin(t)$ we require

$$\begin{cases} a_0 + 3b_0 &= 0 \\ -3a_0 + b_0 &= 1 \end{cases} \implies a_0 = -0.3,\ b_0 = 0.1.$$

5. So the solution will be
$$y_{nh}(t) = -0.3\cos(t) + 0.1\sin(t).$$

6. Therefore, the general solution will be:
$$y = y_{nh} + c_1 e^{-2t} + c_2 e^{-t}.$$

7. But why is it periodic given that damping is involved ($\gamma \neq 0$)? The sinusoidal external force keeps pumping energy into the system.

General result:

Method of Undetermined coefficients

Consider equations
$$ay'' + by' + cy = f(t),$$
where $f(t)$ has the following possible forms:

$$f_1(t) := Ct^m e^{r_* t}, f_2(t) := Ct^m e^{\alpha} \cos(\beta t), f_3(t) := Ct^m e^{\alpha t} \sin(\beta t).$$

Then their corresponding solutions are of the form:

- If $f = Ct^m e^{r_* t}$ then we make the ansatz (assume the solution to be of the form)
$$y_{nh}(t) = t^s(a_0 + a_1 t + ... + a_m t^m)e^{r_* t}.$$
Now the way we pick the exponent s, depends on whether or not r_* is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.
  1. If r_* is not a root, then we set $s := 0$.
  2. If r_* is a simple root, then we set $s := 1$.
  3. If r_* is a double root, then we set $s := 2$.
- If $f = ct^m e^{\alpha} \cos(\beta t)$ or $ct^m e^{\alpha t} \sin(\beta t)$ then we make the ansatz
$$y_{nh}(t) = t^s e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m)\cos(\beta t) + (b_0 + a_1 t + ... + b_m t^m)\sin(\beta t)].$$
Now the way we pick the exponent s, depends on whether or not $\alpha + i\beta$ is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.
  1. If $\alpha + i\beta$ is not a root, then we set $s := 0$.
  2. If $\alpha + i\beta$ is a root, then we set $s := 1$.

Proof. First we will work with

$$ay'' + by' + cy = Ct^m e^{r_* t}.$$

We assume the solution is of the form:

$$y_{nh}(t) = (a_0 + a_1 t + ... + a_n t^n)e^{rt}.$$

for some yet undetermined n. Then plugging it into our ODE we obtain:

$$\begin{aligned} ay''_{nh} + by'_{nh} + cy_{nh} &= a_n\big(ar^2 + br + c\big)t^n e^{rt} + \big(a_n n(2ar + b) + a_{n-1}\big(ar^2 + br + c\big)\big)t^{n-1}e^{rt} \\ &+ \big[a_n n(n-1)a + a_{n-1}(n-1)(2ar + b) + a_{n-2}\big(ar^2 + br + c\big)\big]t^{n-2}e^{rt} \\ &+ \text{lower order terms} \end{aligned}$$

Case 1: If r is not a root of the characteristic equation $ar^2 + br + c$, then the leading term $t^n e^{rt}$ remains and so to obtain $t^m e^{rt}$ we must set $n := m$ giving:

$$y_{nh}(t) = (a_0 + a_1 t + ... + a_n t^m)e^{rt}.$$

Case 2: If r is a simple root, then $ar^2 + br + cy = 0$ and we are left with $t^{n-1}e^{rt}$ being the leading order term and so we set $n - 1 := m$ giving:

$$y_{nh}(t) = \big(a_0 + a_1 t + ... + a_n t^{m+1}\big)e^{rt}.$$

Moreover, since r is a root, then $y_0 := a_0 e^{rt}$ will solve the homogeneous equation $ay'' + by' + cy = 0$ and so we can ignore it (due to additivity of solutions of the homogeneous equation). Thus,

$$y_{nh}(t) = \big(a_1 t + ... + a_n t^{m+1}\big)e^{rt} = t(a_1 + ... + a_n t^m)e^{rt}.$$

Case 3: If r is a double root, then $ar^2 + br + cy = 0$, $2ar + b = 0$ and we are left with $t^{n-2}e^{rt}$ being the leading term and so we set $n - 2 := m$ giving:

$$y_{nh} = (a_0 + a_1 t + ... + a_n t^{m+2})e^{rt}.$$

Moreover, since r is a repeated root, then e^{rt}, te^{rt} are both solutions of the homogeneous equation $ay'' + by' + cy = 0$ and so we can ignore. Thus,

$$y_{nh} = (a_2 t^2 + ... + a_n t^{m+2})e^{rt} = t^2(a_2 + ... + a_n t^m)e^{rt}.$$

Next we will work with

$$ay'' + by' + cy = Ct^m e^{\alpha t}\sin(\beta t) = \frac{1}{2i}Ct^m e^{\alpha t + i\beta t} - \frac{1}{2i}Ct^m e^{\alpha t - i\beta t}.$$

where we used that

$$\sin(\beta t) = \frac{e^{i\beta t} - e^{-i\beta t}}{2i}.$$

Therefore, from the previous we make the guess

$$\begin{aligned} y_{nh} &= (a_0 + a_1 t + ... + a_n t^n)e^{(\alpha + i\beta)t} + (b_0 + b_1 t + ... + b_n t^n)e^{(\alpha - i\beta)t}. \\ &= e^{\alpha t}(c_0 + c_1 t + ... + c_n t^n)\cos(\beta t) + (d_0 + d_1 t + ... + d_n t^n)e^{\alpha t}\sin(\beta t). \end{aligned}$$

So as above we check whether $r_* = \alpha + i\beta$ is a root and the same analysis shows the result. Note that $\alpha + i\beta$ cannot occur as a double root since the characteristic equation, $ar^2 + br + c$, has real coefficients. In fact, if $\alpha + i\beta$ is a root then the other root must be $\alpha - i\beta$.

By representing $\cos(\beta t)$ as $\frac{e^{i\beta t} + e^{-i\beta t}}{2}$ and applying similar logic to the $\sin(\beta t)$ case we complete the proof. □

Examples

- Consider the spring system governed by

$$y'' + 2y' - 3y = 3te^t.$$

Find the solution and its asymptotic behaviour.

1. For this equation we have $m = 1$, $r_* = 1$ and so our guess is:

$$y = t^s(a_0 + a_1 t)e^t.$$

2. To decide on the value of s, we have to check whether 1 is a root and what kind. The characteristic equation is

$$r^2 + 2r - 3 = 0 \implies r = -3,\, 1.$$

3. Therefore, we set $s = 1$ and our guess is:

$$y_{nh} = t(a_0 + a_1 t)e^t.$$

4. Next we determine a_i by plugging them into the equation and equating to $3te^t$:

$$\begin{aligned} 3te^t = y'' + 2y' - 3y &= \big(t(a_0 + a_1 t)e^{r_* t}\big)'' + 2\big(t(a_0 + a_1 t)e^{r_* t}\big)' - 3\big(t(a_0 + a_1 t)e^{r_* t}\big) \\ &\implies 2e^t(2a_0 + 4a_1 t + a_1) = 3te^t. \end{aligned}$$

This implies the following equation

$$\begin{aligned} &2(2a_0 + 4a_1 t + a_1) = 3t \\ \implies &a_1 = 3/8 \text{ and } 4a_0 + 2a_1 = 0 \\ \implies &a_1 = 3/8 \text{ and } a_0 = -\frac{3}{16}. \end{aligned}$$

Therefore, the general solution is

$$y = c_1 e^{-3t} + c_2 e^t + y_{nh} = c_1 e^{-3t} + c_2 e^t + t\left(-\frac{3}{16} + \frac{3}{8}t\right)e^t.$$

5. Therefore, as $t \to +\infty$, we have $y(t) \to +\infty$. Physically this means that the mass will get displaced towards the positive direction because of the external force $3te^t$.

- Consider the spring system governed by

$$y'' + 2y' - 3y = 2te^t \sin(t).$$

Determine what form the solution will take.

1. For this equation we have $m = 1$ and $\alpha = \beta = 1$, so our ansatz will be

$$y_{nh} = t^s e^t[(a_0 + a_1 t)\cos(t) + (b_0 + b_1 t)\sin(t)].$$

2. To decide on s, we have to check whether $1+i$ is a root for our characteristic equation:

$$r^2 + 2r - 3 = 0 \implies r = -3, 1.$$

3. So we put $s = 0$:

$$y_{nh} = e^t[(a_0 + a_1 t)\cos(t) + (b_0 + b_1 t)\sin(t)].$$

4. Therefore, the general solution is

$$y = c_1 e^{-3t} + c_2 e^t + y_{nh} = c_1 e^{-3t} + c_2 e^t + e^t[(a_0 + a_1 t)\cos(t) + (b_0 + b_1 t)\sin(t)].$$

2.3 Method 3: Variation of parameters

We will now consider non-homogeneous equations with coefficients of the form

$$ay'' + by' + cy = f(t),$$

where $f(t)$ is any continuous function and a, b, c are also functions with $a(t) \neq 0$.

Method formal steps

1. First, we obtain two linearly independent solutions y_1, y_2 for the homogeneous problem

$$ay'' + by' + cy = 0.$$

2. Second, we make a guess

$$y_g = v_1(t)y_1 + v_2(t)y_2$$

and plug it into our ODE. This will gives one equation for v_1, v_2.

3. Third; we have two unknowns, so we will need one more equation in order to solve for both. So we impose another condition for v_1, v_2 to obtain another equation:

$$v_1' y_1 + v_2' y_2 = 0.$$

This equation is helpful because it simplifies the first equation (proved in detail below)

$$y_g' = y_1' v_1 + y_2' v_2 + 0 \;\Rightarrow\; a y_g'' + b y_g' + c y_g = a(y_1' v_1' + y_2' v_2').$$

4. Therefore, we can obtain v_1, v_2 from the system

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ y_1' v_1' + y_2' v_2' = \frac{f}{a} \end{cases}$$

Example-presenting the method

Returning to the spring example, suppose that it is damping free $\gamma = 0$ and the exernal force is $f(t) = \tan(t)$:

$$y'' + y = \tan(t).$$

1. First we find independent solutions for the homogeneous problem:

$$y'' + y = 0.$$

2. One can easily check that $\cos(t)$, $\sin(t)$ are solutions for it and computing their Wronskian gives:

$$W(\cos(t), \sin(t), t) = \cos^2(t) + \sin^2(t) = 1 \neq 0.$$

3. Therefore, we make a guess

$$y_g = v_1 \cos(t) + v_2 \sin(t).$$

4. Using our system of equations

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ y_1' v_1' + y_2' v_2' = \frac{f}{a} \end{cases}$$

we obtain

$$\begin{cases} v_1' \cos(t) + v_2' \sin(t) = 0 \\ -\cos(t) v_1' + \cos(t) v_2' = \tan(t) \end{cases}$$
$$\Longrightarrow \begin{cases} v_1' = -\tan(t)\sin(t) \\ v_2' = \tan(t)\cos(t) = \sin(t) \end{cases}$$

Therefore, by integrating we obtain

$$\begin{aligned} v_1 &= -\int \tan(t)\sin(t)\mathrm{d}t = -\int \frac{\sin^2(t)}{\cos(t)}\mathrm{d}t \\ &= \int \left(\cos(t) - \frac{1}{\cos(t)}\right)\mathrm{d}t \\ &= \sin(t) - \ln\left|\frac{1+\sin(t)}{\cos(t)}\right| + c_1 \\ \text{and} \\ v_2 &= \int \sin(t)\mathrm{d}t = -\cos(t) + c_2 \end{aligned}$$

For simplicity we take $c_1 = c_2 = 0$ and we get:

$$\begin{aligned} y_g &= \left(\sin(t) - \ln\left|\frac{1+\sin(t)}{\cos(t)}\right|\right)\cos(t) - \cos(t)\sin(t) \\ &= \cos(t)\ln\left|\frac{\cos(t)}{1+\sin(t)}\right|. \end{aligned}$$

General result:

The equations $y'' + p(t)y' + q(t)y = g(t)$ for continuous p, q, g have solutions of the form

$$Y = y_1 v_1 + y_2 v_2,$$

where y_1, y_2 are fundamental solutions for the homogeneous problem $y'' + p(t)y' + q(t)y = 0$ and

$$v_1 := -\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2, s)}\mathrm{d}s \text{ and } v_2 := \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2, s)}\mathrm{d}s.$$

Proof. We start by making the guess

$$y_g := y_1 v_1 + y_2 v_2.$$

We have

$$y_g' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'.$$

So we note that if we set

$$v_1' y_1 + v_2' y_2 = 0$$

then the second derivative will not contain any v_1'', v_2'' terms:

$$\begin{aligned} y_g' &= 0 + v_1 y_1' + v_2 y_2' \\ \implies y_g'' &= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''. \end{aligned}$$

Therefore, the ODE for y_g becomes

$$\begin{aligned} y_g'' + p y_g' + q y_g &= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' + p(v_1 y_1' + v_2 y_2') \\ &+ q(y_1 v_1 + y_2 v_2) \\ &= v_1(y_1'' + p(t) y_1' + q(t) y_1) + v_2(y_2'' + p(t) y_2' + q(t) y_2) \\ &+ v_1' y_1' + v_2' y_2' \\ &= 0 + v_1' y_1' + v_2' y_2' \end{aligned}$$

because y_1, y_2 are solutions to the homogeneous problem. Therefore, for y_g to be a solution we need

$$v_1' y_1' + v_2' y_2' = g(t).$$

Our second equation was

$$v_1' y_1 + v_2' y_2 = 0.$$

Together they give

$$\begin{aligned} v_1' = \frac{-y_2 g}{W(y_1, y_2, t)} &\text{ and } v_2' = \frac{y_1 g}{W(y_1, y_2, t)} \\ v_1 := C_1 - \int_{t_0}^{t} \frac{y_2(s) g(s)}{W(y_1, y_2, s)} \mathrm{d}s &\text{ and } v_2 := C_2 + \int_{t_0}^{t} \frac{y_1(s) g(s)}{W(y_1, y_2, s)} \mathrm{d}s. \end{aligned}$$

Observe that the constants of integration can be ignored since including them leads to

$$v_1(t) y_1 + v_2 y_2 = \left(- \int_{t_0}^{t} \frac{y_2(s) g(s)}{W(y_1, y_2, s)} \mathrm{d}s \right) y_1 + \left(\int_{t_0}^{t} \frac{y_1(s) g(s)}{W(y_1, y_2, s)} \mathrm{d}s \right) y_2 + \underbrace{C_1 y_1 + C_2 y_2}_{\text{solution to homogeneous}}.$$

□

Examples

- Consider the equation

  $$t y'' - (1+t) y' + y = t^2 e^{2t}$$

  with given fundamental solutions $y_1 = 1 + t, y_2 = e^t$ for the homogeneous problem $t y'' - (1+t) y' + y = 0$.

- We have the system

$$\begin{aligned} &\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ y_1' v_1' + y_2' v_2' = \frac{f}{a} \end{cases} \\ \implies &\begin{cases} v_1'(1+t) + v_2' e^t = 0 \\ v_1' + e^t v_2' = t e^{2t} \end{cases} \\ \implies &v_1' = -e^{2t} \text{ and } v_2' = (1+t) e^t \end{aligned}$$

$$\implies v_1 = -\frac{1}{2}e^{2t} \text{ and } v_2 = te^t.$$

Therefore, the solution for the nonhomogeneous problem will be

$$y = v_1y_1 + v_2y_2 = \left(-\frac{1}{2}e^{2t}\right)(1+t) + \left(te^t\right)e^t = \frac{1}{2}(t-1)e^{2t}.$$

- Consider the equation
$$x^2y'' - 3xy' + 4y = x^2\ln(x)$$
with given fundamental solutions $y_1 = x^2$, $y_2 = x^2\ln(x)$ for the homogeneous problem $x^2y'' - 3xy' + 4y = 0$.
We have the system
$$\begin{aligned} &\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ y_1'v_1' + y_2'v_2' = \frac{f}{a} \end{cases} \\ \implies &\begin{cases} v_1'x^2 + v_2'x^2\ln(x) = 0 \\ 2xv_1' + (2x\ln(x) + x)v_2' = \ln(x) \end{cases} \\ \implies &v_1' = -\ln^2(x)/x \text{ and } v_2' = \ln(x)/x \\ \implies &v_1 = -\frac{\ln^3(x)}{3} \text{ and } v_2 = \frac{\ln^2(x)}{2}. \end{aligned}$$
Therefore, the solution for the nonhomogeneous problem will be
$$y = v_1y_1 + v_2y_2 = -\frac{\ln^3(x)}{3}x^2 + \frac{\ln^2(x)}{2}x^2\ln(x).$$

2.4 Method 4: Reduction of order

For homogeneous equations of the form

$$y'' + p(t)y' + q(t)y = 0 \tag{2.4.1}$$

if we have one solution y_1, we can obtain a second one by setting

$$y_2 := v(t)y_1$$

and identifying an ODE for v. Plugging in y_2 into our ODE we obtain

$$y_1v'' + (2y_1' + py_1)v' = 0$$

where we have used that y_1 satisfies (2.4.1).

Example-presenting the method

Consider the equation

$$x^2y'' - 5xy' + 9y = 0 \implies y'' - 5x^{-1}y' + 9x^{-2} = 0.$$

1. One solution is $y_1 = x^3$. One would guess this solution by observing that this equation preserves the "order" of monomials. That is y'' decreases the power of x by 2 but in the equation y'' is multiplied by x^2. The same phenomenon occurs for xy'. As a result, we obtain a characteristic equation if we guess $y = x^r$ for an r to be determined.

2. Assuming a second solution of the form $y_2 = v(x)x^3$ we obtain

$$0 = y_1 v'' + (2y_1' + py_1)v' = x^3 v'' + \big(6x^2 - 5x^{-1}x^3\big)v' = x^2(xv'' + v') \Rightarrow xv'' + v' = 0.$$

3. This gives us

$$v(x) = c\ln(x).$$

So the general solution will be

$$y = ax^3 + b\ln(x)x^3.$$

2.5 Nonlinear into second order

2.5.1 Riccati

The non-linear Riccati equation can always be reduced to a second order linear ordinary differential equation (ODE): If y satisfies

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2$$

then, wherever q_2 is never zero and differentiable,

$$v = yq_2$$

satisfies a Riccati equation of the form

$$v' = v^2 + R(x)v + S(x),$$

where

$$S = q_2 q_0 \text{ and } R = q_1 + \frac{q_2'}{q_2}$$

because

$$\begin{aligned} v' &= (yq_2)' = y'q_2 + yq_2' \\ &= \big(q_0 + q_1 y + q_2 y^2\big)q_2 + yq_2' \\ &= q_0 q_2 + \left(q_1 + \frac{q_2'}{q_2}\right)q_2 y + (q_2 y)^2 \\ &= q_0 q_2 + \left(q_1 + \frac{q_2'}{q_2}\right)v + v^2. \end{aligned}$$

Substituting

$$v = -\frac{u'}{u},$$

it follows that u satisfies the linear 2nd order ODE

$$u'' + R(x)u' - S(x)u = 0$$

since

$$\begin{aligned} u'' &= (uv)' = u'v + uv' \\ &= u'v + u\big(v^2 + R(x)v + S(x)\big) \\ &= u'\left(\frac{-u'}{u}\right) + u\left(\left(\frac{-u'}{u}\right)^2 + R(x)\left(\frac{-u'}{u}\right) + S(x)\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{(u')^2}{u} + \frac{(u')^2}{u} - R(x)u' + S(x)u \\
&= -R(x)u' + S(x)u
\end{aligned}
$$

and hence

$$u'' + Ru' - Su = 0.$$

A solution of this equation will lead to a solution

$$y = \frac{-u'}{q_2 u}$$

of the original Riccati equation.

2.6 Problems

- **Real distinct roots**
  1. Find the solution, do a rough sketch, and describe its asymptotic behaviour
     (a) $y'' + y' - 2y = 0,\ y(0) = 1,\ y'(0) = 1,$
     (b) $y'' + 3y' = 0,\ y(0) = -2,\ y'(0) = 3.$
  2. Solve
     $$y'' - y' - 2y = 0, y(0) = a,\ y'(0) = 2$$
     and determine for which a, the solution goes to zero as $t \to +\infty$.
- **Wronskian**
  1. Consider the equation $y'' - y' - 2y = 0$
     (a) Show that $y_1(t) := e^{-t}$, $y_2(t) := e^{2t}$ form a set of fundamental solutions.
     (b) Show that each of $y_3(t) := -2e^{2t}$, $y_4(t) := y_1(t) + 2y_2(t)$, and $y_5(t) := 2y_1(t) - 2y_3(t)$ are solutions to the above ode.
     (c) Which of the following pairs give rise to a fundamental pair of solutions:
     $$\{y_1, y_3\}, \{y_2, y_3\}, \{y_1, y_4\}, \{y_4, y_5\}$$
- **Complex roots**
  1. Imagine a spring satisfying the following equations. Find the solution, do a rough sketch, and describe its asymptotic behaviour (steady/growing/decaying oscillation). Finally, explain the asymptotic behaviour based on the coefficients (see notes on damping effect).
     (a) $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 1,$
     (b) (*)$y'' + 2y' + 2y = 0,\ y(\pi/4) = 2,\ y'(\pi/4) = -2.$
- **Repeated roots**
  1. Find the solution, do a rough sketch, and describe its asymptotic behaviour
     (a) $9y'' - 12y' + 4y = 0,\ y(0) = 2,\ y'(0) = -1,$
     (b) $y'' + 4y' + 4y = 0,\ y(-1) = 2,\ y'(-1) = 1.$
  2. Consider the problem
     $$y'' - y' + \frac{y}{4} = 0,\ y(0) = 2,\ y'(0) = b$$
     Find the solution and determine for which b, the solution remains positive for all $t > 0$.

- **Demand and Supply Problems**

  Let the demand and supply functions be, respectively,

  $$D(P) = 9 - P + P' + 3P'' \text{ and } S(P) = -1 + 4P + 2P' + 5P''$$

  with $P(0) = 4,\ P'(0) = 4$.

  1. Derive the price ODE (, and find the price solution.
  2. Does it have a globally stable solution as $t \to +\infty$? What does the stability result tell you?

- **Method of undetermined coefficients**

  Find the general form of the solution (with abstract coefficients) and use the stability result to determine whether they will have a globally stable solution.

  1. $y'' - 2y' - 3y = 3e^{2t}$,
  2. $y'' - y' - 2y = -2t + 4t^2$,
  3. $y'' + 2y' = 3 + 4\sin(2t)$

- Find the solution of the given IVP:

  $$y'' + y' - 2y = 2t,\ y(0) = 0,\ y'(0) = 1.$$

- **Variation of parameters**

  Below you are given the fundamental solutions y_1, y_2 of the homogeneous problem. Use them to find a solution of the nonhomogeneous one.

  1. $t^2y'' - 2y = 3t^2 - 1$ with $y_1 = t^2,\ y_2 = t^{-1}$,
  2. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$ with $y_1 = t,\ y_2 = te^t$.

Chapter 3

Systems of ODEs

Consider system of equations:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \ldots + p_{1n}x_n + g_1(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \ldots + p_{nn}x_n + g_n(t) \end{aligned}$$

where $p_{ij}(t), g_i(t)$ are continuous functions. The continuity ensures that we have existence and uniqueness of solutions. Equivalently we can rewrite this system as

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t).$$

where $\mathbf{P}(t)$ denote the matrix where the entry in the ij^{th} position is $p_{ij}(t)$ and $\mathbf{g}(t)$ is the n-vector with entries $g_k(t)$ for $1 \leq k \leq n$. For the homogeneous problem (i.e. $\mathbf{g} \equiv 0$) we can see by linearity that if $\mathbf{x}_1, \mathbf{x}_2$ are both solutions to that system, then any linear combination $\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also a solution. In fact, as with second order ODEs, we will show that any solution to the system is of that form if $\{\mathbf{xn}_i\}$, for $1 \leq i \leq n$ are linearly independent solutions to the system (that is, any solution can be expressed as a linear combination of the solutions $\mathbf{x}_1, \ldots, \boldsymbol{x}_n$ when they are linearly independent). Analogously to second order, a collection of solutions $\mathbf{x}_1, ..., \mathbf{x}_n$ is called *linearly independent* if there exists t_* s.t.

$$\det[\mathbf{X}(t)] := \det \begin{bmatrix} x_{11}(t_*) & x_{12}(t_*) & \cdots & x_{1n}(t_*) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t_*) & x_{n2}(t_*) & \cdots & x_{nn}(t_*) \end{bmatrix} \neq 0,$$

where $\mathbf{x}_i = i^{\text{th}}\ \textit{row} = (x_{1i}, ..., x_{ni})$.

General solution

Consider $\mathbf{x}_1, \ldots, \mathbf{x}_n$ linearly independent solutions of the system $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$ where $\mathbf{P}(t)$ is an $n \times n$ matrix and $\mathbf{v}(t)$ is any solution for the nonhomogeneous problem $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t)$. Then for any other solution $\mathbf{y}$ of the nonhomogeneous problem, there exist unique constants $\{c_i\}$ s.t.

$$\mathbf{y} = c_1\mathbf{x}_1 + ... + c_n\mathbf{x}_n + \mathbf{v}(t).$$

Proof. We will follow the same ideas as in the analogous second order result. We begin by proving an auxiliary result about the homogeneous equation. Let $\boldsymbol{\varphi}$ be a solution for the above homogeneous problem and $\mathbf{K} := \boldsymbol{\varphi}(t_*)$. Consider the system of equations given by

$$\mathbf{X}(t_*)\mathbf{c} := \begin{bmatrix} x_{1,1}(t_*) & x_{1,2}(t_*) & \cdots & x_{1,n}(t_*) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}(t_*) & x_{n,2}(t_*) & \cdots & x_{n,n}(t_*) \end{bmatrix} \mathbf{c} = \mathbf{K}$$

for unknown vector $\mathbf{c}$. Then, by linear independence of the solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$, the matrix, $\mathbf{X}(t_*)$ is invertible and so we can solve for $\mathbf{c}$ by inverting $\mathbf{X}(t_*)$ and multiplying on the right by $\mathbf{K}$ (i.e $\mathbf{c} = \mathbf{X}^{-1}(t_*)\mathbf{c}$). Let $\boldsymbol{\zeta}(t) := \mathbf{X}(t)\mathbf{c}$ where $\mathbf{c}$ is the vector obtained from the above discussion. Then we have that $\boldsymbol{\zeta}(t_*) = \mathbf{K} = \boldsymbol{\varphi}(t_*)$. Therefore, by existence and uniqueness $\boldsymbol{\zeta}(t) = \boldsymbol{\varphi}(t)$ for all t.

Next let $\mathbf{y}$ be a solution for the nonhomogeneous problem. Then $\mathbf{y} - \mathbf{v}$ is a solution of the homogeneous problem and thus, by the above discussion, $\exists\, \mathbf{a} \in \mathbb{R}^n$ s.t.

$$\mathbf{y} = \mathbf{a} \cdot \mathbf{x} + \mathbf{v}.$$

$\square$

3.1 Homogeneous linear systems with constant coefficients

Consider the homogeneous system of n-ODEs

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

where $\mathbf{A}$ is $n \times n$ matrix with constant real entries. As with second order we make the ansatz $\boldsymbol{x}(t) = \boldsymbol{\xi} e^{\lambda t}$ where $\boldsymbol{\xi}$ is a fixed n-vector (to be chosen precisely later). Then, we observe that if $\boldsymbol{\xi}$ is chosen so that $\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$ (i.e $\boldsymbol{\xi}$ is an eigenvector of $\boldsymbol{A}$) we get

$$\mathbf{A}\mathbf{x}(t) = \mathbf{A}\boldsymbol{\xi} e^{\lambda t} = \lambda\boldsymbol{\xi} e^{\lambda t} = \boldsymbol{\xi}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\right) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t).$$

Such $\boldsymbol{\xi}, \lambda$ are called $\mathbf{A}'$s eigenvector and eigenvalue respectively (as noted earlier). We will now obtain the general solution. First we will assume that all eigenvalues $\{\lambda_i\}_{i=1}^n$ of $\mathbf{A}$ are real and distinct from each other; in the other sections we study the other cases. Let $\{\boldsymbol{\xi}_i\}_{i=1}^n$ be the corresponding eigenvectors. Then the solutions $\{\boldsymbol{\xi}_i e^{\lambda_i t}\}_{i=1}^n$ are linearly independent:

$$\det \begin{bmatrix} \xi_{11}e^{\lambda_1 t} & \cdots & \xi_{1n}e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ \xi_{n1}e^{\lambda_1 t} & \cdots & \xi_{nn}e^{\lambda_n t} \end{bmatrix} = e^{(\lambda_1 + \ldots + \lambda_n)t} \det \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & \ddots & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} \neq 0$$

where the last step follows because when all the eigenvalues of a matrix are distinct, then its eigenvectors will be linearly independent. Thus, from the above result we obtained the general solution.

Example —presenting the method

Consider two connected tanks A and B containing 1000L of well-mixed salt-water with $x(t), y(t)$ kilogram amounts of salt respectively. Let IP,OP denote the L/min-rate of salt-free water entering and exiting the two tanks and $P1, P2$ the L/min-rate of saltwater getting exchanged between the two tanks.

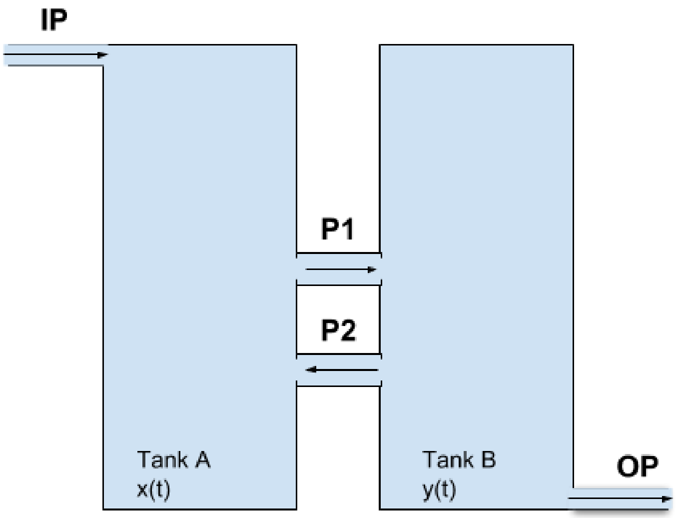


Figure 3.1.1: Tanks A and B containing salt

To keep the volume of water constant in the two tanks we set IP = OP = $1(L/\text{min})$. Let the rates $P1 = 1(L/\text{min})$ and $P2 = 2(L/\text{min})$ be constant in time. The concentration of salt in each tank is $\frac{x(t)}{1000}$kg/L,$\frac{y(t)}{1000}$kg/L respectively. Therefore, for tank A the rate of change of the amount of salt:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = \text{Input rate} - \text{Output rate} = 2 \cdot \frac{y(t)}{1000} - 1 \cdot \frac{x(t)}{1000}$$

and for tank B we must also subtract the draining of salt from pipe OP

$$\frac{\mathrm{d}y}{\mathrm{dt}} = \text{Input rate} - \text{Output rate} = 1 \cdot \frac{x(t)}{1000} - 2 \cdot \frac{y(t)}{1000} - 1 \cdot \frac{y(t)}{1000}.$$

In matrix form our system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{1000} \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

1. First we compute the eigenvalues

$$\det \begin{bmatrix} -1-\lambda & 2 \\ 1 & -3-\lambda \end{bmatrix} = 0 \Rightarrow (-1-\lambda)(-3-\lambda) - 2 \cdot 1 = 0$$
$$\Rightarrow \lambda_1 = -2 + \sqrt{3},\ \lambda_2 = -2 - \sqrt{3}.$$

2. Second we find the corresponding eigenvectors. To find $\boldsymbol{\xi}_1 := \binom{\xi_{1,1}}{\xi_{2,1}}$ we solve the system (up to multiples):

$$\begin{bmatrix} -1-\lambda_1 & 2 \\ 1 & -3-\lambda_1 \end{bmatrix} \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By solving the system directly we obtain the solution (up to multiples). For example, we rewrite the above to get:

$$\begin{cases} (1-\sqrt{3})\xi_{1,1} + 2\xi_{2,1} = 0 \\ \xi_{1,1} + (-1-\sqrt{3})\xi_{2,1} = 0 \end{cases} \implies \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}.$$

Similarly to obtain $\boldsymbol{\xi}_2$ we have to solve

$$\begin{bmatrix} -1-(-2-\sqrt{3}) & 2 \\ 1 & -3-(-2-\sqrt{3}) \end{bmatrix} \begin{pmatrix} \xi_{1,2} \\ \xi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we get

$$\boldsymbol{\xi}_2 = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}.$$

3. Therefore, by the discussion above the general solution will be

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{1000} c_1 \cdot \boldsymbol{\xi}_1 e^{\lambda_1 t} + \frac{1}{1000} c_2 \cdot \boldsymbol{\xi}_2 e^{\lambda_2 t} = c_1 \cdot \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix} \frac{e^{(-2+\sqrt{3})t}}{1000} + c_2 \cdot \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix} \frac{e^{(-2-\sqrt{3})t}}{1000}.$$

4. Since $2 > \sqrt{3}$, both the eigenvalues are negative and in turn the salt concentrations $x(t), y(t)$ will go to zero as $t \to +\infty$. This is reasonable because through pipe IP we are injecting salt-free water that over time transports the tanks' salt out through pipe OP.

5. Next we study the stability. Since $-2+\sqrt{3} > -2-\sqrt{3}$, we get $e^{(-2+\sqrt{3})t} > e^{(-2-\sqrt{3})t}$ and so as $t \to +\infty$ the first eigenvector $\binom{1+\sqrt{3}}{1}$ will dominate.

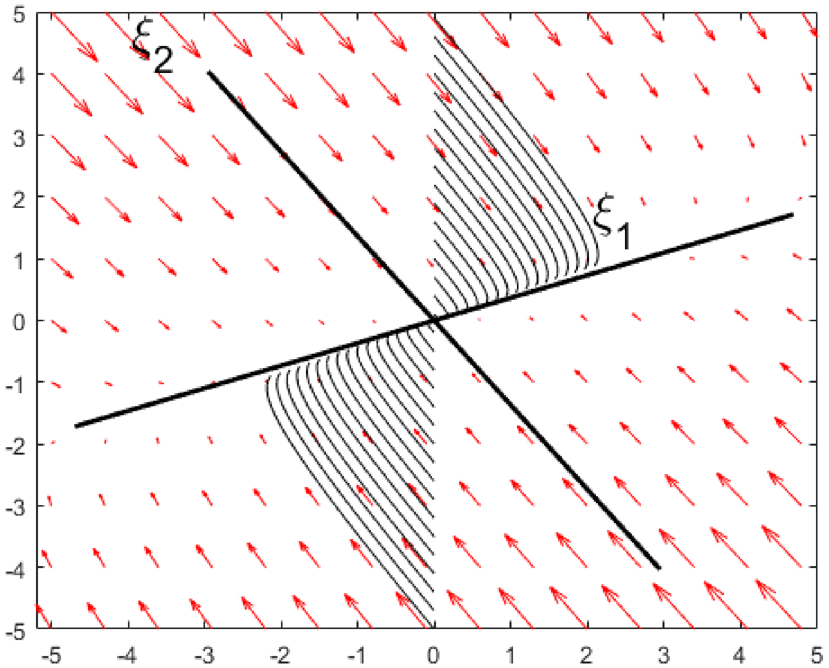


Figure 3.1.2: Solutions converge to line defined by vector $\xi_1 = \binom{1+\sqrt{3}}{1}$ and then to $(0, 0)$.

In other words, for large t we will have

$$x(t) \approx (1+\sqrt{3}) \cdot e^{(-2+\sqrt{3})t} > 1 \cdot e^{(-2+\sqrt{3})t} \approx y(t).$$

This is reasonable because $P2 > P1$ and so as $t \to +\infty$ the salt concentration x(t) in tank A will be greater than that of tank B.

Method formal steps

1. Starting with a matrix $\mathbf{A}$ we first compute its eigenvalues $\{\lambda_i\}$ eg. for matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have two eigenvalues:

$$det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\Rightarrow \lambda = \frac{(a+d)}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)} = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4det(A)}.$$

2. Second for each eigenvalue, we find the corresponding eigenvector. Continuing the example from the previous step, we find the eigenvector $\boldsymbol{\xi}_1 := \binom{\xi_{1,1}}{\xi_{2,1}}$ by solving the system:

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here are the general formulas for eigenvectors for 2D systems:

- If $c \neq 0$ then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}.$$

- If $b \neq 0$ then
$$\boldsymbol{\xi}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}.$$
- If $b = c = 0$ then
$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3. Then the general solution will be of the form
$$x(t) = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + ... + c_n \boldsymbol{\xi}_n e^{\lambda_n t}.$$

4. Finally we study the stability for 2D systems:

   - If $\lambda_1 \neq \lambda_2$ and both positive then (0,0) will be a **nodal source** and solutions will be moving away from it (**unstable**).
   - If $\lambda_1 \neq \lambda_2$ and both negative then (0,0) will be a **nodal sink** and solutions will be moving towards it (**asymptotically stable**).
   - If $\lambda_1 \neq \lambda_2$ and with opposite signs then (0,0) will be a **saddle point** and solutions will be moving away from it along one eigenvector and towards it along the other eigenvector (**unstable**).
   - If one of the eigenvalues is zero eg. $\lambda_1 = 0$ and $\lambda_2 < 0$ then the line defined by ξ_1 will be a nodal source (**asymptotically stable**).
   - If one of the eigenvalues is zero eg. $\lambda_1 = 0$ and $\lambda_2 > 0$ then the line defined by ξ_1 will be a nodal sink (**asymptotically unstable**).

Examples

- We will exhibit each of the above stability cases by studying the IVP problem
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mathbf{x}$$
with $\mathbf{x}(0) = \binom{2}{2}$.

   1. First we find the eigenvalues. For diagonal matrices this is immediate: $\lambda_1 = a, \lambda_2 = b$.
   2. Next we find the corresponding eigenvectors:
   $$\begin{bmatrix} a - \lambda_1 & 0 \\ 0 & b - \lambda_1 \end{bmatrix} \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \xi_{2,1} = 0$$
   therefore $\boldsymbol{\xi}_1 = \binom{1}{0}$. Similarly, we obtain $\boldsymbol{\xi}_2 = \binom{0}{1}$.
   3. Therefore, the general solution is
   $$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{bt}.$$
   4. Finally, using the initial condition we obtain
   $$\mathbf{x}(t) = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{bt}.$$
   5. Next we study the stability. The origin is a special point for dynamics because if $\mathbf{x}(t_*) = 0$ then $\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}(t_*) = A \cdot \binom{0}{0} = \binom{0}{0}$ and so it is a critical point.

(a) If $a \neq b$ and both positive we obtain that the solutions diverge to infinity

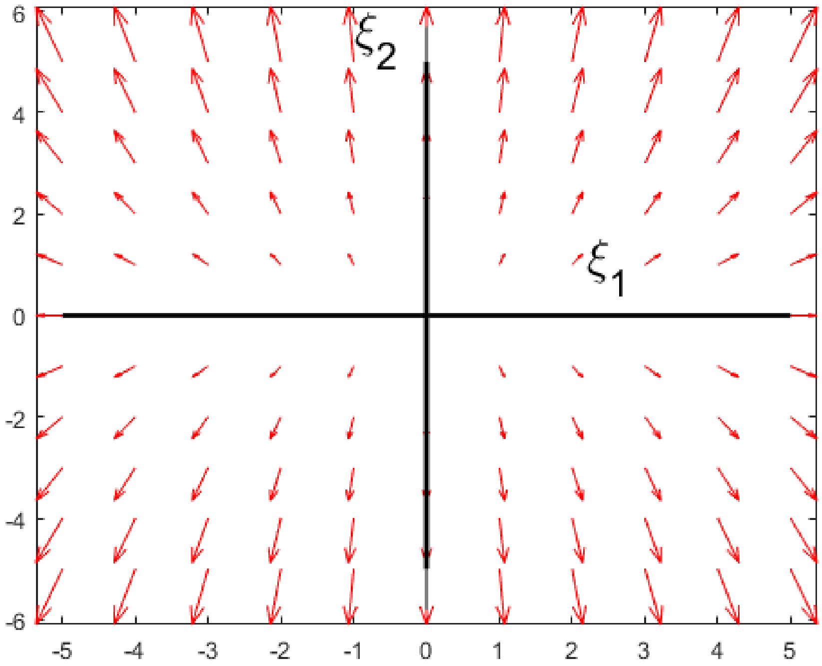


Figure 3.1.3: $a = 1$, $b = 3$

(b) If $a \neq b$ and both negative we obtain that they both converge to the source (0,0)

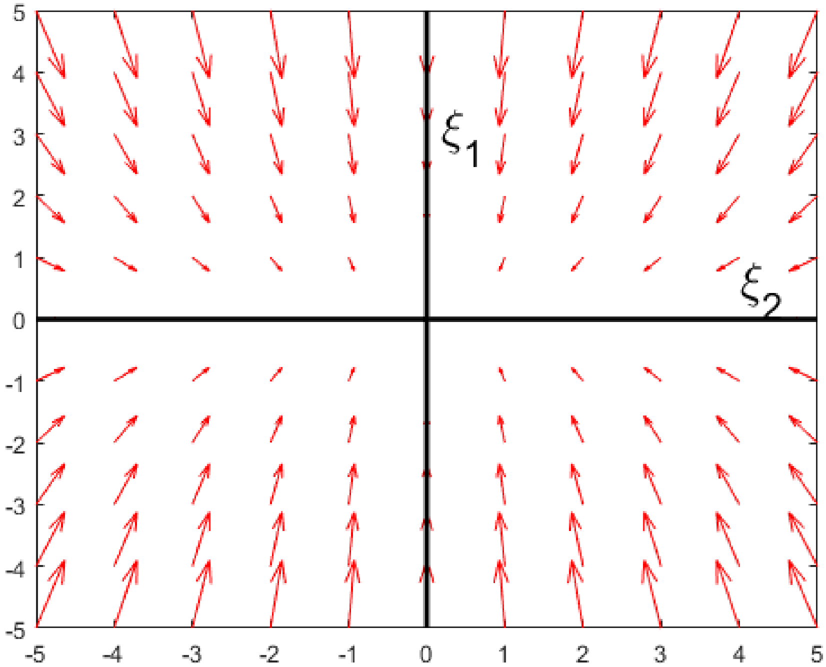
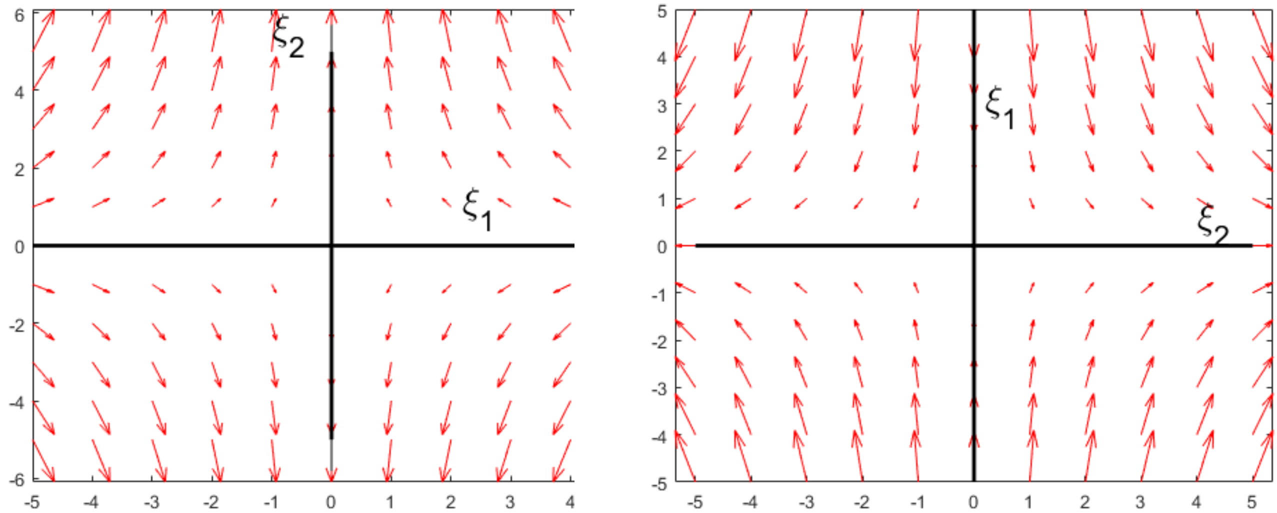


Figure 3.1.4: $a = -1$, $b = -3$

(c) If $a \neq b$ with different signs we obtain that (0,0) is a saddle point: if $a = -1, b = 3$ then the solutions converge to the origin if they start on the linear span of ξ_1 (x-axis) otherwise they diverge to infinity.



(a) $a = -1$, $b = 3$

(b) $a = 1$, $b = -3$

(d) If $a = 0$ and $b < 0$ then the linear span of ξ_1 (x-axis) will attract all the solutions:

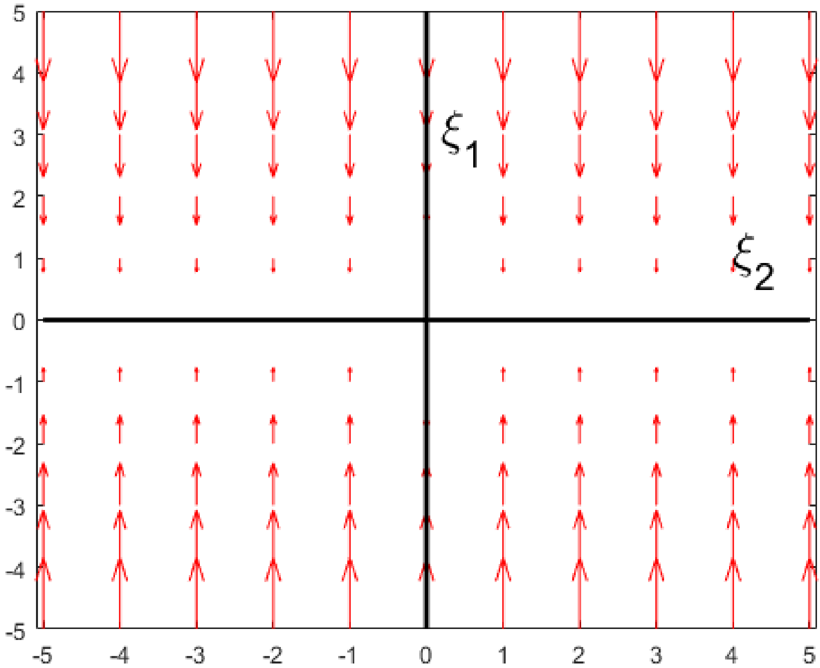


Figure 3.1.6: $a = 0$, $b = -3$

(e) If $a = 0$ and $b > 0$ then the linear span of ξ_1 (x-axis) will repel all the solutions:

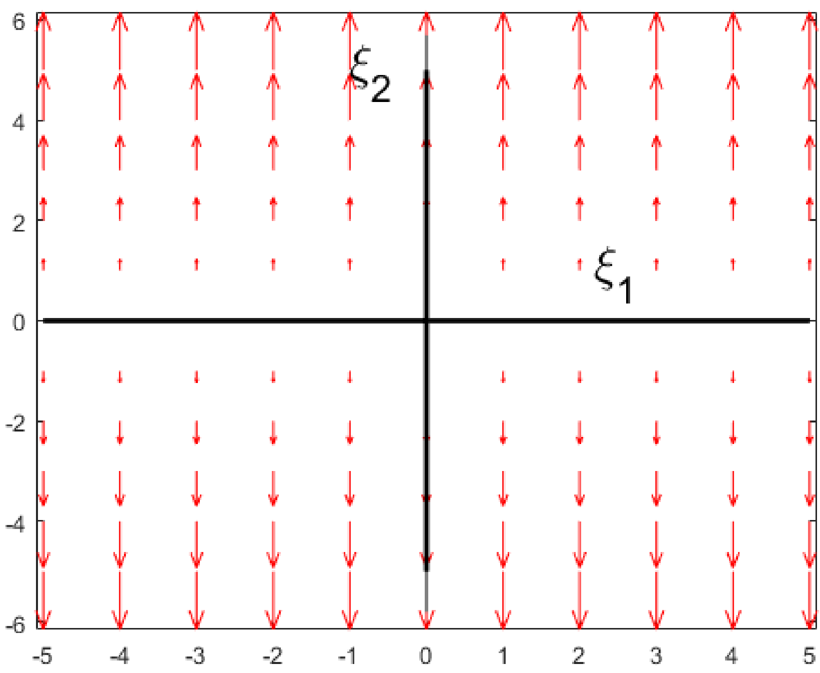


Figure 3.1.7: $a = 0$, $b = 3$

Applied examples

- **Richardson Arms race model**: Consider countries A, B with $x(t), y(t)$ amount of weaponry respectively. The model for the rate of change of weaponry is:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -a \cdot x + b \cdot y + e_1$$
$$\frac{\mathrm{d}y}{\mathrm{dt}} = c \cdot x - d \cdot y + e_2$$

The constants a, b, c, d are nonnegative. The constants b, c represent the fear magnitude eg. when $y(t)$ goes up then country A will increase its rate of weapon production by $b \cdot y(t)$. The constants a, d represent the fatigue factor because some countries decide on a lower rate of production given the amount of weapons they currently possess. For simplicity the constant e_1 represents the distrust country A has for country B and the reverse for e_2. But they can represent other factors not accounted for such as revenge, degradation of weapons, etc. So if we have no interaction i.e. $a = b = 0$ but positive amount of distrust $e_1 > 0$ then we still have a steady rate of weapon production $x'(t) = e_1 > 0$.
For simplicity $e_1 = -1, e_2 = -1$ consider the following matrix system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

1. A constant solution for this nonhomogeneous problem is $v(t) = \binom{1}{1}$, which we obtained by setting $\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \binom{0}{0}$ and solving for $\mathbf{x}$. Therefore, as explained above the general solution will be:

$$\mathbf{x} = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\xi}_2 e^{\lambda_2 t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

2. The eigenvalues for this matrix are the solutions to

$$0 = \lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 4\lambda - 5$$
$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -5.$$

3. The corresponding eigenvectors are $\boldsymbol{\xi}_1 = \binom{1}{1}$ and $\boldsymbol{\xi}_2 = \binom{-1}{2}$. So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-5t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4. Therefore, $\binom{1}{1}$ becomes a saddle point. That is, if a solution starts from $\binom{1}{1}$ in a direction parallel to $\binom{-1}{2}$ (i.e. choose $c_1 = 0$), then the solution will converge to the constant $\binom{1}{1}$ at an exponential rate (like e^{-5t}). For example, this happens if $\mathbf{x}(0) = \binom{\frac{1}{2}}{2}$. However for $c_1 \neq 0$ the solution will diverge to infinity like e^t in the direction $\binom{1}{1}$ away from the starting point $\binom{1}{1}$.

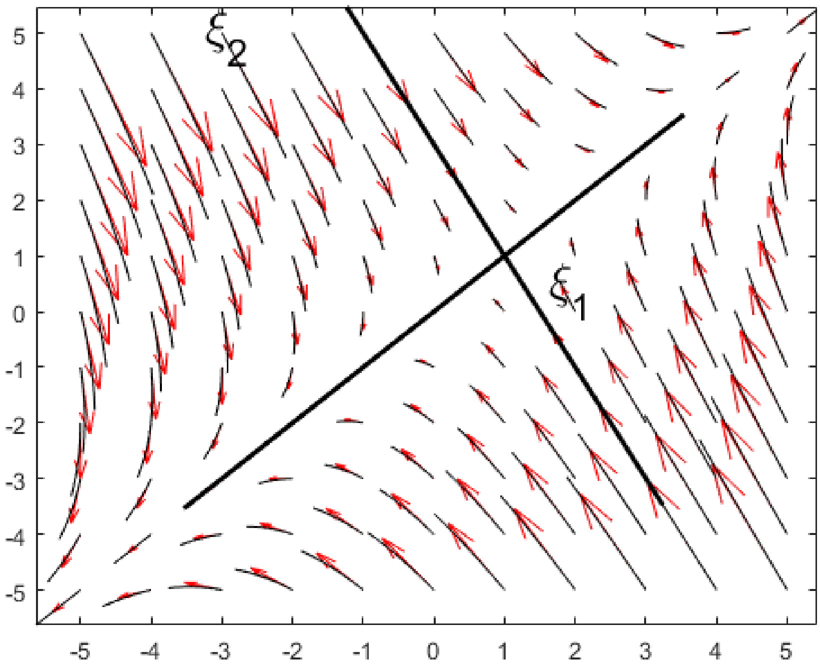


Figure 3.1.8: The $\binom{1}{1}$ is a saddle point

5. This is reasonable because if the amount of distrust is negative $e_1 = -1, e_2 = -1 < 0$ (i.e. positive trust), then for appropriate initial conditions the solutions will converge to peaceful coexistence $(x(t), y(t)) \to (1, 1)$.

6. To make sense of the special role $c_1 = 0$ plays we have to study the critical level sets. We have $x'(t) \geq 0$ and $y'(t) \geq 0$ when $-x + 2y - 1 \geq 0$ and $4x - 3y - 1 \geq 0$ respectively; call these lines as L_1, L_2. The first inequality happens when we are above L_1 and the second when we are below L_2. This is the region enclosed between the lines on the right. So in the direction of ξ_1 both countries are increasing their rate without stop . On the other hand, in the direction of ξ_2, which is above or below both lines, the rates will have opposite signs (In this case, one country is increasing their amount of weaponry while the other is decreasing it. The effect of one country decreasing their stock of weaponry interferes with the other countries desire to have more weapons. This is because the country that was originally increasing its stock of weapons will see the other country deplete its stock and so will have less incentive to create more.).

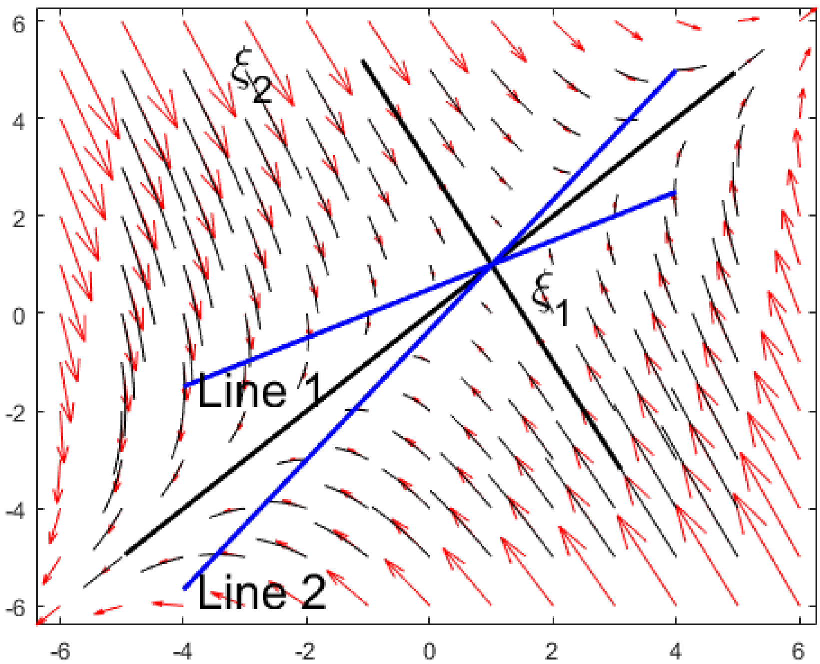


Figure 3.1.9: The lines L_1, L_2 separate into regions of stability and instability.

As a reference the general solution for

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} -a & b \\ c & -d \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.1.1}$$

is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \frac{\lambda_1 + a}{b} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ \frac{\lambda_2 + a}{b} \end{pmatrix} e^{\lambda_2 t} + \begin{pmatrix} \frac{e_1 d + e_2 b}{ad - bc} \\ \frac{e_1 a + e_2 c}{ad - bc} \end{pmatrix}. \tag{3.1.2}$$

- Consider the parallel circuit displayed in Figure 3.1.10 *capacitance* C (eg. battery), *resistance* R (eg. light bulb) and *inductance* L (eg. coil used for storing energy). Note that there is no voltage source.

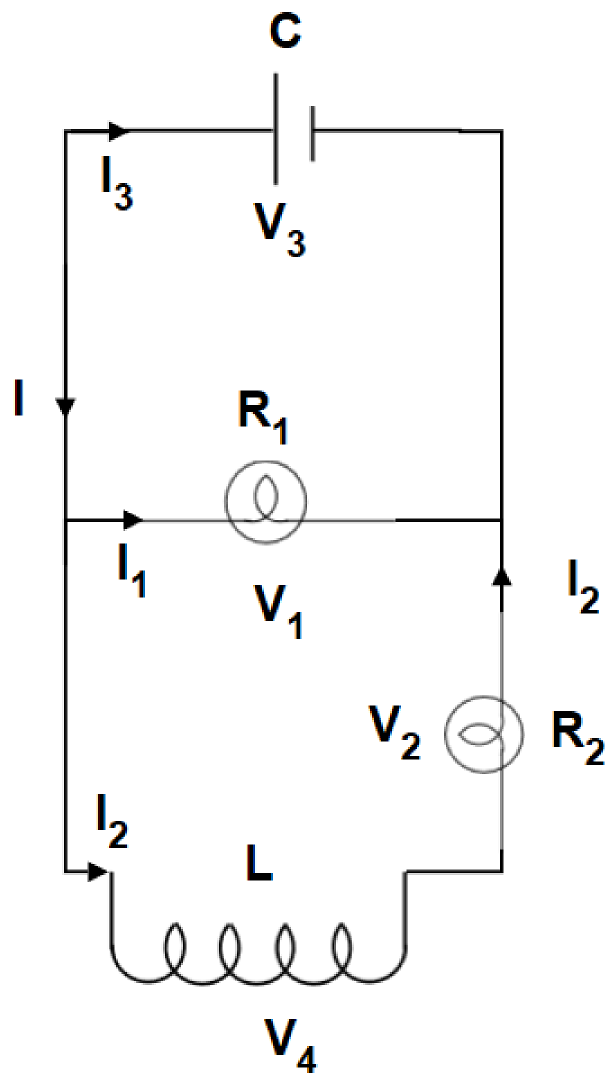


Figure 3.1.10: Parallel LRC circuit.

Let V be the *voltage drop* and I the *current* passing through the circuit. Here is a quick summary of the laws governing such systems:

- **Ohm's law**(OL): For the resistance we have $V = R \cdot I$.
- For the capacitance we have $I_3 = C \cdot \frac{\mathrm{d}V_3}{\mathrm{dt}}$.
- **Faraday's law** and **Lenz's law**(FLL)**:** For the inductance we have $V_4 = L \cdot \frac{\mathrm{d}I_2}{\mathrm{dt}}$.
- **Kirchhoff's current law**(KCL): $-I_3 = I = I_1 - I_2$; this is the conservation of energy law for circuits.
- **Kirchhoff's Voltage law**(KVL): The sum of voltages in a loop is zero. Thus, in the upper loop $V_3 = V_1$ and in the lower loop $V_2 + V_4 + V_1 = 0$.

We can express all these relations in a system of odes that will describe the above circuit system. We have

$$CV_1' \overset{(KVL)}{=} CV_3' = I_3 \overset{(KCL)}{=} -I \overset{(KCL)}{=} -I_1 + I_2 \overset{(OL)}{=} -\frac{V_1}{R_1} + I_2$$

where we first applied Kirchhoff's voltage law and then current law. We also have

$$LI_2' \overset{(FLL)}{=} V_4 \overset{(KVL)}{=} -V_1 - V_2 \overset{(OL)}{=} -V_1 - R_2 I_2.$$

Therefore, we have a system for I_2, V_1:

$$\begin{aligned} CV_1' &= I_2 - \frac{V_1}{R_1} \\ LI_2' &= -R_2 I_2 - V_1. \end{aligned}$$

We rewrite this in matrix form:

$$\begin{pmatrix} V_1' \\ I_2' \end{pmatrix} = \begin{bmatrix} -\frac{1}{CR_1} & \frac{1}{C} \\ \frac{-1}{L} & \frac{-R_2}{L} \end{bmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix}.$$

Suppose, for example, $R_1 = \frac{3}{5}, R_2 = 1, L = 1, C = \frac{1}{3}$ then

$$\begin{pmatrix} V_1' \\ I_2' \end{pmatrix} = \begin{bmatrix} -5 & 3 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix}.$$

1. First we find the eigenvalues. $\lambda_1 = -4, \lambda_2 = -2$.

2. Second we find the eigenvectors. By solving the system or using the formulas we obtain
$$\boldsymbol{\xi}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3. Therefore, the general solution is
$$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}.$$

4. Therefore the current and voltage will go to zero as $t \to +\infty$. This is reasonable because there is no voltage source and so eventually electricity will dissipate by passing through the light bulbs.

3.1.1 Complex eigenvalues

Consider the system
$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$
where now matrix $\mathbf{A}$ will have at least one pair of complex eigenvalues.

Example - Presenting the method

Consider the following parallel circuit with capacitance C (eg. battery), resistance R (eg. light bulb) and inductance L (eg. coil used for storing energy).

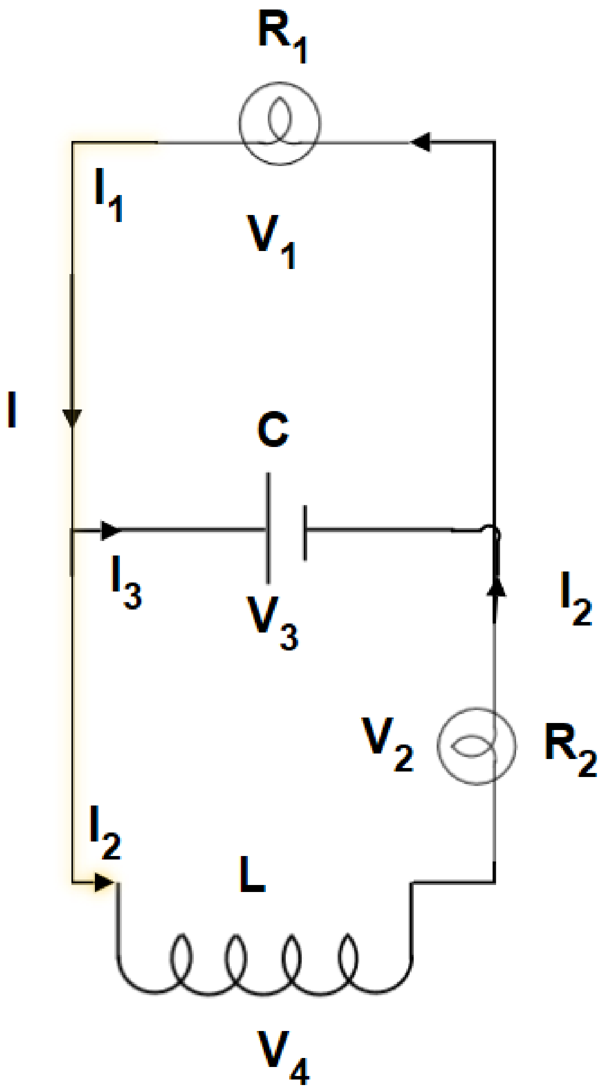


Figure 3.1.11: Parallel LRC circuit.

Let V be the voltage drop and I the current passing through the circuit. We also define the counterclockwise orientation as the positive one. Here is a quick summary of the laws governing such systems:

- **Ohm's law**(OL): $V = R \cdot I$.
- $C \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = I$.
- **Faraday's law** and **Lenz's law**(FLL): $L \cdot \frac{\mathrm{d}I}{\mathrm{d}t} = V$.
- **Kirchhoff's current law**(KCL): $I = I_2 + I_3$; this is the conservation of energy law for circuits.
- **Kirchhoff's Voltage law**(KVL): sum of voltages in a loop is zero. Thus, $V_1 = V_3$ and $V_3 + V_2 + V_4 = 0$.

We can express all these relations in a system of odes that will describe the circuit system depicted in Figure 3.1.11. We have

$$CV_1' \overset{(KVL)}{=} CV_3' = -I_3 = -I_1 + I_2 = -\frac{V_2}{R} + I_2 = -\frac{V_1}{R_1} + I_2$$

where we first applied Kirchhoff's voltage and current law and then Ohm's law. Note that we used $C(-V_3)' = I_3$ because V_3 is in the clockwise direction. We also have $LI_2' = V_4 = -V_2 - V_1 = -R_2 I_2 - V_1$. Therefore, we have a system for I_2, V_1:

$$CV' = I - \frac{V}{R_1}$$
$$LI' = -V + R_2 I.$$

We rewrite this in matrix form:

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -\frac{1}{CR_1} & -\frac{1}{C} \\ \frac{1}{L} & \frac{R_2}{L} \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

Let $R_1 = R_2 = 4, L = 8, C = \frac{1}{2}$ then

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -\frac{1}{2} & 2 \\ -\frac{1}{8} & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

1. First we find the eigenvalues. $\lambda_1 = \frac{-1+i}{2}, \lambda_2 = \frac{-1-i}{2}$.

2. Second we find the eigenvectors. By solving the system or using the formulas we obtain

$$\boldsymbol{\xi}_1 = \begin{pmatrix} -4i \\ 1 \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} 4i \\ 1 \end{pmatrix}.$$

3. Therefore, the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 \begin{pmatrix} -4i \\ 1 \end{pmatrix} e^{(\frac{-1+i}{2})t} + c_2 \begin{pmatrix} 4i \\ 1 \end{pmatrix} e^{(\frac{-1-i}{2})t}.$$

To obtain a real valued general solution it suffices to take real and imaginary parts of one of the basis elements:

$$\begin{pmatrix} \frac{-1+i}{2} \\ 1 \end{pmatrix} e^{(\frac{-1+i}{2})t} = e^{-t/2} \begin{pmatrix} -4ie^{it/2} \\ e^{it/2} \end{pmatrix})$$
$$= e^{-t/2} \begin{pmatrix} -4i(cos(t/2) + isin(t/2)) \\ (cos(t/2) + isin(t/2) \end{pmatrix})$$
$$= e^{-t/2} (\begin{pmatrix} 4sin(t/2) \\ cos(t/2) \end{pmatrix} + i \begin{pmatrix} -4cos(t/2) \\ sin(t/2) \end{pmatrix}).$$

So we take

$$u(t) = e^{-t/2} (\begin{pmatrix} 4sin(t/2) \\ cos(t/2) \end{pmatrix} \text{ and } v(t) = e^{-t/2} \begin{pmatrix} -4cos(t/2) \\ sin(t/2) \end{pmatrix}.$$

Indeed by computing their wronskian we get

$$W(u, v) = \begin{bmatrix} e^{-t/2} 4sin(t/2) & -e^{-t/2} 4cos(t/2) \\ e^{-t/2} cos(t/2) & e^{-t/2} sin(t/2) \end{bmatrix} = 4e^{-t/2}(cos^2(t/2) + sin^2(t/2)) = 4e^{-t/2} \neq 0.$$

4. Since there is no voltage source, as expected the solutions will converge to the origin.

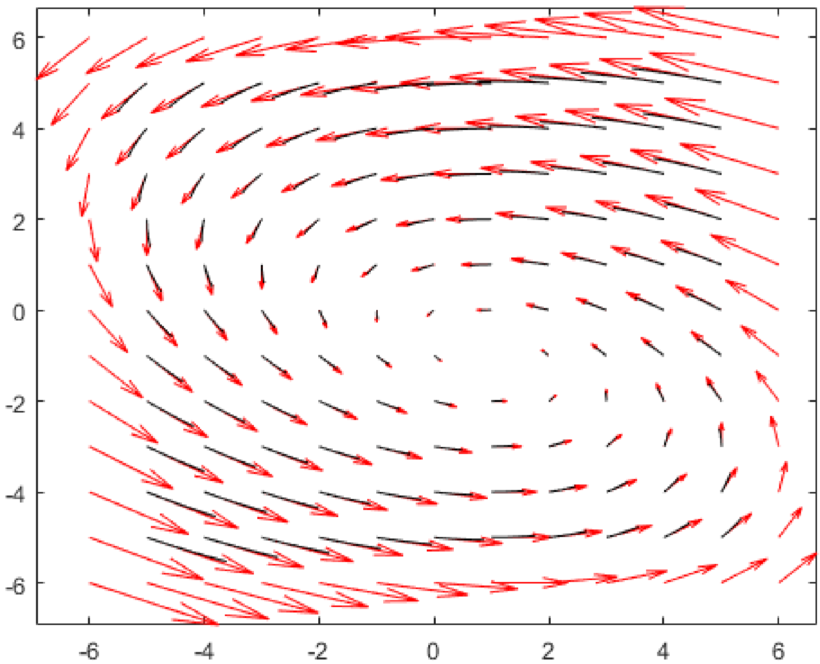


Figure 3.1.12: Phase portrait for (V, I).

5. To understand the periodicity we will describe the circuit's analogy with a mass spring.
   (a) When the spring is compressed we are storing energy in the form of atomic-bond energy or potential energy (i.e. the spring tries to regain its original position).
   (b) Then that energy is released into kinetic energy.
   (c) When the spring mass returns, it compress the springs and so the cycle begins again.

   In a circuit, a charged capacitor (battery) is analogous to a compressed spring and an inductor is analogous to the inertia mass.

   (a) The charged capacitor releases the electrical energy into the circuit which the inductor converts into magnetic field energy (analogous to kinetic energy).
   (b) When the capacitor is fully discharged, the magnetic field energy creates a counter current (by Faraday's law), which then charges the capacitor in the opposite direction.
   (c) The oppositely charged capacitor starts releasing a current in the opposite direction and so the cycle starts again.

Method formal steps

1. Let $\lambda_1 = -a + ib, \lambda_2 = -a - ib$ be the complex eigenvalues and $\xi_1 = \binom{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}, \xi_2 = \binom{\varrho_1 e^{i\varphi_1}}{\varrho_2 e^{i\varphi_2}}$ the corresponding eigenvectors. Then the solution is
$$\mathbf{x} = e^{-at}(c_1 \boldsymbol{\xi}_1 e^{ibt} + c_2 \boldsymbol{\xi}_2 e^{-ibt}).$$

2. To obtain a real-valued solution (not all) it suffices to pick one of the terms above, say $e^{-at}\boldsymbol{\xi}_1 e^{ibt}$. Then its real and imaginary parts will also be solutions:
$$e^{-at}\boldsymbol{\xi}_1 e^{ibt} = e^{-at}\begin{pmatrix} r_1 e^{i(\theta_1+b)} \\ r_2 e^{i(\theta_2+b)} \end{pmatrix} = e^{-at}\begin{pmatrix} r_1 cos(\theta_1+b) \\ r_2 cos(\theta_2+b) \end{pmatrix} + ie^{-at}\begin{pmatrix} r_1 sin(\theta_1+b) \\ r_2 sin(\theta_2+b) \end{pmatrix} =: u + iv.$$

3. Sometimes we can even obtain a general solution. By computing the wronskian we obtain:

$$W[u,v] = e^{-at} r_1 r_2 (cos(\theta_1 + b) sin(\theta_2 + b) - cos(\theta_2 + b) sin(\theta_1 + b)).$$

So as we see depending on the choice of θ_1, θ_2, the solutions u,v might be linearly independent or dependent.

4. Stability results. We note that the crucial role of stability is played by the factor e^{-at}.

   - If $a > 0$, then the solutions will converge to the node sink $(0,0)$ and the spiral will be inward.
   - If $a < 0$, then the solutions will diverge away from the node source $(0,0)$ and the spiral will be outward.
   - If $a = 0$, then the solutions will be concentric circles centered at $(0,0)$.

3.1.2 Examples

- Consider the system

$$\mathbf{x}' = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

1. We first compute its eigenvalues: $\lambda^2 - a\lambda + 1 = 0 \Rightarrow \lambda = \frac{a \pm \sqrt{a^2-4}}{2}$. So to explore the complex case we assume that $a^2 < 4$.
2. Let ξ_1, ξ_2 be the corresponding eigenvectors: $\xi_1 = \binom{\lambda_1}{-1}, \xi_2 = \binom{\lambda_2}{-1}$
   - Assume $\alpha = -1 < 0$, then $\lambda = \frac{-2 \pm i\sqrt{3}}{2}$ and so the general solution will be

$$\mathbf{x} = e^{-t}(c_1 \boldsymbol{\xi}_1 e^{i\frac{\sqrt{3}}{2}t} + c_2 \boldsymbol{\xi}_2 e^{-i\frac{\sqrt{3}}{2}t}).$$

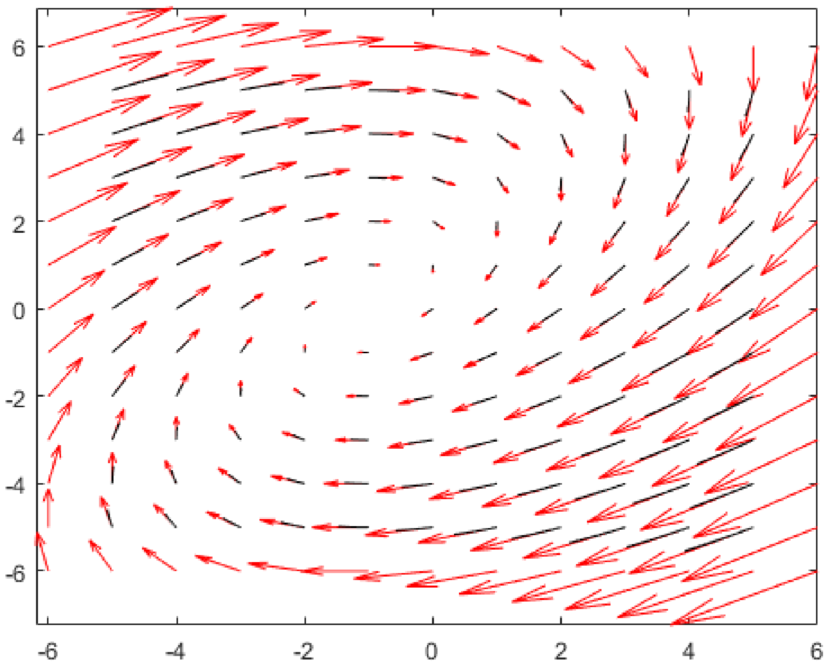


Figure 3.1.13: The solutions are converging towards the node sink (0,0).

– Assume $\alpha = 1 > 0$, then the general solution will be

$$\mathbf{x} = e^t(c_1\boldsymbol{\xi}_1 e^{i\frac{\sqrt{3}}{2}t} + c_2\boldsymbol{\xi}_2 e^{-i\frac{\sqrt{3}}{2}t}).$$

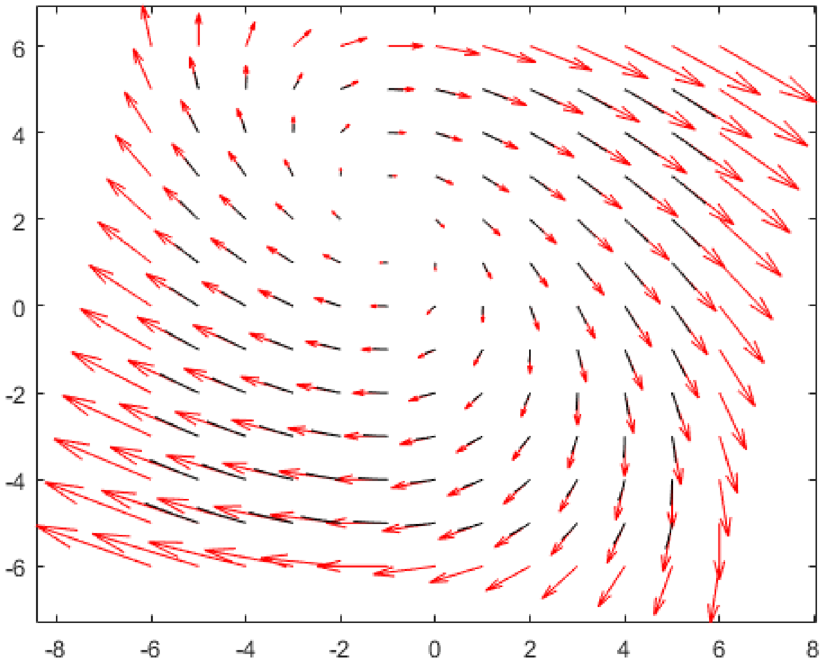


Figure 3.1.14: The solutions are spiraling outward from the node source $(0, 0)$.

– Assume $\alpha = 0$, then the general solution will be

$$\mathbf{x} = c_1\boldsymbol{\xi}_1 e^{i\frac{\sqrt{3}}{2}t} + c_2\boldsymbol{\xi}_2 e^{-i\frac{\sqrt{3}}{2}t}.$$

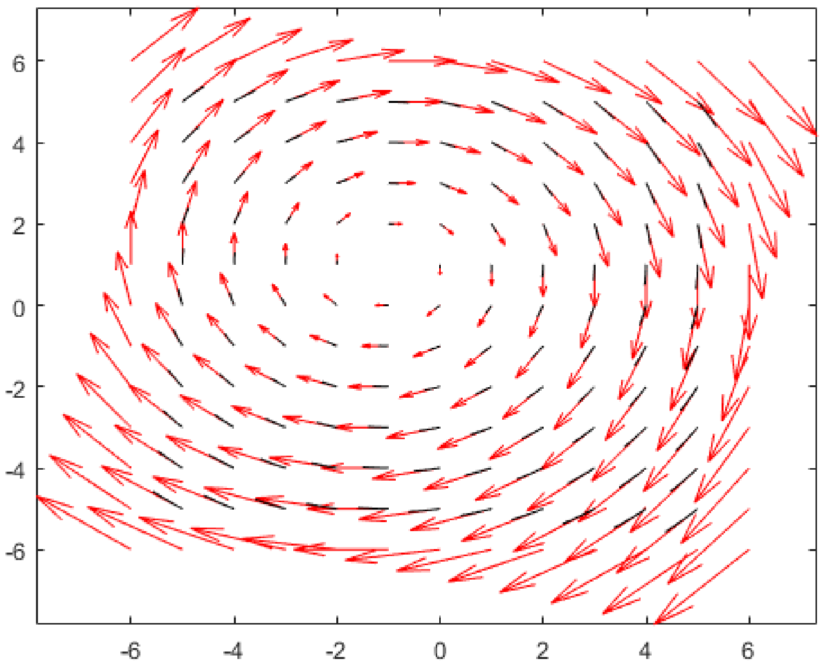


Figure 3.1.15: The solutions are concentric circles centered at $(0, 0)$.

3.1.3 Repeated eigenvalues

Consider the system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

where now matrix $\mathbf{A}$ will have at least two duplicated eigenvalues.

Example-presenting the method

Consider the following LRC circuit.

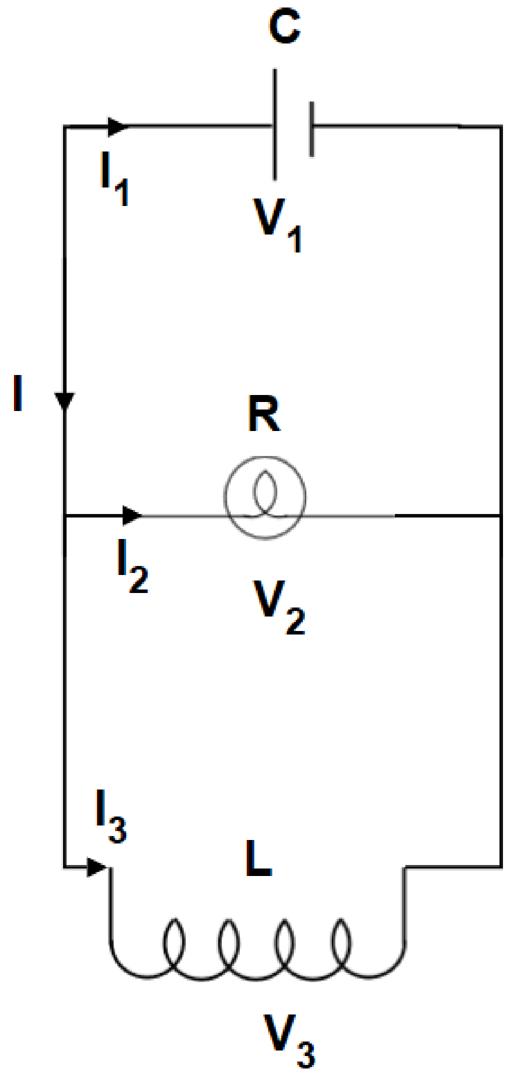


Figure 3.1.16: Parallel LRC circuit.

We have $CV_1' = -I = -I_1 - I_2 = -\frac{V_2}{R} - I_2$ and $LI_2' = V_3 = V_2 = V_1$. Therefore, the matrix system is:

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -\frac{1}{CR} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

1. The eigenvalues are $\lambda = -\frac{1}{2CR} \pm \frac{1}{2}\sqrt{(\frac{1}{CR})^2 - 4\frac{1}{CL}}$ and so if $(\frac{1}{CR})^2 - 4\frac{1}{CL} = 0 \Leftrightarrow L = 4R^2C$, we get a repeated eigenvalue $\lambda_1 = \lambda_2 = -\frac{1}{2CR}$. So assume $R = C = 1$ and $L = 4$, then the system is
$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -1 & -1 \\ \frac{1}{4} & 0 \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

2. Then the eigenvalue is $\lambda = \frac{-1}{2}$ and the eigenvector is $\boldsymbol{\xi} = \binom{-2}{1}$. So we obtain the first term of the solution $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.

3. Similarly to second order odes with repeated roots, we make the ansatz
$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}),$$
where $\boldsymbol{\eta}$ is a yet undetermined vector.

4. Plugging in $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$ we obtain:
$$\lambda \boldsymbol{\xi} e^{\lambda t} t + (\boldsymbol{\xi} + \lambda \boldsymbol{\eta}) e^{\lambda t} = \mathbf{A}(\boldsymbol{\xi} e^{\lambda t} t + \boldsymbol{\eta} e^{\lambda t}).$$

5. Equating the coefficients of $e^{\lambda t} t$ and $e^{\lambda t}$ we get
$$\begin{aligned} \lambda \boldsymbol{\xi} &= \mathbf{A}\boldsymbol{\xi} \\ \boldsymbol{\xi} + \lambda \boldsymbol{\eta} &= \mathbf{A}\boldsymbol{\eta}. \end{aligned}$$

6. The first equation is always true by virtue of $\boldsymbol{\xi}$ being an eigenvector. We will use the second system to determine $\boldsymbol{\eta}$. In other words, we must solve the system
$$(\mathbf{A} - \lambda I)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

7. In our case we have
$$\begin{bmatrix} -1 & -1 \\ \frac{1}{4} & -0 \end{bmatrix} + \frac{1}{2} I) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{2} & -1 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

8. By solving the system we obtain
$$\eta_2 = 2 - \frac{\eta_1}{2}.$$
Therefore, for any $\eta_1 = k$ we obtain
$$\boldsymbol{\eta} = \begin{pmatrix} k \\ 2 - \frac{k}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + k \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}.$$

9. Returning above the ansatz solution will be:
$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} \cdot t + \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} + k \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \right\} e^{-t/2} \right] \\ &= a_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} \cdot t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t/2} \right], \end{aligned}$$
where $a_1 := c_1 - k/2$.

10. Therefore, the (Voltage,Current) pair presents no periodicity and it simply goes to $(0, 0)$. However, for as $t \to +\infty$ the term $\binom{-2}{1} e^{-t/2} \cdot t$ will dominate and so the solutions will converge to the linear span of $\binom{-2}{1}$ and then along that line to the node sink $(0, 0)$.

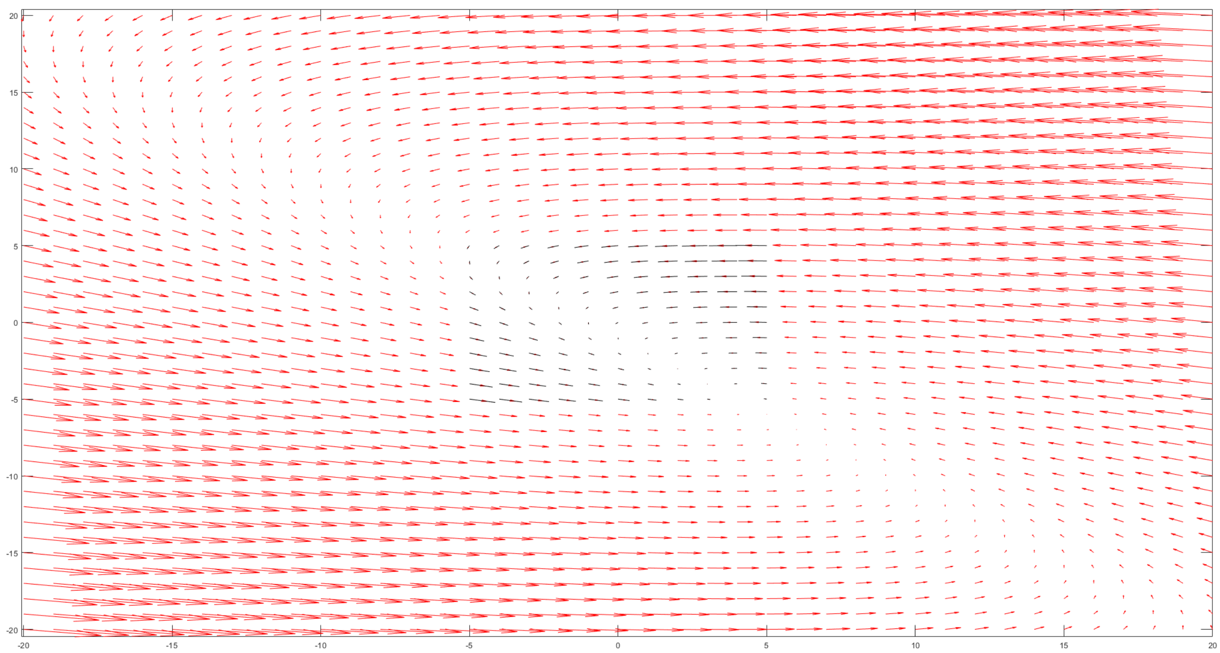


Figure 3.1.17: The linear span of $\binom{-2}{1}$ is a node and the origin will be the node sink.

Method formal steps

1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.

2. For the second term we make the ansatz

$$\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}.$$

3. Plugging this into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ we obtain the stystem:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

4. By determining $\boldsymbol{\eta}$ we obtain:

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}).$$

Examples

- Consider system

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

1. We first find the eigenvalues:

$$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})} = 2$$

and so we have a repeated eigenvalue.

2. Second, we find the corresponding eigenvector:

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

3. Assuming the solution is of the form $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ and plugging into our ODE we obtain:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi} \Longrightarrow \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   Solving this system gives us:

$$\eta_1 + \eta_2 = -1 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} k \\ -k-1 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

   where k is any real number. We can rewrite $\boldsymbol{\eta}$ as:

$$\boldsymbol{\eta} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

4. Therefore, the general solution is:

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \\ &= c_1 e^{2t}\boldsymbol{\xi} + c_2(\boldsymbol{\xi} e^{2t} \cdot t + \boldsymbol{\eta} e^{2t}) \\ &= c_1 e^{2t}\boldsymbol{\xi} + c_2\left[\boldsymbol{\xi} e^{2t} \cdot t + \left\{k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} e^{2t}\right] \\ &= e^{2t}\left[(c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}\right]. \end{aligned}$$

5. The vector $\boldsymbol{\xi}_1 = \binom{1}{-1}$ dominates the long term behaviour due to the extra term $\binom{1}{-1}t$ (provided we do not choose $c_2 = 0$). So we see that, essentially, all solutions are diverging away from the linear span of $\binom{1}{-1}$.

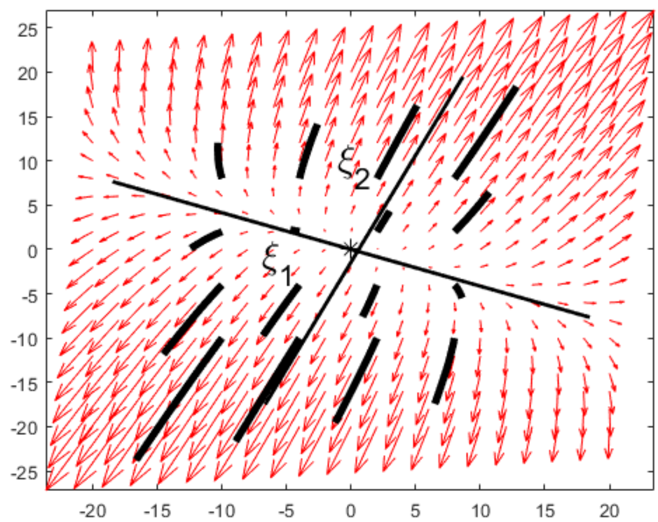


Figure 3.1.18: The phase portrait for $\mathbf{x} = (x_1, x_2)$ with $\boldsymbol{\xi}_1 = \binom{1}{-1}$, $\boldsymbol{\xi}_2 = \binom{0}{-1}$.

- Consider the system

$$\mathbf{x}' = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \mathbf{x}$$

with $\mathbf{x}(0) = \binom{1}{3}$.

1. First we find the eigenvalues:

$$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})} = \frac{3}{2}$$

and so we have a repeated eigenvalue.

2. Second, we find a corresponding eigenvector:

$$\begin{bmatrix} 2-\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1-\frac{3}{2} \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

3. Assuming the solution is of the form $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ and plugging into our ODE we obtain:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi} \Longrightarrow \begin{bmatrix} 2-\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1-\frac{3}{2} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solving this system gives us:

$$\eta_1 + \eta_2 = 2 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} k \\ -k+2 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

where k is any real number. We can rewrite $\boldsymbol{\eta}$ as:

$$\boldsymbol{\eta} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Therefore, the general solution is:

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \\ &= c_1 e^{\frac{3t}{2}}\boldsymbol{\xi} + c_2(\boldsymbol{\xi} e^{\frac{3t}{2}} \cdot t + \boldsymbol{\eta} e^{\frac{3t}{2}}) \\ &= c_1 e^{\frac{3t}{2}}\boldsymbol{\xi} + c_2\left[\boldsymbol{\xi} e^{\frac{3t}{2}} \cdot t + \left\{k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\} e^{\frac{3t}{2}}\right] \\ &= e^{\frac{3t}{2}}\left[(c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}\right]. \end{aligned}$$

If we now use the initial condition we obtain the system of equations

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = (c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

which can be solved to obtain that $c_1 = 1 - 2k$ and $c_2 = 2$.

4. The vector $\boldsymbol{\xi}_1 = \binom{1}{-1}$ dominates the long term behaviour due to the extra term $\binom{1}{-1}t$. So we see that all solutions to the IVP are diverging away from the linear span of $\binom{1}{-1}$.

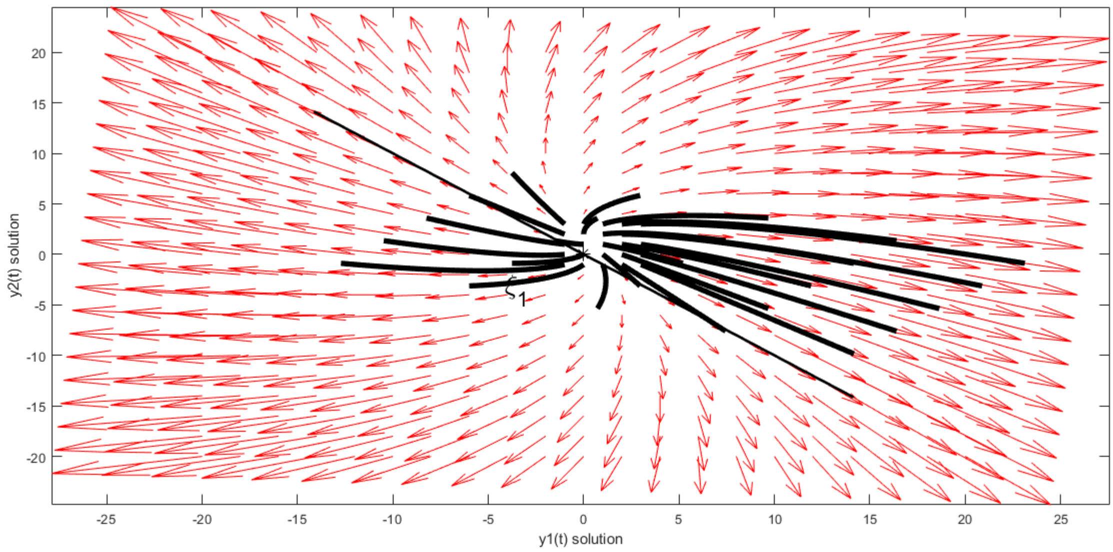
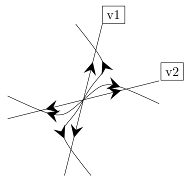
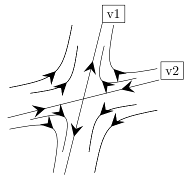
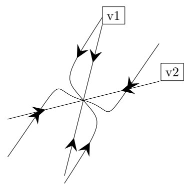
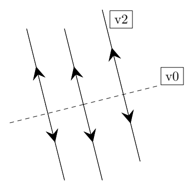
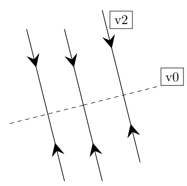
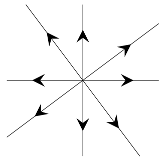
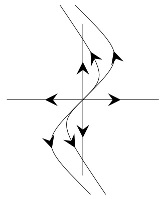
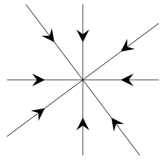
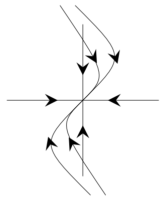
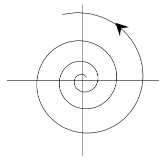
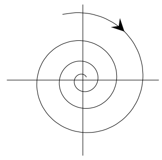
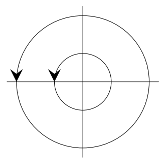


Figure 3.1.19: The phase portrait for $\mathbf{x} = (x_1, x_2)$ with $\boldsymbol{\xi}_1 = \binom{1}{-1}$, $\boldsymbol{\xi}_2 = \binom{0}{2}$.

3.1.4 Stability

Summary of the stability results.

Eigenvalues	**Type of critical point**	**Stability**	**Sample phase portrait**
$\lambda_1 > \lambda_2 > 0$	Nodal source	Unstable	
$\lambda_1 > 0 > \lambda_2$	Saddle point	Unstable	
$0 > \lambda_1 > \lambda_2$	Nodal sink	Asymptotically stable	
$\lambda_1 = 0, \lambda_2 > 0$	v_0 line	Unstable and source	
$\lambda_1 = 0, \lambda_2 < 0$	v_0 line	Unstable and sink	

$\lambda = \lambda_1 = \lambda_2 > 0$ and $A = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$	Proper node	Unstable	
$\lambda_1 = \lambda_2 > 0$ and A is not diagonal	Improper node	Asymptotically unstable	
$\lambda = \lambda_1 = \lambda_2 < 0$ and $A = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$	Proper node	Stable	
$\lambda_1 = \lambda_2 < 0$ and A is not diagonal	Improper node	Asymptotically stable	
$\lambda_1 = a + ib, \lambda_2 = a - ib$	Spiral		
a>0		Unstable	
0>a		Stable	
$\lambda_1 = ib, \lambda_2 = -ib$	Center	Stable	

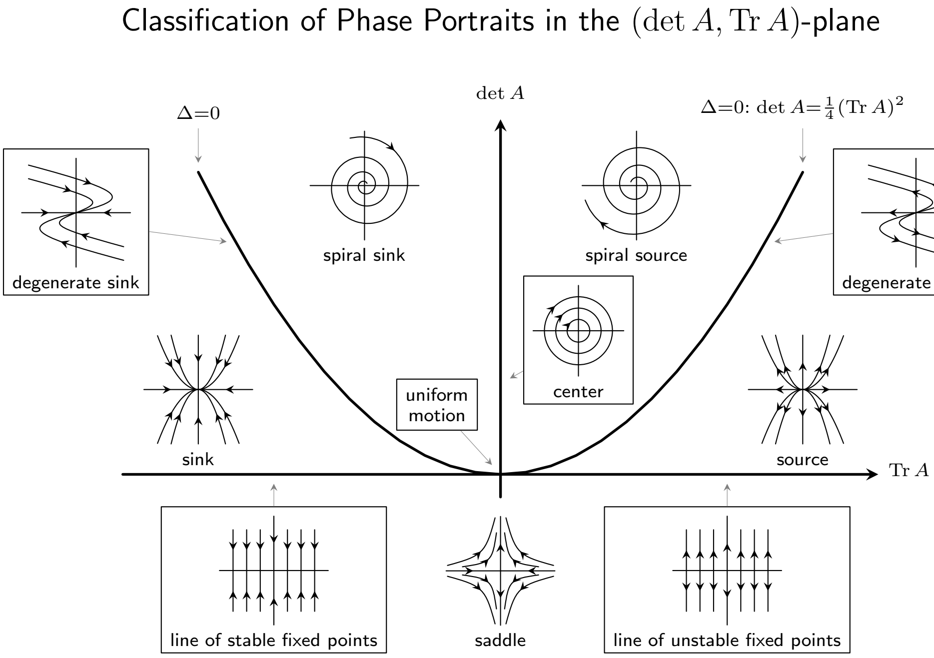


Figure 3.1.20: Classification of phase portraits.

Stability of Eigenvalue Dependence

In this section we study the limiting behaviour of solutions as distinct eigenvalues of $\mathbf{A}$ become repeated. Specifically, we demonstrate that the solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has distinct eigenvalues converges pointwise, after being suitably prepared, to the solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has repeated eigenvalues. We focus our analysis on the 2×2 case. Consider the problem

$$\mathbf{x}'(t) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mathbf{x}(t) \tag{3.1.3}$$

where $\lambda \in \mathbb{R}$ as well as the perturbed problem

$$\mathbf{x}'(t) = \begin{pmatrix} \lambda+\epsilon & 1 \\ 0 & \lambda-\epsilon \end{pmatrix} \mathbf{x}(t) \tag{3.1.4}$$

where $\epsilon > 0$. Notice that the perturbed problem has a matrix with distinct eigenvalues $\lambda + \epsilon$ and $\lambda - \epsilon$. One might hope that if we take a sequence of solutions to the perturbed problems as $\epsilon \to 0^+$ then, in the limit, we obtain a solution to the limiting problem (3.1.3). As we will see, this only works if we choose the sequence of solutions appropriately. To begin, we note that the solution $\mathbf{x}_\epsilon(t)$ to the perturbed problem (3.1.4) is

$$\mathbf{x}_\epsilon(t) = c_1(\epsilon)e^{(\lambda+\epsilon)t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(\epsilon)e^{(\lambda-\epsilon)t}\begin{pmatrix} 1 \\ -2\epsilon \end{pmatrix}$$

for each $\epsilon > 0$, where $c_1(\epsilon)$ and $c_2(\epsilon)$ are constants that may depend on ϵ (note that we are considering a sequence of solutions with no initial conditions and so we are free to choose these constants as we please). We now wish to show that as $\epsilon \to 0^+$ the above family of solutions tends to the solution of (3.1.3). Observe that we can write this solution as

$$\mathbf{x}_\epsilon(t) = \begin{pmatrix} c_1(\epsilon)e^{(\lambda+\epsilon)t} + c_2(\epsilon)e^{(\lambda-\epsilon)t} \\ -2\epsilon c_2(\epsilon)e^{(\lambda-\epsilon)t} \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_1(\epsilon)e^{\epsilon t} + c_2(\epsilon)e^{-\epsilon t} \\ -2\epsilon c_2(\epsilon)e^{-\epsilon t} \end{pmatrix}.$$

To ensure that the second component converges to an interesting value, $k \in \mathbb{R}$, of our choosing we see that we must require that $c_2(\epsilon) = \frac{-k}{2\epsilon}$. Updating the family of solutions with this choice of $c_2(\epsilon)$ we obtain

$$\mathbf{x}_\epsilon(t) = e^{\lambda t}\begin{pmatrix} c_1(\epsilon)e^{\epsilon t} - \frac{k}{2\epsilon}e^{-\epsilon t} \\ ke^{-\epsilon t} \end{pmatrix}.$$

Observe that the first component now has a term that diverges as $\epsilon \to 0^+$. Thus, we must choose $c_1(\epsilon)$ in a way that combats this divergent term. In accordance with the above logic we choose $c_1(\epsilon) = \frac{k}{2\epsilon} + c_3(\epsilon)$ where we will decide later how to choose $c_3(\epsilon)$. Updating our family of solutions we obtain

$$e^{\lambda t}\begin{pmatrix} \left(\frac{k}{2\epsilon} + c_3(\epsilon)\right)e^{\epsilon t} - \frac{k}{2\epsilon}e^{-\epsilon t} \\ ke^{-\epsilon t} \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_3(\epsilon)e^{\epsilon t} + k\left(\frac{e^{\epsilon t} - e^{-\epsilon t}}{2\epsilon}\right) \\ ke^{-\epsilon t} \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_3(\epsilon)e^{\epsilon t} + kt\left(\frac{e^{\epsilon t} - e^{-\epsilon t}}{2\epsilon t}\right) \\ ke^{-\epsilon t} \end{pmatrix}.$$

We now choose $c_3(\epsilon) = a$ for some constant $a \in \mathbb{R}$ so that the first term in the first component converges. Observe also that, by L'Hôpital's rule we have

$$\lim_{\epsilon \to 0^+} \frac{e^{\epsilon t} - e^{-\epsilon t}}{2\epsilon t} = \lim_{\epsilon \to 0^+} \frac{2\epsilon}{2\epsilon} = 1.$$

Thus, we conclude that by letting ϵ tend to 0 from the right we obtain, for each $t \in \mathbb{R}$

$$\lim_{\epsilon \to 0^+} \mathbf{x}_\epsilon(t) = e^{\lambda t}\begin{pmatrix} a + kt \\ k \end{pmatrix} = ae^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ke^t\begin{pmatrix} t \\ 1 \end{pmatrix}$$

which is a solution to (3.1.3). Notice that a and k were arbitrary choices and so we can obtain any such solution. Notice, however, that we had to choose very specific behaviour for the coefficients to get this convergence.

3.2 Nonhomogeneous linear systems

In this section we study nonhomogeneous first order systems of equations building off of the previous work on homogeneous first order systems.

3.2.1 Diagonalization Method

Consider nonhomogeneous linear first order systems:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a vector of continuous functions and $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with eigenvalues $\{\lambda_i\}_{i=1,\dots,n}$. The latter assumption means that if $\mathbf{T}$ has the eigenvectors of $\mathbf{A}$ as columns, then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$ is a diagonal matrix.

Using diagonalization Plugging in $\mathbf{x} = \mathbf{T}\mathbf{y}$ for some yet unknown $\mathbf{y}$ we obtain

$$\begin{aligned} \mathbf{T}\mathbf{y}' &= \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t) \\ \implies \mathbf{y}' &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t). \end{aligned}$$

As a result, we decoupled the system. From this decoupled system we obtain the first order equations:

$$y_i' = \lambda_i y_i(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_i \quad \text{for } i = 1, ..., n.$$

For $h_i(t) := (\mathbf{T}^{-1}\mathbf{g}(t))_i$ we have (by the method of integrating factors)

$$y_i(t) = e^{\lambda_i t}\left[\int_0^t e^{-\lambda_i s} h_i(s) ds + c_i\right].$$

Therefore, we found the solution $\mathbf{x} = \mathbf{T}\mathbf{y}$.

Using Jordan form

Method formal steps

1. As usual we first find the eigenvalues λ_1, λ_2 and corresponding eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

2. Form the change of basis matrix $\mathbf{T} := [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$ and find the solution to the following two first order odes

$$\begin{aligned} y_1' &= \lambda_1 y_1(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_1 \\ y_2' &= \lambda_2 y_2(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_2. \end{aligned}$$

3. By integrating factor the solutions are

$$\begin{aligned} y_1(t) &= e^{\lambda_1 t}\left[\int_0^t e^{-\lambda_1 s}(\mathbf{T}^{-1}\mathbf{g}(s))_1 ds + c_1\right] \\ y_2(t) &= e^{\lambda_1 t}\left[\int_0^t e^{-\lambda_2 s}(\mathbf{T}^{-1}\mathbf{g}(s))_2 ds + c_2\right]. \end{aligned}$$

4. We obtain the original solution by undoing the change of basis:

$$\mathbf{x} = \mathbf{T}\mathbf{y}.$$

Example-Presenting the method Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

1. First we find the eigenvalues

$$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\big(\mathrm{Tr}(\mathbf{A})\big)^2 - 4\det(\mathbf{A})} \Longrightarrow \lambda_1 = -1,\ \lambda_2 = 1.$$

2. The corresponding eigenvectors are, respectively, $v_1 = (1,3)^{\mathrm{T}}, v_2 = (1,1)^{\mathrm{T}}$ and so the change of basis matrix $\mathbf{T}$ that diagonalizes our matrix is:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}.$$

3. Therefore, as argued above, the solution will be

$$\mathbf{x} = \mathbf{T}\mathbf{y},$$

where

$$y_1(t) = e^{-t}\int_0^t e^s h_1(s)\mathrm{d}s + c_1 e^{-t} \text{ and } y_2(t) = e^t \int_0^t e^{-s} h_2(s)\mathrm{d}s + c_2 e^t$$

with

$$h_1(t) := \frac{1}{2}(t - e^t) \text{ and } h_2(t) := \frac{1}{2}(3e^t - t).$$

First we find the y_i (note that $\sinh(t)$ is $\frac{e^t - e^{-t}}{2}$):

$$y_1 = c_1 e^{-t} + \frac{1}{2}e^{-t}\big[e^t\{t - \sinh(t) - 1\} + 1\big] \text{ and } y_2 = c_2 e^t + \frac{1}{2}e^t\big[3t + e^{-t}(t+1) - 1\big].$$

4. Therefore, we obtain, since $\mathbf{x} = \mathbf{T}\mathbf{y}$

$$\begin{cases} x_1 = c_1 e^{-t} + c_2 e^t + \frac{1}{2}e^{-t}[1 + e^t\{-1 + t - \sinh(t)\}] + \frac{1}{2}e^t[-1 + 3t + e^{-t}(1+t)] \\ x_2 = 3c_1 e^{-t} + c_2 e^t + \frac{3}{2}e^{-t}[1 + e^t\{-1 + t - \sinh(t)\}] + \frac{1}{2}e^t[-1 + 3t + e^{-t}(1+t)] \end{cases}$$

$$\Longrightarrow \mathbf{x} = \left[c_1 e^{-t} + \frac{1}{2}e^{-t}\{1 + e^t(-1 + t - \sinh(t))\}\right]\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \left[c_2 e^t + \frac{1}{2}e^t\{-1 + 3t + e^{-t}(1+t)\}\right]\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Examples

- Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}.$$

1. First we find the eigenvalues
$$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\left(\mathrm{Tr}(\mathbf{A})\right)^2 - 4\det(\mathbf{A})} \implies \lambda_1 = -2 + i\sqrt{5},\ \lambda_2 = -2 - i\sqrt{5}.$$

2. The corresponding eigenvectors are, respectively, $v_1 = (i\sqrt{5}, 1)^{\mathrm{T}}, v_2 = (-i\sqrt{5}, 1)^{\mathrm{T}}$ and so the change of basis matrix $\mathbf{T}$ that diagonalizes our matrix is:
$$\mathbf{T} = \begin{pmatrix} i\sqrt{5} & -i\sqrt{5} \\ 1 & 1 \end{pmatrix}.$$

3. Therefore, as argued above, the solution will be
$$\mathbf{x} = \mathbf{T}\mathbf{y},$$
where
$$y_1(t) = e^{(-2+i\sqrt{5})t}\int_0^t e^{-(-2+i\sqrt{5})s}h_1(s)\mathrm{d}s + c_1 e^{(-2+i\sqrt{5})t} \text{ and } y_2(t) = e^{(-2-i\sqrt{5})t}\int_0^t e^{-(-2-i\sqrt{5})s}h_2(s)\mathrm{d}s$$
with
$$h_1(t) := -2i\cos(t) \text{ and } h_2(t) := \cos(t).$$

4. First we find the y_i. We will do y_1 and y_2 is similar.
$$\begin{aligned}\int_0^t e^{-(-2+i\sqrt{5})s}h_1(s)\mathrm{d}s + c_1 &= -2i\int_0^t e^{-(-2+i\sqrt{5})s}\cos(s)\mathrm{d}s + c_1 \\ &= \frac{-2i}{1+(2-i\sqrt{5})^2}\left\{(e^{-t(-2+i\sqrt{5})}(\sin(t) + (2-i\sqrt{5})\cos(t)) - (2-i\sqrt{5})\right\} + c_1,\end{aligned}$$
where we used the formula
$$\int e^{(a+ib)t}\cos(s)ds = \frac{1}{1+(a+ib)^2}(e^{t(a+ib)}(\sin(t) + (a+ib)\cos(t)) - (a+ib)).$$
we simplify by setting $c_1 := \frac{-2i(2-i\sqrt{5})}{1+(2-i\sqrt{5})^2}$ to get
$$\begin{aligned} y_1 &= e^{t(-2+i\sqrt{5})}\frac{-2i}{1+(2-i\sqrt{5})^2}\left\{e^{-t(-2+i\sqrt{5})}(\sin(t) + (2-i\sqrt{5})\cos(t))\right\} \\ &= \frac{-2i}{1+(2-i\sqrt{5})^2}\left\{\sin(t) + (2-i\sqrt{5})\cos(t)\right\} \\ &= \frac{-2i}{-4i\sqrt{5}}\left\{2\cos(t) + \sin(t) - i\sqrt{5}\cos(t)\right\} \\ &= \frac{1}{2\sqrt{5}}\left\{2\cos(t) + \sin(t) - i\sqrt{5}\cos(t)\right\} \\ &=: u(t) + iv(t)\end{aligned}$$
For y_2 we have
$$\begin{aligned} y_2(t) &= \frac{1}{1+(2+i\sqrt{5})^2}\left\{\sin(t) + (2+i\sqrt{5})\cos(t)\right\} \\ &= \frac{1}{4i\sqrt{5}}\left\{2\cos(t) + \sin(t) + i\sqrt{5}\cos(t)\right\}\end{aligned}$$
using that $i^{-1} = -i$ we obtain
$$= \frac{1}{4\sqrt{5}}\sqrt{5}\cos(t) - \frac{i}{4\sqrt{5}}(2\cos(t) + \sin(t))$$

$$= \frac{1}{4}\cos(t) - \frac{i}{4\sqrt{5}}(2\cos(t) + \sin(t))$$
$$=: \widetilde{u}(t) + i\widetilde{v}(t).$$

5. Undoing the change of basis we obtain:

$$\begin{aligned}
\mathbf{x}_{nh} &= \mathbf{T}\mathbf{y} \\
&= \begin{pmatrix} i\sqrt{5} & -i\sqrt{5} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u + iv \\ \widetilde{u} + i\widetilde{v} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{5}(\widetilde{v} - v + i(u - \widetilde{u})) \\ \widetilde{v} - v + i(u - \widetilde{u}) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{5}(\widetilde{v} - v) \\ \widetilde{v} - v \end{pmatrix} + i \begin{pmatrix} \sqrt{5}(u - \widetilde{u}) \\ u - \widetilde{u} \end{pmatrix}
\end{aligned}$$

we have that

$$\begin{aligned}
\widetilde{v} - v &= \frac{-1}{4\sqrt{5}}(2\cos(t) + \sin(t)) + \cos(t)\frac{1}{2} \\
&= \cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{u} - u &= \frac{1}{4}\cos(t) - \frac{1}{2\sqrt{5}}(2\cos(t) + \sin(t)) \\
&= \cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{2\sqrt{5}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&= \begin{pmatrix} \sqrt{5}(\cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}}) \\ \cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}} \end{pmatrix} \\
&\qquad + i \begin{pmatrix} \sqrt{5}(\cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{2\sqrt{5}}) \\ \cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{2\sqrt{5}} \end{pmatrix}
\end{aligned}$$

6. Since we are only looking for a particular solution we only take the real part:

$$\mathbf{x}_{nh} = \begin{pmatrix} \sqrt{5}(\cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}}) \\ \cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}} \end{pmatrix}$$

7. Therefore, the general solution is:

$$\begin{aligned}
\mathbf{x} = &\left[c_1 e^{(-2+i\sqrt{5})t} \begin{pmatrix} i\sqrt{5} \\ 1 \end{pmatrix} + c_2 e^{(-2-i\sqrt{5})t} \begin{pmatrix} -i\sqrt{5} \\ 1 \end{pmatrix} \right] \\
&\qquad + \begin{pmatrix} \sqrt{5}(\cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}}) \\ \cos(t)\frac{1}{2}(1 - \frac{1}{\sqrt{5}}) + \sin(t)\frac{-1}{4\sqrt{5}} \end{pmatrix}
\end{aligned}$$

- Consider the 2nd order equation

$$\theta'' - 2\theta' - 3\theta = e^{-t} + 3$$

then by setting $x = \theta, y = \theta'$ we obtain the system:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ e^{-t} + 3 \end{pmatrix}.$$

1. First we solve the second order equation. The homogeneous part is
$$x_h = c_1 e^{3t} + c_2 e^{-t}.$$
2. Then we apply our guess: $x_{nh} = at^s e^{-t}$. Since -1 is a simple root, we have $s = 1$ and so it remains to find a:
$$a(-2e^{-t} + te^{-t} - 2e^{-t} + 2te^{-t} - 3te^{-t}) = e^{-t} \Rightarrow a = \frac{-1}{4}.$$
3. So the solution is:
$$x_{gen} = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{4} t e^{-t} + 1.$$
4. Next we solve the corresponding nonhomogeneous system. The eigenpairs are $(3, \binom{1}{3}), (-1, \binom{-1}{1})$ and so the homogeneous part is:
$$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$
5. So the change of basis is matrix $\mathbf{T} = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$ and we obtain:
$$\mathbf{h}(t) = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}.$$
6. Thus, we obtain
$$\begin{aligned} y_1(t) =& e^{3t} \int e^{-3s} (\frac{e^{-s}}{4}) ds \\ =& \frac{e^{-t}}{-16} \end{aligned}$$
and
$$\begin{aligned} y_2(t) =& e^{-t} \int e^{s} (\frac{e^{-s}}{4}) ds \\ =& \frac{e^{-t} t}{4}. \end{aligned}$$
7. Therefore, the nonhomogeneous part is
$$\mathbf{x}_{nh,1} = \mathbf{T}\mathbf{y} = \frac{e^{-t}}{4} \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4} \\ t \end{pmatrix} = \frac{e^{-t}}{4} \begin{pmatrix} -\frac{1}{4} \\ \frac{-3}{4} \end{pmatrix} + \frac{e^{-t} t}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
8. For the system $\mathbf{x}' = Ax + \binom{0}{3}$, we have
$$\mathbf{x}_{nh,2} = A^{-1} \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

9. The general solution will be:

$$\mathbf{x}_{gen} = c_1 e^{3t}\begin{pmatrix}1\\3\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + \frac{-e^{-t}}{16}\begin{pmatrix}1\\3\end{pmatrix} + \frac{e^{-t}t}{4}\begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}.$$

10. Indeed the $x = \theta$ component is of the form:

$$\theta = x = c_1 e^{3t} + c_2 e^{-t} - \frac{e^{-t}t}{4} + 1,$$

which agrees with our second order solution.

- Consider the equation

$$\theta'' + 2\theta' + 2\theta = 3 + e^{-t} sin(t)$$

and its related system

$$\mathbf{x}'(t) = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\mathbf{x} + \begin{pmatrix}0\\ 3 + e^{-t}sin(t)\end{pmatrix}.$$

1. First we solve the second order equation. The homogeneous part is

$$x_h = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}.$$

2. Then we apply our guess: $x_{nh} = t^s e^{-t}(a\cos(t) + b\sin(t))$. Since -1+i is a simple root, we have $s = 1$ and so it remains to find a,b:

$$\begin{aligned}&2(e^{-t} - e^{-t}t)(b\cos(t) - a\sin(t))\\ &+ e^{-t}t(-a\cos(t) - b\sin(t)) + 2e^{-t}t(a\cos(t) + b\sin(t))\\ &+ (e^{-t}t - 2e^{-t})(a\cos(t) + b\sin(t))\\ &+ 2(e^{-t}t(b\cos(t) - a\sin(t))\\ &+ e^{-t}(a\cos(t) + b\sin(t)) - e^{-t}t(a\cos(t) + b\sin(t)))\\ &= e^{-t}sin(t).\end{aligned}$$

Isolating the term $e^{-t}sin(t)$ we obtain

$$-2a - b = 1 \Rightarrow a = -\frac{1}{2} \text{ and } b = 0$$

and so the nonhomogeneous part is

$$x_{nh} = te^{-t}(-\frac{1}{2}\cos(t)).$$

3. So the solution is:

$$x_{gen} = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} + te^{-t}(-\frac{1}{2}\cos(t)) + \frac{3}{2}.$$

4. Next we solve the corresponding nonhomogeneous system. The eigenpairs are $(-1 + i, \binom{-1-i}{2}), (-1+i, \binom{1-i}{-2})$ and so the homogeneous part is:

$$\mathbf{x} = c_1 e^{(-1+i)t}\begin{pmatrix}-1-i\\2\end{pmatrix} + c_2 e^{(-1-i)t}\begin{pmatrix}-1+i\\2\end{pmatrix}.$$

5. So the change of basis is matrix $\mathbf{T} = \begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix}$ and we obtain:

$$\mathbf{h}(t) = \begin{pmatrix} -i-1 & i-1 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^{-t}sin(t) \end{pmatrix} = \frac{e^{-t}sin(t)}{4} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.$$

6. Thus, we obtain

$$\begin{aligned} y_1(t) =& e^{(-1+i)t} \int e^{-(-1+i)s} (\frac{e^{-s}\sin(s)(1+i)}{4})ds \\ =& \frac{e^{(-1+i)t}(1+i)}{4}(\frac{-it}{2} - \frac{e^{-2it}}{4}) \\ =& \frac{e^{(-1+i)t}(-1-i)}{8}(it + \frac{e^{-2it}}{2}) \end{aligned}$$

and

$$\begin{aligned} y_2(t) =& e^{(-1-i)t} \int e^{-(-1-i)s} (\frac{e^{-s}\sin(s)(1-i)}{4})ds \\ =& \frac{e^{(-1-i)t}(1-i)}{8}(it - \frac{e^{2it}}{2}). \end{aligned}$$

7. Therefore, the nonhomogeneous part is

$$\mathbf{x}_{nh,1} = \mathbf{Ty} = \begin{pmatrix} \frac{ie^{(-1-i)t}}{4}(it - \frac{e^{2it}}{2}) + \frac{e^{(-1+i)t}}{4}(it + \frac{e^{-2it}}{2}) \\ (1+i)\frac{e^{(-1+i)t}}{4}(-it - \frac{e^{-2it}}{2}) + (1-i)\frac{e^{(-1-i)t}}{4}(it - \frac{e^{2it}}{2}) \end{pmatrix}$$

using the Euler formula we obtain

$$\begin{aligned} &= \begin{pmatrix} \frac{-e^{-t}}{4}t2\cos(t) + i\frac{ie^{-t}}{4}\sin(t) \\ \frac{e^{-t}}{4}(2t-1)(\sin(t)+\cos(t)) \end{pmatrix} \\ &= \frac{e^{-t}}{4}t2\cos(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{e^{-t}}{4}\sin(t)\begin{pmatrix} i \\ -1 \end{pmatrix} + \frac{e^{-t}}{4}(2t\sin(t) - \cos(t))\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

8. For the system $\mathbf{x}' = Ax + \binom{0}{3}$, we have

$$\mathbf{x}_{nh,2} = A^{-1}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$

9. The general solution will be:

$$\begin{aligned} \mathbf{x}_{gen} =& c_1 e^{(-1+i)t}\begin{pmatrix} i+1 \\ -2 \end{pmatrix} + c_2 e^{(-1-i)t}\begin{pmatrix} i-1 \\ -2 \end{pmatrix} \\ &+ \frac{e^{-t}}{4}t2\cos(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &+ \frac{e^{-t}}{4}\sin(t)\begin{pmatrix} i \\ -1 \end{pmatrix} + \frac{e^{-t}}{4}(2t\sin(t) - \cos(t))\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}. \end{aligned}$$

10. Indeed the $x = \theta$ component is of the form:

$$\theta = x = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} + te^{-t}(-\frac{1}{2}\cos(t)) + \frac{3}{2},$$

which agrees with our second order solution.

3.2.2 Using method of undetermined coefficients

If the components of $\mathbf{g}(t)$ are linear combinations of polynomial, exponential, or sinusoidal functions, then as before we assume that the solution $\mathbf{x}$ is a linear combination of the same type of functions.

Example-Presenting the method

Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1. Given $\mathbf{g}(t) = \binom{e^t}{t}$ we assume that the solution is of the form:

$$\mathbf{x}(t) = \mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}.$$

for some vectors **a, b, c,** and **d** to be found.

2. Plugging this into our system we obtain

$$\mathbf{a}(te^t + e^t) + \mathbf{b}e^t + \mathbf{c} = \mathbf{x}' = \mathbf{A}\big(\mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}\big) + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, we obtain algebraic equations for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$:

$$\begin{aligned} \mathbf{a} &= \mathbf{A}\mathbf{a}, \\ \mathbf{a} + \mathbf{b} &= \mathbf{A}\mathbf{b} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ 0 &= \mathbf{A}\mathbf{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \mathbf{c} &= \mathbf{A}\mathbf{d}. \end{aligned}$$

The first equation implies that **a** is an eigenvector for **A** with associated double eigenvalue $\lambda_1 = 1$ and so we can assume that

$$\mathrm{a} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

because it will solve the second equation:

$$\mathrm{b} = \begin{pmatrix} k \\ k - \frac{1}{2} \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By solving the remaining systems we obtain

$$\mathrm{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore, the solution is

$$\mathrm{x} = c_1 e^{-t}\begin{pmatrix}1\\3\end{pmatrix} + \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}te^t + \left\{c_2\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\right\}e^t + \begin{pmatrix}1\\2\end{pmatrix}t - \begin{pmatrix}0\\1\end{pmatrix}.$$

3.2.3 Integrating Factor

We now present an alternative method for solving nonhomogeneous first order systems. Specifically, we wish to solve the first order system described by

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)$$

where $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ is a column vector, $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ is continuous. Observe that, by moving the term $\mathbf{A}\mathbf{x}(t)$ to the other side we obtain

$$\mathbf{x}'(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{g}(t). \tag{3.2.1}$$

One might be reminded of the one-dimensional nonhomogeneous differential equation given by

$$x'(t) - ax(t) = g(t).$$

You may recall that to solve such an ODE we multiplied through by an integrating factor, $\mu(t)$, chosen carefully so that we may view the left hand side as the derivative of a product of functions. Specifically, we choose μ to satisfy $\mu'(t) = -a\mu(t)$. Multiplying through by μ and applying the above strategy leads to the solution, as one can check,

$$x(t) = e^{at}\left\{C + \int_0^t e^{-as}g(s)\mathrm{d}s\right\}.$$

Motivated by the philosophy that ODEs that look similar are probably solved by similar techniques we attempt to use a similar strategy for the first order system. Note that since we are working with matrices we have to be careful with what we multiply through since not all matrices can be multiplied together (recall that matrix multiplication only makes sense if the row and column sizes are appropriate). After some consideration we may anticipate that the function we desire is a function of the form $\boldsymbol{\mu} : \mathbb{R} \to M_{n\times n}(\mathbb{R})$. That is, a function such at each time t we obtain an $n \times n$ matrix $\boldsymbol{\mu}(t)$. Multiplying our equation on the left by $\boldsymbol{\mu}(t)$ gives

$$\boldsymbol{\mu}(t)\mathbf{x}'(t) - \boldsymbol{\mu}(t)\mathbf{A}\mathbf{x}(t) = \boldsymbol{\mu}(t)\mathbf{g}(t).$$

If we now choose $\boldsymbol{\mu}$ such that $\boldsymbol{\mu}'(t) = -\boldsymbol{\mu}(t)\mathbf{A}$, where this derivative is understood componentwise as in the case of $\mathbf{x}'$, then we could rewrite the equation as

$$\big(\boldsymbol{\mu}\boldsymbol{x}\big)'(t) = \boldsymbol{\mu}(t)\mathbf{g}(t)$$

and then integrating[1] we obtain

$$\boldsymbol{\mu}(t)\mathbf{x}(t) = \mathbf{C} + \int_0^t \boldsymbol{\mu}(s)\mathbf{g}(s)\mathrm{d}s$$

[1] It is worth pondering what integration would mean here since the product of $\boldsymbol{\mu}$ and $\mathbf{g}$ is a column vector and not a scalar. One is usually taught that integrating a continuous scalar function f is the result of taking a limit of Riemann sums. That is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x.$$

where $\mathbf{C}$ is a vector of constants of integration. Finally, if we are lucky enough that $\boldsymbol{\mu}(t)$ is invertible for all t then we can solve for $\mathbf{x}$ to obtain

$$\mathbf{x}(t) = \big(\boldsymbol{\mu}(\boldsymbol{t})\big)^{-1}\left(\mathbf{C} + \int_0^t \boldsymbol{\mu}(s)\mathbf{g}(s)\mathrm{d}s\right). \tag{3.2.2}$$

We now try to find our candidate $\boldsymbol{\mu}$. Recall that we needed to solve the equation

$$\boldsymbol{\mu}'(t) = -\boldsymbol{\mu}(t)\mathbf{A}. \tag{3.2.3}$$

Given that the one-dimensional case had $\mathbf{A}$ on the left hand side of the candidate function one might find it odd to have the matrix $\mathbf{A}$ on the right for this computation. To fix this, we define $\mathbf{y}(t) = \big(\boldsymbol{\mu}(\boldsymbol{t})\big)^T$. Transposing equation (3.2.3) we obtain

$$\mathbf{y}'(t) = \big(\boldsymbol{\mu}'(t)\big)^T = \big(-\boldsymbol{\mu}(t)\mathbf{A}\big)^T = -\mathbf{A}^T\big(\boldsymbol{\mu}(\boldsymbol{t})\big)^T = -\mathbf{A}^T\mathbf{y}(t).$$

Observe that if we write $\mathbf{y}$ is columns then the above can be understood as

$$\begin{bmatrix}\mathbf{y}_1'(t) & \cdots & \mathbf{y}_n'(t)\end{bmatrix} = \mathbf{A}^T\begin{bmatrix}\mathbf{y}_1(t) & \cdots & \mathbf{y}_n(t)\end{bmatrix} = \begin{bmatrix}-\mathbf{A}^T\mathbf{y}_1(t) & \cdots & -\mathbf{A}^T\mathbf{y}_n(t)\end{bmatrix}$$

which means that each column $\mathbf{y}_i$ satisfies the first order homogeneous system

$$\mathbf{y}_i'(t) = -\mathbf{A}^T\mathbf{y}_i(t).$$

Suppose now we choose the columns of $\mathbf{y}$ to be n-linearly independent solutions (i.e fundamental solutions) of this first order homogeneous system. Then we will have found $\mathbf{y}$ which means we have found $\boldsymbol{\mu}$ by transposing. Specifically, choose $\mathbf{y}_1 \ldots, \mathbf{y_n}$ to be n-linearly independent solutions. Now we let $\mathbf{y}(t) = \begin{bmatrix}\mathbf{y}_1(t) & \cdots & \mathbf{y}_n(t)\end{bmatrix}$. This tells us that

$$\boldsymbol{\mu}(t) = \big(\mathbf{y}(t)\big)^T = \begin{bmatrix}\big(\mathbf{y}_1(t)\big)^T \\ \vdots \\ \big(\mathbf{y}_n(t)\big)^T\end{bmatrix}.$$

In view of this, we notice that if $\mathbf{f} : [a,b] \to \mathbb{R}^m$ is now a vector-valued function then $\mathbf{f}$ is simply a column of scalar-valued functions $\big(f_1, \ldots, f_m\big)^T$. Observe that in this case it makes sense to write

$$\sum_{i=1}^n \mathbf{f}(x_i)\Delta x = \sum_{i=1}^n \begin{pmatrix} f_1(x_i) \\ \vdots \\ f_m(x_i)\end{pmatrix}\Delta x = \begin{pmatrix}\sum_{i=1}^n f_1(x_i)\Delta x \\ \vdots \\ \sum_{i=1}^n f_m(x_i)\Delta x.\end{pmatrix}$$

Now taking limits suggests the definition

$$\int_a^b \mathbf{f}(x)\Delta x := \begin{pmatrix}\int_a^b f_1(x)\mathrm{d}x \\ \vdots \\ \int_a^b f_m(x)\mathrm{d}x.\end{pmatrix}$$

One can check that equation (3.2.3) is satisfied. Since we chose that the solutions $\mathbf{y}_i$ are all linearly independent then $\boldsymbol{\mu}(t)$ is invertible for all t. In particular the formula given in equation (3.2.2) is valid.[2] Note that the above technique results in a more general answer than the technique given by using diagonalization since we did not assume anything about the matrix $\mathbf{A}$. However, we can see that the cost of generality is that obtaining the solution is more challenging.

To make the above construction more notationally clear we use the concept of a *matrix exponential* in the following section outlining the steps to implementing the above construction. The matrix exponential, $e^{t\mathbf{A}}$, (whose formula for a 2×2 matrix depends on whether its eigenvalues are complex, real and repeated, or real and distinct) is the matrix whose columns are the fundamental solutions to the problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and whose value at $t=0$ is the identity. Notice that the notation was deliberately chosen to remind you of the scalar ODE $x' = ax$ whose solution (up to a constant) is e^{ta}.

Method formal steps

1. As usual we first find the eigenvalues λ_1, λ_2 of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
2. Identifying the exponential of $\mathbf{A}$:
   - If the eigenvalues are distinct then
   $$e^{t\mathbf{A}} := e^{\lambda_1 t}\frac{1}{\lambda_1-\lambda_2}(\mathbf{A}-\lambda_2\mathbf{I}_2) - e^{\lambda_2 t}\frac{1}{\lambda_1-\lambda_2}(\mathbf{A}-\lambda_1\mathbf{I}_2).$$
   - If $\lambda = \lambda_1 = \lambda_2$ then
   $$e^{t\mathbf{A}} := e^{\lambda t}\mathbf{I}_2 + e^{\lambda t}t(\mathbf{A}-\lambda\mathbf{I}_2).$$
   - If $\lambda_1 = a+ib, \lambda_2 = a-ib$ then
   $$e^{t\mathbf{A}} := \frac{e^{at}}{b}\{b\cos(bt)\mathbf{I}_2 + \sin(bt)(\mathbf{A}-a\mathbf{I}_2)\}.$$
3. We compute $e^{-t\mathbf{A}}$ by inverting the matrix $e^{t\mathbf{A}}$ (see the linear algebra appendix)
4. Finally we obtain the general solution for our system (using identities proven in the linear algebra appendix):
$$\mathbf{x}(t) = \big(\boldsymbol{\mu}(\boldsymbol{t})\big)^{-1}\left(\mathbf{C} + \int_0^t \boldsymbol{\mu}(s)\mathbf{g}(s)\mathrm{d}s\right) = exp\{t\mathbf{A}\}\left(\mathbf{C} + \int_0^t exp\{-s\mathbf{A}\}\mathbf{g}(s)\mathrm{d}s\right) \tag{3.2.4}$$

Example-Presenting the method

Consider the system
$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}. \tag{3.2.5}$$
Notice that the matrix $\mathbf{A}$ in equation (3.2.5) has only 1 as an eigenvalue but the only solution to
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{v} = \mathbf{v}$$

[2] One might try to check that the formula for the solution given in equation (3.2.2) is in fact correct. Note, however, that this computation is actually somewhat sophisticated since you have to differentiate the function $\big(\boldsymbol{\mu}(t)\big)^{-1}$ which requires computing the derivative of the function that assigns the inverse of a matrix.

is

$$\mathbf{v} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $a \in \mathbb{R}$. So this matrix is not diagonalizable and hence the diagonalization method does not apply. However, the integrating factor technique will still work. Following step 1 we notice that we have a repeated eigenvalue $\lambda = 1$. By step 2 we obtain that

$$e^{t\mathbf{A}} = e^t\mathbf{I}_2 + e^t t(\mathbf{A} - \mathbf{I}_2) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

By step 3 we learn that

$$(e^{t\mathbf{A}})^{-1} = e^{-t\mathbf{A}} = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

Finally, by equation $(3.2.4)$ we learn that the general solution is:

$$\begin{aligned}\mathbf{x}(t) &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\left(\mathbf{C} + \int_0^t \begin{bmatrix} e^{-s} & -se^{-s} \\ 0 & e^{-s} \end{bmatrix}\begin{bmatrix} e^s \\ 1 \end{bmatrix}\mathrm{d}s\right) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\left(\mathbf{C} + \int_0^t \begin{bmatrix} 1 - se^{-s} \\ e^{-s} \end{bmatrix}\mathrm{d}s\right) \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\left(\mathbf{C} + \begin{bmatrix} t - 1 + e^{-t} + te^{-t} \\ 1 - e^{-t} \end{bmatrix}\right) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\mathbf{C} + \begin{bmatrix} 2te^t - e^t + 1 \\ e^t - 1 \end{bmatrix}\end{aligned}$$

One can check that this solves equation $(3.2.5)$ since

$$\mathbf{x}'(t) = \begin{bmatrix} e^t & te^t + e^t \\ 0 & e^t \end{bmatrix}\mathbf{C} + \begin{bmatrix} 2te^t + e^t \\ e^t \end{bmatrix}$$

while

$$\mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) = \begin{bmatrix} e^t & te^t + e^t \\ 0 & e^t \end{bmatrix}\mathbf{C} + \begin{bmatrix} 2te^t \\ e^t - 1 \end{bmatrix} + \begin{bmatrix} e^t \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t + e^t \\ 0 & e^t \end{bmatrix}\mathbf{C} + \begin{bmatrix} 2te^t + e^t \\ e^t \end{bmatrix}.$$

3.2.4 Variation of Parameters

Given the complexity of the process to solve the inhomogeneous first order system found in section $(3.2.3)$ one might wish to find a simpler way of obtaining the solution. This is possible if one is more clever about how they proceed. Recall that for the scalar inhomogeneous equation $x'(t) = ax(t) + g(t)$ the general solution is

$$x(t) = e^{at}\left(C + \int_0^t e^{-as}g(s)\mathrm{d}s\right)$$

where C is a constant. Observe that the term Ce^{at} actually solves the homogeneous equation $x'(t) = ax(t)$. Thus, the part of this solution that is needed to solve the inhomogeneous equation is

$$e^{at}\int_0^t e^{-as}g(s)\mathrm{d}s.$$

Observe that this looks like the solution to the homogeneous equation multiplied by a new function. Inspired by this we might try to solve the system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t). \tag{3.2.6}$$

by using the ansatz

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{y}(t)$$

where $\mathbf{X}(t)$ is the $n \times n$ matrix consisting of n linearly independent solutions to the homogeneous equation and $\mathbf{y}$ is to be determined. Note that $\mathbf{X}$ plays the role of e^{at} from the scalar case. Thus, we desire that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) = \mathbf{A}\mathbf{X}(t)\mathbf{y}(t) + \mathbf{g}(t)$$

but we have

$$\mathbf{x}'(t) = \mathbf{X}'(t)\mathbf{y}(t) + \mathbf{X}(t)\mathbf{y}'(t) = \mathbf{A}\mathbf{X}(t)\mathbf{y}(t) + \mathbf{X}(t)\mathbf{y}'(t).$$

Comparing these two equations we see that we must demand

$$\mathbf{X}(t)\mathbf{y}'(t) = \mathbf{g}(t)$$

which becomes after solving for $\mathbf{y}'(t)$, since $\mathbf{X}(t)$ is invertible,

$$\mathbf{y}'(t) = \big(\mathbf{X}(t)\big)^{-1}\mathbf{g}(t).$$

Integrating then gives

$$\mathbf{y}(t) = \mathbf{C} + \int_0^t \big(\mathbf{X}(s)\big)^{-1}\mathbf{g}(s)\mathrm{d}s.$$

Thus, we have found the solution

$$\mathbf{x}(t) = \mathbf{X}(t)\left(\mathbf{C} + \int_0^t \big(\mathbf{X}(s)\big)^{-1}\mathbf{g}(s)\mathrm{d}s\right).$$

There are a few of advantages to this solution over the one found in section $\big($3.2.3$\big)$. First, notice that it is not too hard to verify that this does in fact solve $\big($3.2.6$\big)$. Second, unlike the solution found in section $\big($3.2.3$\big)$ this formula makes reference directly to the matrix of fundamental solutions to the homogeneous system. As a result of this, less computations are needed. In particualr, exponential matrix identities are not needed to make this expression simpler.

Method formal steps

1. Solve the homogeneous systems to find two linearly independent solutions $\mathbf{x}_1(t) = \binom{x_{1,1}(t)}{x_{1,2}(t)}$ and $\mathbf{x}_2(t) = \binom{x_{2,1}(t)}{x_{2,2}(t)}$ to form the fundamental matrix:

$$\mathbf{\Psi}(t) := \begin{bmatrix} x_{1,1}(t) & x_{2,1}(t) \\ x_{1,2}(t) & x_{2,2}(t) \end{bmatrix},$$

   which satisfies $\mathbf{\Psi}' = \mathbf{A}\mathbf{\Psi}$

2. We make the ansatz we have $\mathbf{x}_{nh}(t) = \mathbf{\Psi}(t) \cdot \mathbf{v}(t) = \mathbf{\Psi}(t) \cdot \binom{v_1(t)}{v_2(t)}$.

3. Plugging this guess to the equation we obtain the system:

$$\mathbf{\Psi}(t) \cdot \mathbf{v}'(t) = \mathbf{g}(t) \Rightarrow$$

$$\begin{bmatrix} x_{1,1}(t) & x_{2,1}(t) \\ x_{1,2}(t) & x_{2,2}(t) \end{bmatrix} \cdot \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \Rightarrow$$

we obtain the system

$$\begin{cases} x_{1,1}(t)v_1' + x_{1,2}(t)v_2' = g_1(t) \\ x_{2,1}(t)v_1' + x_{2,2}(t)v_2' = g_2(t) \end{cases}$$

4. Solving this system for v_1', v_2' we then integrate to obtain v_1, v_2 and finally obtain the $\mathbf{x}_{nh}(t) = \mathbf{\Psi}(t) \cdot \mathbf{v}(t)$.

Examples

- Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}.$$

1. First we find the fundamental matrix $\mathbf{\Psi}$: the eigenpairs are $(-3, \binom{-1}{4}), (2, \binom{1}{1})$ and so the fundamental matrix is:

$$\mathbf{\Psi} = \begin{bmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{bmatrix}.$$

2. The system is

$$\begin{cases} -e^{-3t}v_1' + e^{2t}v_2' = e^{-2t} \\ 4e^{-3t}v_1' + e^{2t}v_2' = -2e^t \end{cases}$$

and we obtain

$$\mathbf{v}'(t) = \begin{bmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{bmatrix}^{-1} \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix} = \frac{1}{-5} \begin{pmatrix} e^t + 2e^{4t} \\ -4e^{-4t} + 2e^{-t} \end{pmatrix} \Rightarrow$$

$$\mathbf{v}(t) = \frac{1}{-5} \begin{pmatrix} e^t + \frac{1}{2}e^{4t} \\ e^{-4t} - 2e^{-t} \end{pmatrix}.$$

Therefore, the nonhomogeneous solution is

$$\mathbf{x}_{nh}(t) = \mathbf{\Psi}\mathbf{v} = \begin{pmatrix} \frac{e^t}{2} \\ -e^{-2t} \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 4 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{e^t}{2} \\ -e^{-2t} \end{pmatrix}.$$

3. From the above solution we note that the dominating term is $e^{2t}\binom{1}{1}$ so we expect the solution to converge to the linear span of $\binom{1}{1}$.

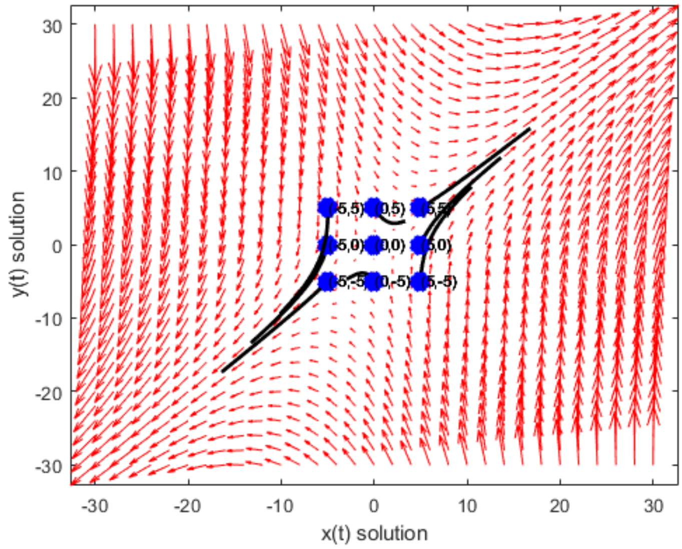


Figure 3.2.1: Phase portrait

- Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

1. First we find the fundamental matrix $\mathbf{\Psi}$: the eigenvalue is $(-2, \binom{1}{1})$ and $\eta = k\binom{1}{1} - \binom{0}{1}$, therefore, the solution is

$$\begin{aligned} \mathbf{x}(t) =& c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}) \Rightarrow \\ \mathbf{\Psi}(t) =& \begin{bmatrix} e^{-2t} & te^{-2t} \\ e^{-2t} & e^{-2t}(t-1) \end{bmatrix}. \end{aligned}$$

2. The system is

$$\begin{cases} e^{-2t}v_1' + te^{-2t}v_2' = e^{-2t} \\ e^{-2t}v_1' + e^{-2t}(t-1)v_2' = -2e^{t} \end{cases}$$

and we obtain

$$\begin{aligned} \mathbf{v}'(t) &= \begin{pmatrix} 1 - t(1+2e^{3t}) \\ 1 + 2e^{3t} \end{pmatrix} \Rightarrow \\ \mathbf{v}(t) &= \begin{pmatrix} t - \frac{1}{2}t^2 - \frac{2}{3}te^{3t} + \frac{1}{9}e^{3t} \\ t + \frac{2}{3}e^{3t} \end{pmatrix}. \end{aligned}$$

Therefore, the nonhomogeneous solution is

$$\mathbf{x}_{nh}(t) = \begin{pmatrix} e^{-2t}(-t^2/2 - 2/3e^{3t}t + t + e^{3t}/9) + e^{-2t}t(t + (2e^{3t})/3) \\ e^{-2t}(-t^2/2 - 2/3e^{3t}t + t + e^{3t}/9) + e^{-2t}(t-1)(t + (2e^{3t})/3) \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}) + \mathbf{x}_{nh}(t).$$

- Consider the 2nd order equation

$$\theta'' - 2\theta' + \theta = 3 + e^{-t}$$

then by setting $x = \theta, y = \theta'$ we obtain the system:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 3 + e^{-t} \end{pmatrix}.$$

1. First we solve the second order equation. The homogeneous part is

$$x_h = c_1 e^t + c_2 e^t t.$$

2. Then we apply our guess: $x_{nh} = at^s e^{-t}$. Since -1 is not a root, we have $s = 0$ and so it remains to find a:

$$a(e^{-t} + 2e^{-t} + e^{-t}) = e^{-t} \Rightarrow a = \frac{1}{4}.$$

3. So the solution is:

$$x_{gen} = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{4} e^{-t} + 3.$$

4. Next we solve the corresponding nonhomogeneous system. The eigenpair is $(1, \binom{1}{1})$ and $\eta = k\binom{1}{1} + \binom{0}{1}$ and so the homogeneous part is:

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t (t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

5. So the fundamental matrix is

$$\mathbf{\Psi} = \begin{pmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{pmatrix}.$$

6. The system becomes

$$\begin{cases} x_{1,1}(t)v_1' + x_{1,2}(t)v_2' = g_1(t) \\ x_{2,1}(t)v_1' + x_{2,2}(t)v_2' = g_2(t) \end{cases} \Rightarrow$$
$$\begin{cases} e^t v_1' + te^t v_2' = 0 \\ e^t v_1' + (t+1)e^t v_2' = e^{-t} \end{cases} \Rightarrow$$
$$\begin{cases} v_1' = -te^{-2t} \\ v_2' = e^{-2t} \end{cases} \Rightarrow$$
$$\begin{cases} v_1 = \frac{1}{4} e^{-2t}(2t+1) \\ v_2 = \frac{e^{-2t}}{-2} \end{cases}$$

7. Therefore, the nonhomogeneous part is

$$\mathbf{x}_{nh,1} = \mathbf{\Psi v} = \frac{e^{-t}}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

8. For the system $\mathbf{x}' = Ax + \binom{0}{3}$, we have

$$\mathbf{x}_{nh,2} = A^{-1}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

9. The general solution will be:

$$\mathbf{x}_{gen} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t (t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) + \frac{e^{-t}}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

10. Indeed the $x = \theta$ component is of the form:

$$\theta = x = c_1 e^t + c_2 t e^t + \frac{e^{-t}}{4} + 3,$$

which agrees with our second order solution.

3.3 Problems

3.3.1 Real eigenvalues

- Find the general solution of the system. Describe the asymptotic behaviour (what is the dominating term and the limit). Draw the two eigenvector's spans and draw arrows towards the dominating term. Is it a saddle or a sink to the origin?

  1. $$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}.$$

  2. $$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}.$$

- Find the particular solution of the system. Describe the asymptotic behaviour (what is the dominating term and the limit). Draw the two eigenvector's spans and draw arrows towards the dominating term.

  1. $$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

  2. $$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- For some $a \in [\frac{1}{2}, 2]$ consider the system

  $$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -a & -1 \end{pmatrix}\mathbf{x}.$$

  Find the general solution in terms of a. Determine the asymptotic behaviour for $a = \frac{1}{2}$ and for 2, and find the $a_* \in [\frac{1}{2}, 2]$,called the bifurcation value, where the asymptotic behaviour changes.

- (*)The amounts of salt $x_1(t), x_2(t)$ in the two tanks satisfy the equations

  $$\frac{\mathrm{d}x_1}{\mathrm{dt}} = -k_1 x_1,\ \frac{\mathrm{d}x_2}{\mathrm{dt}} = k_1 x_1 - k_2 x_2 \text{ with } x_1(0) = 15,\ x_2(0) = 0,$$

  where $k_1 = \frac{r}{V_1} = \frac{1}{5}$, $k_2 = \frac{r}{V_2} = \frac{2}{5}$. Find the particular solution and determine the asymptotic behaviour. What does it tell you about the tank's salt concentration?

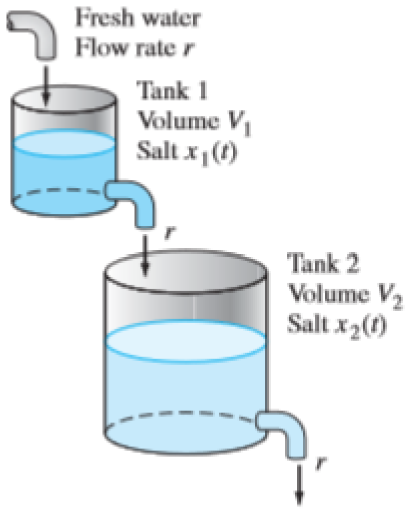


Figure 3.3.1: The two brine tanks.

3.3.2 Complex eigenvalues

- Find the general solution of the system. Describe the asymptotic behaviour. Are the trajectories forming a spiral source, a spiral sink or concentric circles?

  1. $$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\mathbf{x}.$$

  2. $$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

  3. $$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\mathbf{x}.$$

- (*) Find the general solution of the system. Find the bifurcation value or values of α where the qualitative nature of the phase portrait for the system changes. Draw a phase portrait for a value of α slightly below, and for another value slightly above, each bifurcation value.

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}\mathbf{x}.$$

- (*)Consider the circuit

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & \frac{-1}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

  Solve and determine long term behaviour. Is it asymptotically stable?

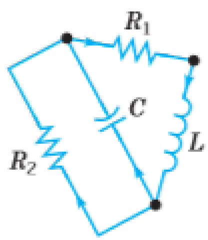


Figure 3.3.2: The circuit with complex eigenvalues.

3.3.3 Repeated eigenvalues

- Find the general solution of the system. Describe the asymptotic behaviour. Are the trajectories forming a source or sink behaviour wrt the origin?

1.
$$\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x}.$$

2.
$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}.$$

3. Find the particular solution and determine the asymptotic behaviour as above:
$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

3.3.4 Differential inequalities

1. In this question we will show the following result: Suppose $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a function that is continuous on $[a, b]$, differentiable on (a, b), and satisfies, for $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ a diagonal matrix,
$$\mathbf{x}'(t) \leq \mathbf{A}\mathbf{x}(t)$$
for all $t \in (a, b)$, where the inequality means that each component of the left hand side is smaller than the corresponding component on the right hand side. Then
$$\mathbf{x}(t) \leq e^{(t-a)\mathbf{A}}\mathbf{x}(a).$$[3]

   (a) First show that if $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ then $\mathbf{A}\mathbf{x} \geq \mathbf{0}_{n\times 1}$ whenever $\mathbf{x} \geq \mathbf{0}_{n\times 1}$ if and only if $\mathbf{A}$ is a non-negative matrix (all entries in the matrix are non-negative).

   (b) Next show that if $t \geq 0$ then $e^{t\mathbf{A}}$ is a non-negative matrix if and only if $\mathbf{A}$ has non-negative off diagonal entries.[4]

   (c) Consider the function $\mathbf{w} : [a, b] \to \mathbb{R}$ defined by[5]
$$\mathbf{w}(t) = e^{(a-t)\mathbf{A}}\mathbf{x}(t).$$
   Show, using (1*b*), that $\mathbf{w}'(t) \leq \mathbf{0}_{n\times 1}$ for $t \in (a, b)$.

   (d) Conclude that each component of $\mathbf{w}$ is decreasing on $[a, b]$.

   (e) Finally, conclude that $e^{(a-t)\mathbf{A}}\mathbf{x}(t) = \mathbf{w}(t) \leq \mathbf{w}(a) = \mathbf{x}(a)$ which can be rewritten as
$$\mathbf{x}(a) - e^{(a-t)\mathbf{A}}\mathbf{x}(t) \geq \mathbf{0}_{n\times 1}.$$
   Use (1*b*) to conclude the desired inequality.

2. In this question we show that solutions to the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for $t \in (a, b)$ with $\mathbf{x}(a) = \mathbf{x}_0$ are unique using Grönwall type inequalities. Suppose $\mathbf{x}, \mathbf{y} : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that they satisfy $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ on (a, b) as well as $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$ on (a, b).

   (a) First show that, for $t \in (a, b)$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{x}(t) - \mathbf{y}(t)\|^2\big) = \big(\mathbf{x}(t) - \mathbf{y}(t)\big)^T\mathbf{A}\big(\mathbf{x}(t) - \mathbf{y}(t)\big).$$

[3]This is known as Grönwall's inequality. The principle involved here is that if a function grows no quicker than $\mathbf{A}\mathbf{x}(t)$ then the value of the function should not exceed the solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ which maximizes its growth.

[4]Such matrices are called Metzler matrices.

[5]Some properties of the matrix exponential will be used here. Please refer to the linear algebra appendix for proofs of these properties.

(b) Next use equation (3.3.1) and the previous step to conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{x}(t)-\mathbf{y}(t)\|^2\right) \leq \lambda_n(\mathbf{A})\|\mathbf{x}(t)-\mathbf{y}(t)\|^2.$$

(c) Argue that we obtain, for $t \in [a, b]$,

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|^2 \leq e^{2t\lambda_n(\mathbf{A})}\|\mathbf{x}(a)-\mathbf{y}(a)\|^2.$$

(d) Deduce that if $\mathbf{x}(a) = \mathbf{y}(a)$ then $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [a, b]$.

3.3.5 Systems of ODEs and Quadratic forms

1. (a) In this problem we show that if $\mathbf{A}$ has only positive eigenvalues, $\mathbf{x}(0) \neq \mathbf{0}_{n\times 1}$, and if $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ then $\|\mathbf{x}(t)\| \to +\infty$ as $t \to +\infty$.

   i. First, show that

   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{x}(t)\|^2\right) = \big(\mathbf{x}(t)\big)^T\mathbf{A}\mathbf{x}(t).$$

   ii. Next, observe that

   $$\min_{\|\mathbf{x}\|=1}\left\{\mathbf{x}^T\mathbf{A}\mathbf{x}\right\} = \lambda_1(\mathbf{A})$$

   where $\lambda_1(\mathbf{A})$ denotes the smallest eigenvalue of $\mathbf{A}$. To see this, note that $\mathbf{A}$ is diagonalizable and so we can represent $\mathbf{x}$ as a linear combination of orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. So we have

   $$\mathbf{A}\mathbf{x} = \mathbf{A}\left(\sum_{i=1}^{n} c_1\mathbf{u}_i\right) = \sum_{i=1}^{n}\lambda_i(\mathbf{A})c_i\mathbf{u}_i$$

   which means

   $$\mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i=1}^{n}\lambda_i c_i^2 \geq \lambda_1(\mathbf{A})\sum_{i=1}^{n}c_i^2 = \lambda_1(\mathbf{A})\|\mathbf{x}\|^2 = \lambda_1(\mathbf{A})$$

   and observe that $\mathbf{x}$ was an arbitrary unit vector. Note that equality can be obtained.

   iii. Using the previous two questions observe that

   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{x}(t)\|^2\right) \geq \lambda_1(\mathbf{A})\|\mathbf{x}(t)\|^2.$$

   Using the integrating factor $e^{-2t\lambda_1(\mathbf{A})}$ conclude that

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2t\lambda_1(\mathbf{A})}\|\mathbf{x}(t)\|^2\right) \geq 0.$$

   iv. Conclude that

   $$\|\mathbf{x}(t)\|^2 \geq \|\mathbf{x}(0)\|^2 e^{2t\lambda_1(\mathbf{A})}.$$

   (b) In this problem we show that if $\mathbf{A}$ has all negative eigenvalues then $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$ if $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

   i. Show that, as in the previous question,

   $$\max_{\|\mathbf{x}\|=1}\left\{\mathbf{x}^T\mathbf{A}\mathbf{x}\right\} = \lambda_n(\mathbf{A}) \tag{3.3.1}$$

and conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{x}(t)\|^2\right) \le \lambda_n(\mathbf{A})\|\mathbf{x}(t)\|^2.$$

ii. Conclude that

$$\|\mathbf{x}(t)\|^2 \le \|\mathbf{x}(0)\|^2 e^{2t\lambda_n(\mathbf{A})}.$$

2. In this problem we will demonstrate how to find a solution with perpendicular trajectories in $\mathbb{R}^2$. Suppose $\mathbf{x}, \mathbf{y} : \mathbb{R} \to \mathbb{R}^2$ solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ and $\mathbf{y}'(t) = \mathbf{B}\mathbf{y}(t)$ respectively, where $\mathbf{A}, \mathbf{B} \in M_{2\times 2}(\mathbb{R})$. Assume also that, for all $\mathbf{a} \in \mathbb{R}^2$, that if $\mathbf{x}(0) = \mathbf{a} = \mathbf{y}(0)$ then $\mathbf{x}'(0) \perp \mathbf{y}'(0)$.

   (a) Use the conditions given in the problem description to conclude that

   $$\mathbf{a} \cdot \left(\mathbf{A}^T\mathbf{B}\mathbf{a}\right) = 0$$

   for all $a \in \mathbb{R}^2$.

   (b) Conclude that there is a constant $c \in \mathbb{R}$ such that

   $$\mathbf{A}^T\mathbf{B} = c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (c) If $c = 0$ conclude that either $\mathbf{A} = \mathbf{0}_{2\times 2}$, $\mathbf{B} = \mathbf{0}_{2\times 2}$, or $\mathbf{A}^T\mathbf{B} = \mathbf{0}_{2\times 2}$ while $\mathbf{A}$ and $\mathbf{B}$ are both not the zero matrix. In the event that $\mathbf{A}^T\mathbf{B} = \mathbf{0}_{2\times 2}$ even though both $\mathbf{A}$ and $\mathbf{cB}$ are both not the zero matrix show that $\operatorname{im}(\mathbf{A}) \perp \operatorname{im}(\mathbf{B})$.

   (d) Now we may assume that $c \neq 0$. By taking determinants show that both $\mathbf{A}$ and $\mathbf{B}$ are invertible. Finally, conclude that either

   $$\mathbf{A}^T\mathbf{B} = c\mathbf{R}_{\frac{\pi}{2}}$$

   or

   $$\mathbf{A}^T\mathbf{B} = c\mathbf{R}_{\frac{-\pi}{2}}$$

   for some $c > 0$ where $\mathbf{R}_{\pm\frac{\pi}{2}}$ denote rotation matrices at angles $\pm\frac{\pi}{2}$ respectively. Use this to conclude that

   $$\mathbf{B} = c\left(\mathbf{A^{-1}}\right)^T\mathbf{R}_{\pm\frac{\pi}{2}}.$$

3. Notice that for each $t \in \mathbb{R}$ we can define a map $\boldsymbol{\varphi}_t : \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{\varphi}_t(\mathbf{x}) = e^{t\mathbf{A}}\mathbf{x}$. We say that the family of maps $\{\boldsymbol{\varphi}_t\}_{t\in\mathbb{R}}$ is volume preserving if for every $t \in \mathbb{R}$ and for every open set $U \subset \mathbb{R}^n$ we have $\operatorname{vol}(\boldsymbol{\varphi}_t(U)) = \operatorname{vol}(U)$.[6]

   (a) With $\boldsymbol{\varphi}_t$ defined as above for each $t \in \mathbb{R}$ show that, by the change of variables theorem, that

   $$\operatorname{vol}(\boldsymbol{\varphi}_t(U)) = |\det(e^{t\mathbf{A}})|\operatorname{vol}(U).$$

   (b) Use this to show that if $\{\boldsymbol{\varphi}_t\}_{t\in\mathbb{R}}$ is a volume preserving family then $\det(e^{t\mathbf{A}}) = 1$ for all $t \in \mathbb{R}$.

   (c) Recall Liouville's formula, as seen in the linear algebra appendix, says that

   $$\det(e^{t\mathbf{A}}) = e^{t\cdot\operatorname{tr}(\mathbf{A})}.$$

   Use this to conclude that $\operatorname{tr}(\mathbf{A}) = 0$.

   (d) Conversely, show that if $\operatorname{tr}(\mathbf{A}) = 0$ then $\{\boldsymbol{\varphi}_t\}_{t\in\mathbb{R}}$ is a volume preserving family of maps.

[6]This exercise describes a particular instance of a result known as *Liouville's Theorem.*

Chapter 4

Autonomous systems

As in the 1D case we will study the following system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x, y), \; \frac{\mathrm{d}y}{\mathrm{dt}} = G(x, y),$$

where F, G are continuously differentiable functions. Here again we might not be able to obtain explicit solutions, but we can provide a qualitative analysis.

Method formal steps

1. First, we find the critical points by setting

$$F(x, y) = 0 \text{ and } G(x, y) = 0.$$

2. For each critical point we carve out the regions of the phase portrait that converge to it, called *basin regions of attraction.* For example, as we will explain later in the competing species section, we obtain phase portraits of the form:

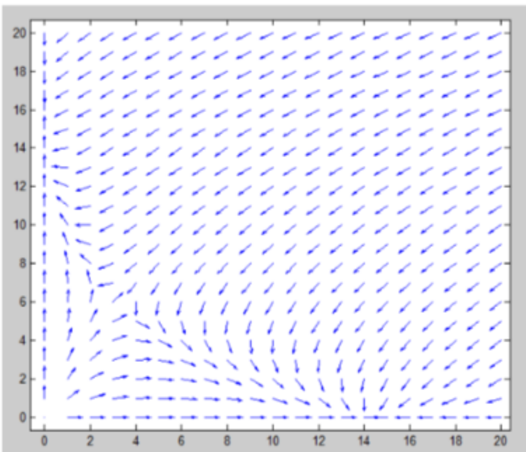


Figure 4.0.1: The critical points $(14, 0)$ and $(0, 14)$ have their own basin regions of attraction

Here the points $(14, 0)$ and $(0, 14)$ have their own basin regions of attraction (arrows pointing towards them) that are separated by curves called the *separatrix.*

3. Sometimes we can even solve such systems by taking their ratio and obtain a parametric

solution:

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{\frac{\mathrm{d}y}{\mathrm{dt}}}{\frac{\mathrm{d}x}{\mathrm{dt}}} = \frac{G(x,y)}{F(x,y)}.$$

This ratio depends only on x and y (and not t), so methods from the first order section could be used.

4. Next we sketch the direction field by doing a *nullcline analysis* around the critical points. That is we study the signs of the pair $(\frac{\mathrm{d}x}{\mathrm{dt}}, \frac{\mathrm{d}y}{\mathrm{dt}})$.

5. Finally, we plot the parametric solution and check whether it agrees with the direction field from the above step.

Example-presenting the method

Consider the following oscillating pendulum: a mass m is attached to one end of a rigid, but weightless, rod of length L which hangs from the pivot point.

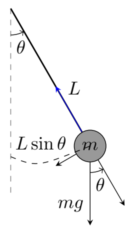


Figure 4.0.2: oscillating pendulum

The gravitational force mg acts downward and the damping force $c|\frac{d\theta}{dt}|$ is always opposite to the direction of motion. A rotational analog of Newton's second law of motion might be written in terms of torques:

$$mg \cdot L\sin(\theta) + \frac{\mathrm{d}\theta}{\mathrm{dt}} \cdot L + m\frac{\mathrm{d}^2\theta}{\mathrm{dt}^2}L^2 = 0 \implies \frac{\mathrm{d}^2\theta}{\mathrm{dt}^2} + \gamma\frac{\mathrm{d}\theta}{\mathrm{dt}} + \omega^2\sin(\theta) = 0$$

where $\gamma = \frac{1}{mL}$ and $\omega^2 = \frac{g}{L}$. This is a nonhomogeneous second order equation, but we can also view it as a system of equations by letting $x := \theta$ and $y := \frac{\mathrm{d}\theta}{\mathrm{dt}}$:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \; \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \omega^2\sin(x),$$

where γ is called the damping constant and, as in the spring problem, it is responsible for removing energy. Notice that this is an autonomous system.

1. First we find the critical points:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 0, \; \frac{\mathrm{d}y}{\mathrm{dt}} = 0 \implies y = 0, \; \sin(x) = 0 \implies (k\pi, 0) \text{ for } k \in \mathbb{Z}.$$

2. Then we numerically draw the solutions

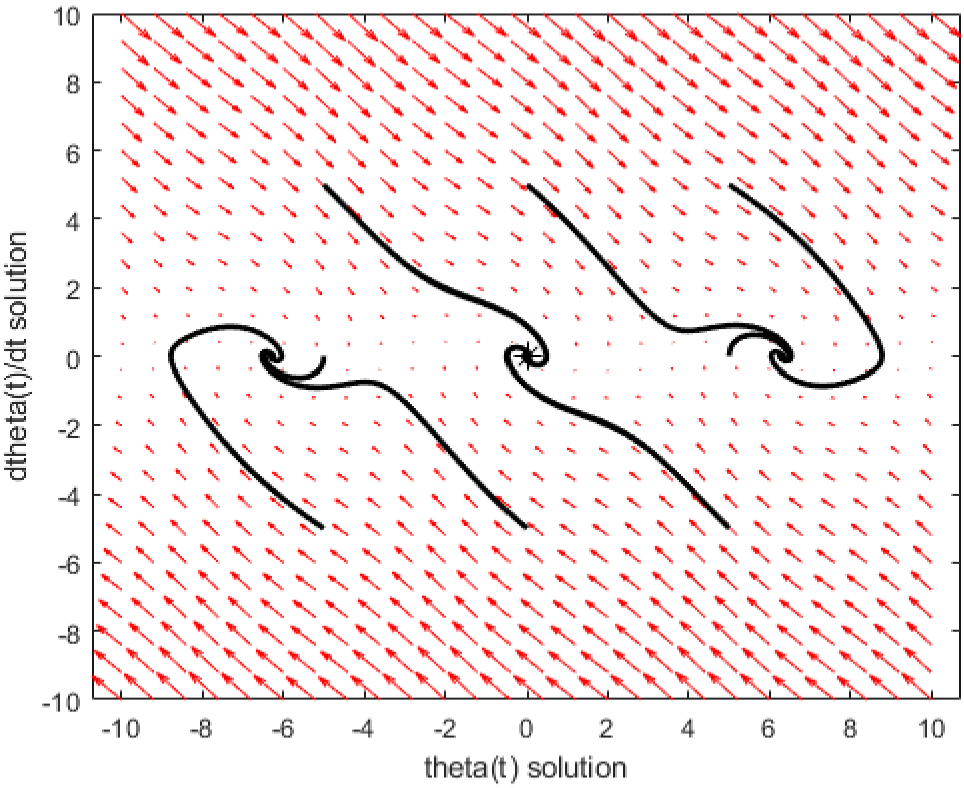


Figure 4.0.3: Phase portrait and solutions for oscillating pendulum.

We see that the basin regions of attraction for each critical point are the areas separated by the black spiral curves.

3. The ratio is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{-\gamma y - \omega^2 \sin(x)}{y},$$

which is not amenable to known methods (eg. see Chini's equation).

4. However, if we set $\gamma = 0$ (undamped pendulum), we get a separable equation and in turn the implicit solution:

$$y^2 = 2(\omega^2 \cos(x) + c) \implies \frac{y^2}{2} - \omega^2 \cos(x) = \text{constant}.$$

An alternative to using the equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$ to obtain the implicit equation is to note that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(y^2\right) = y\frac{\mathrm{d}y}{\mathrm{d}t} = -y\omega^2 \sin(x) = -\omega^2 \sin(x)\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\omega^2 \cos(x)\right).$$

5. Next we do a nullcline analysis around the origin.

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff

$$y > 0,\ -\omega^2 \sin(x) > 0 \iff y > 0,\ \frac{-\pi}{2} < x < 0.$$

- We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff
$$y > 0,\ -\omega^2 \sin(x) < 0 \iff y > 0,\ 0 < x < \frac{\pi}{2}.$$
- We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff
$$y < 0,\ -\omega^2 \sin(x) > 0 \iff y < 0,\ \frac{-\pi}{2} < x < 0.$$
- We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff
$$y < 0,\ -\omega^2 \sin(x) < 0 \iff y < 0,\ 0 < x < \frac{\pi}{2}.$$

6. Therefore, in summary, around the origin we have the sketch:

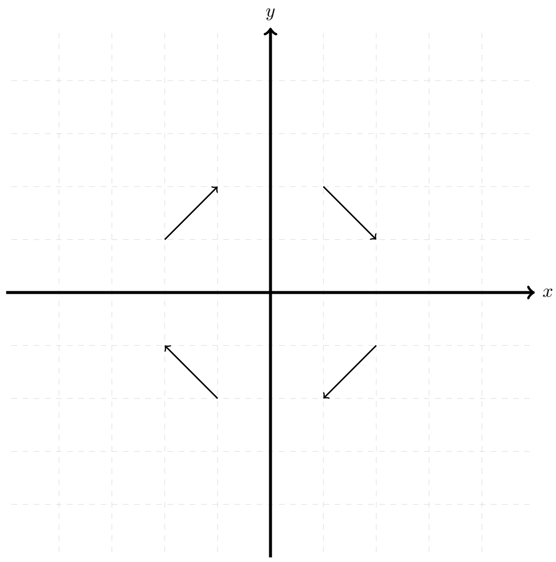


Figure 4.0.4: Phase portrait sketch for an undamped oscillating pendulum.

Indeed, the parametric solution follows the circular behaviour of the direction field in Figure 1.4.

7. The sketch in Figure 1.4 agrees with the numerically generated phase portrait displayed in Figure 1.5:

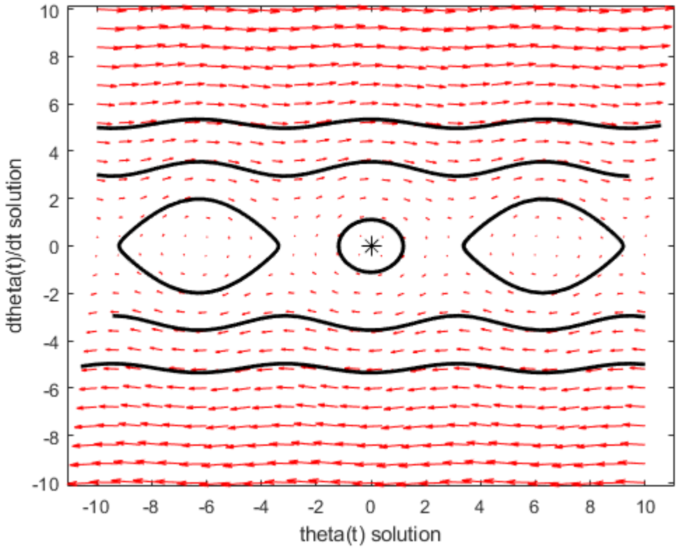


Figure 4.0.5: Phase portrait and solutions for undamped oscillating pendulum.

We see that the basins of attractions are separated by ellipses along the horizontal axis and they are separated from periodic behaviour along the vertical axis. Physically a closed curve around critical point represents the pendulum oscillating periodically since the velocity $y = \dot{\theta}$ oscillates periodically around that critical point. The wavy lines represent the pendulum spinning around the pivot point.

Examples

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 2y, \ \frac{\mathrm{d}y}{\mathrm{dt}} = -8x.$$

1. The critical point is just $(0, 0)$.
2. To determine the solutions we solve:

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{-8x}{2y}.$$

This equation is separable and so we easily obtain:

$$y^2 = -4x^2 + c.$$

3. Therefore, the solutions are ellipses $y^2 + 4x^2 = c$ centered at zero.
4. Next we do a nullcline analysis around the origin.
   - We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff $2y > 0$, $-8x > 0 \iff y > 0$, $x < 0$.
   - We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff $y > 0$, $x > 0$.
   - We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff $y < 0$, $x < 0$.
   - We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff $y < 0$, $x > 0$.
5. Therefore, in summary, around the origin we have the sketch depicted in Figure 1.6:

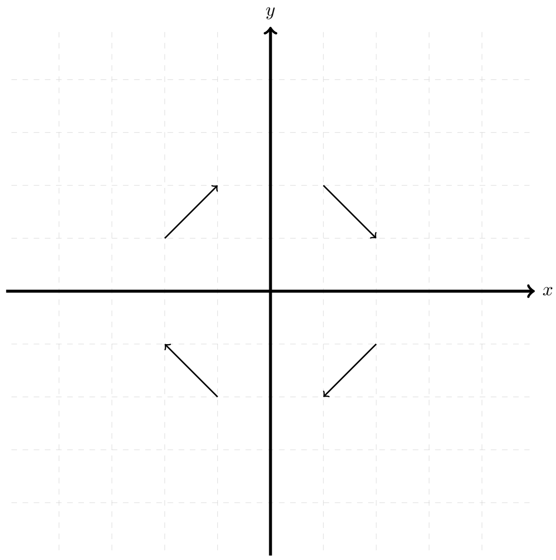


Figure 4.0.6: Phase portrait sketch.

Indeed the parametric solution follows the circular behaviour of the above direction field.

6. The above sketch agrees with the numerically generated phase portrait in Figure 1.7:

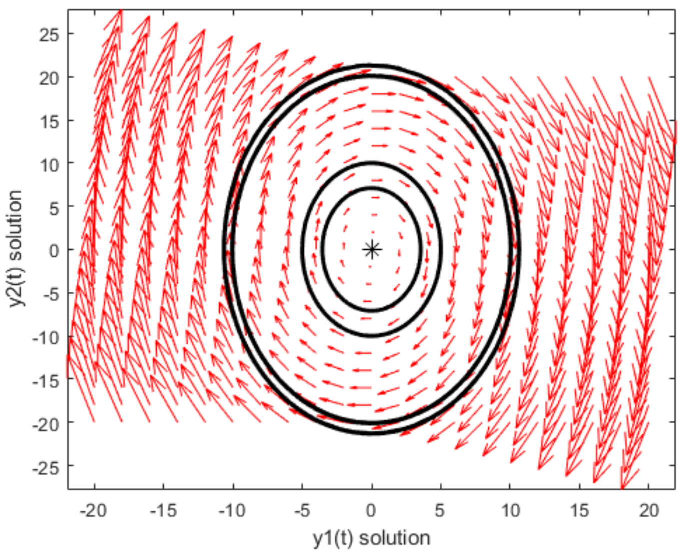


Figure 4.0.7: Phase portrait and solutions

- Consider the system (Duffing's equation)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \ \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \frac{x^3}{6}.$$

It describes the motion of a damped oscillator with a more complex potential than in simple harmonic motion. In physical terms, it models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law. The Duffing equation is an example of a dynamical system that exhibits chaotic behavior.

1. The critical points are $(0, 0)$, $\left(\pm\sqrt{6}, 0\right)$.

2. To determine the solutions we solve:

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{-x + \frac{x^3}{6}}{y}.$$

   This equation is separable and so we easily obtain:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + \frac{x^4}{24} + c.$$

3. Therefore, the solutions are the pairs of hyperbolas and ellipses $y^2 + x^2 - \frac{x^4}{12} = c$ symmetric wrt to the x-axis.
4. Next we do a nullcline analysis around the origin.
   – We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff

$$y > 0,\ -x + \frac{x^3}{6} > 0 \iff y > 0,\ x \in [-\sqrt{6}, 0] \cup [\sqrt{6}, \infty).$$

   – We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff

$$y > 0,\ -x + \frac{x^3}{6} < 0 \iff y > 0,\ x \in (-\infty, -\sqrt{6}] \cup [0, \sqrt{6}].$$

   – We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff

$$y < 0,\ -x + \frac{x^3}{6} > 0 \Leftrightarrow y > 0,\ x \in [-\sqrt{6}, 0] \cup [\sqrt{6}, \infty).$$

   – We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0$, $\frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff

$$y < 0,\ -x + \frac{x^3}{6} < 0 \Leftrightarrow y > 0,\ x \in (-\infty, -\sqrt{6}] \cup [0, \sqrt{6}].$$

5. Therefore, in summary, around the origin we have the sketch depicted in Figure 1.8:

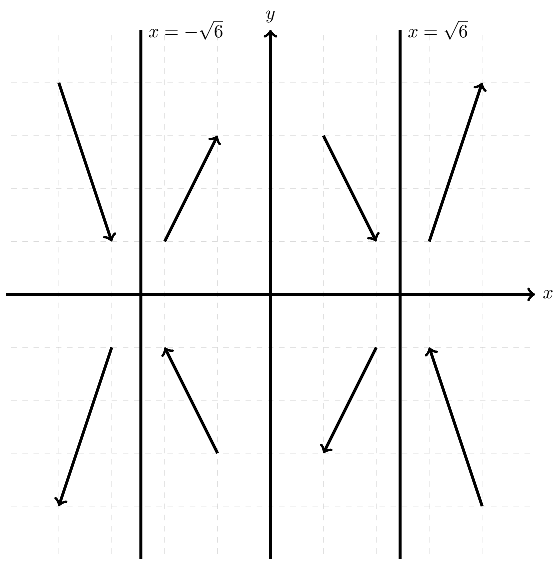


Figure 4.0.8: Phase portrait sketch.

Indeed the parametric solution follows the circular behaviour of the above direction field.

6. The above sketch agrees with the numerically generated phase portrait depicted in Figur 1.9:

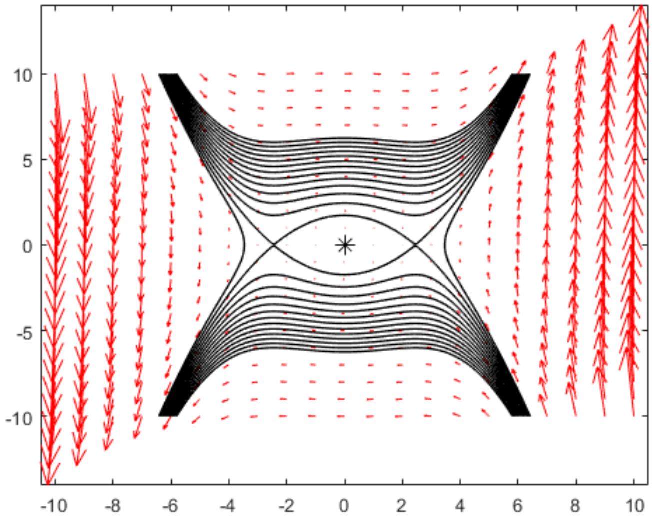


Figure 4.0.9: Phase portrait and solutions of duffing's system

Chapter 5

Locally linear systems

We will study systems

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where the components of $\mathbf{f}$ are C^1 functions so that we are able to Taylor expand them to first order. A system of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$$

is called *locally linear* around a critical point $\mathbf{x}_0$ of $\mathbf{f}$ if

$$\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}-\mathbf{x}_0\|} \to 0 \text{ as } \mathbf{x} \to \mathbf{x}_0.$$

Example-presenting the method

We continue our study with the damped oscillating pendulum system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y - \omega^2 \sin(x),$$

where γ is called the damping constant and as in the spring problem it is responsible for removing energy.

1. First we find the critical points. From the previous section we have:

   $$(k\pi, 0) \text{ for any integer } k.$$

2. Second we Taylor expand the RHS of the system $\mathbf{F}(x,y) := \binom{y}{-\gamma y - \omega^2 \sin(x)}$ around arbitrary critical point (x_0, y_0):

   $$\begin{aligned}\mathbf{F}(x,y) &= \mathbf{F}(x_0,y_0) + \mathbf{J}_{\mathbf{F}}(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + o(\|(x-x_0, y-y_0)\|) \\ &= \begin{pmatrix} 0 & 1 \\ & \\ -\omega^2\cos(x_0) & -\gamma \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + o(\|(x-x_0, y-y_0)\|).\end{aligned}$$

   Here $\mathbf{J}_{\mathbf{F}}(x_0, y_0)$ is the Jacobian matrix at (x_0, y_0) which, for function $\mathbf{F}(x,y) = \binom{\mathbf{F}_1(x,y)}{\mathbf{F}_2(x,y)}$, is defined as:

$$J_{\mathbf{F}}(x_0, y_0) := \begin{pmatrix} \frac{\mathrm{d}\mathbf{F}_1}{\mathrm{d}x}(x_0, y_0) & \frac{\mathrm{d}\mathbf{F}_1}{\mathrm{d}y}(x_0, y_0) \\ \\ \frac{\mathrm{d}\mathbf{F}_2}{\mathrm{d}x}(x_0, y_0) & \frac{\mathrm{d}\mathbf{F}_2}{\mathrm{d}y}(x_0, y_0) \end{pmatrix}$$

3. The linearization around $(x_0, y_0) = (k\pi, 0)$ for an even integer k is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x - k\pi \\ y \end{pmatrix} + o(\|(x - k\pi, y)\|).$$

The eigenvalues of that matrix are:

$$\lambda_1,\ \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}.$$

(a) If $\gamma^2 - 4\omega^2 > 0$, then the eigenvalues are real, distinct, and negative. Therefore, the critical points will be stable nodes.

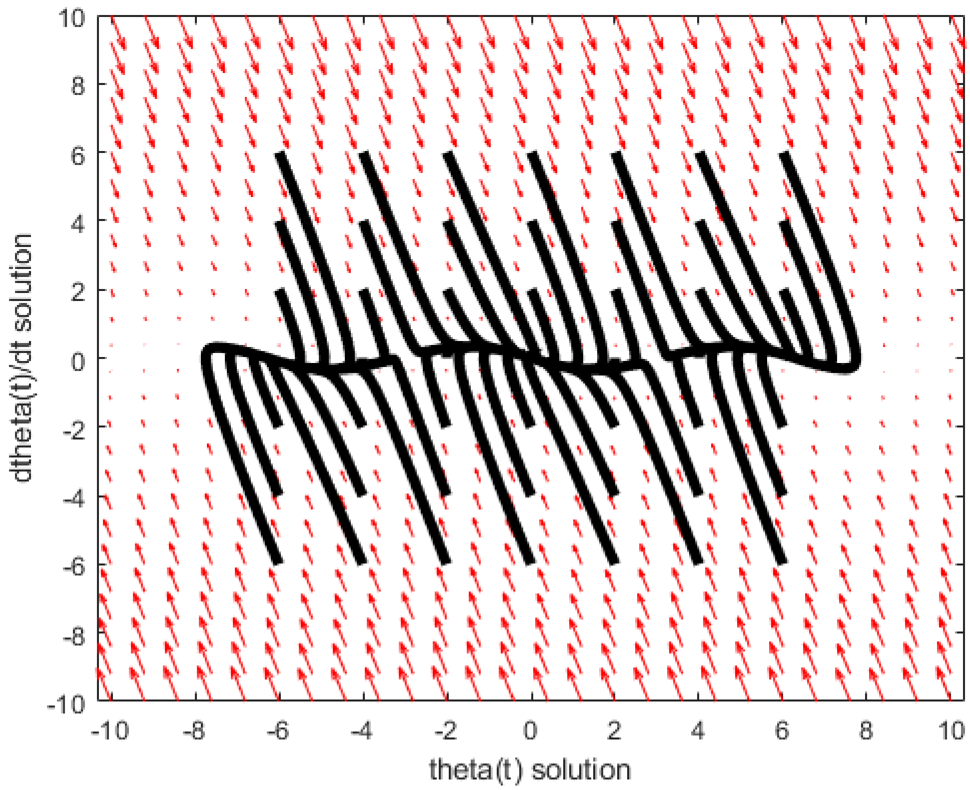


Figure 5.0.1: Stable nodes at even integer k critical points $(k\pi, 0)$ for $k = 0, 2, -2$.

We observe that the basins of attractions for each even-integer critical points are well-separated.

(b) If $\gamma^2 - 4\omega^2 = 0$, then the eigenvalues are repeated, real, and negative. Therefore, the critical points will be stable nodes.

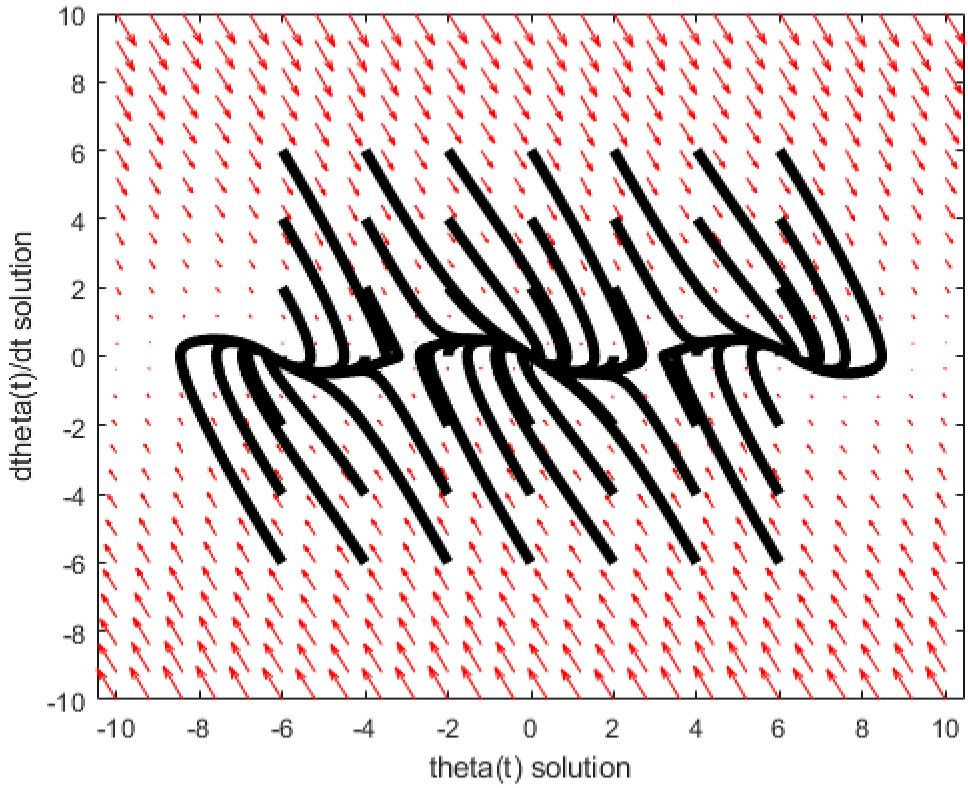


Figure 5.0.2: Stable nodes at even integer k critical points $(k\pi, 0)$.

(c) If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part. Therefore, the critical points will be stable spiral sinks.

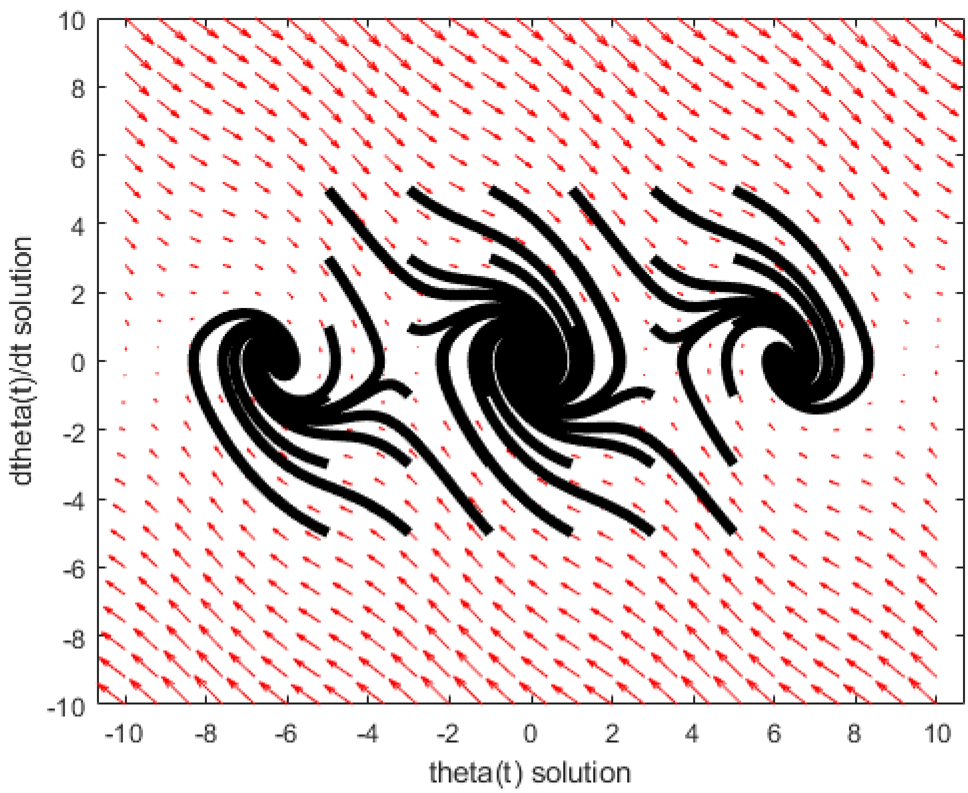


Figure 5.0.3: Stable spiral sinks at even integer k critical points $(k\pi, 0)$.

4. The linearization around $(x_0, y_0) = (k\pi, 0)$ for odd integer k is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x - k\pi \\ y \end{pmatrix} + o(\|(x - k \cdot \pi, y)\|).$$

The eigenvalues of that matrix are:

$$\lambda_1,\, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}.$$

Therefore, it has one negative eigenvalue $\lambda_1 < 0$ and one positive eigenvalue $\lambda_2 > 0$, and so the critical points will be unstable saddle points.

Method formal steps

1. First we obtain the critical points for the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x})$$

i.e. points (x_0, y_0) where $\mathbf{F}(x_0, y_0) = 0$.

2. We Taylor expand $\mathbf{F}$ in higher dimensions around an arbitrary critical point:

$$\begin{aligned}\mathbf{F}(x, y) &= \mathbf{F}(x_0, y_0) + J_{\mathbf{F}}(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(\|(x - x_0, y - y_0)\|) \\ &= \mathbf{J}_{\mathbf{F}}(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(\|(x - x_0, y - y_0)\|).\end{aligned}$$

Here $\mathbf{J}_{\mathbf{F}}(x_0, y_0)$ is the Jacobian matrix for the function $\mathbf{F}(x, y) = \left(\begin{smallmatrix} \mathbf{F}_1(x,y) \\ \mathbf{F}_2(x,y) \end{smallmatrix}\right)$:

$$\mathbf{J}_{\mathbf{F}}(x_0, y_0) := \begin{pmatrix} \frac{\mathrm{d}\mathbf{F}_1}{\mathrm{d}x}(x_0, y_0) & \frac{\mathrm{d}\mathbf{F}_1}{\mathrm{d}y}(x_0, y_0) \\ \frac{\mathrm{d}\mathbf{F}_2}{\mathrm{d}x}(x_0, y_0) & \frac{\mathrm{d}\mathbf{F}_2}{\mathrm{d}y}(x_0, y_0) \end{pmatrix}$$

3. Around each critical point we determine the eigenvalues and identify the type of qualitative behaviour.

General result:

The system

$$x' = F(x, y),\; y' = G(x, y)$$

is locally linear around a critical point (x_0, y_0) if $F, G \in C^2(U)$ where $U \subset \mathbb{R}^2$ is some open neighbourhood around (x_0, y_0).

Proof. First we Taylor expand F and G about (x_0, y_0) to get:

$$F(x, y) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + R_F(x, y),$$

$$G(x,y) = G(x_0,y_0) + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0) + R_G(x,y),$$

where by Taylor's theorem the remainders, R_F and R_G, satisfy

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|R_F(x,y)|}{|(x-x_0,y-y_0)|} = 0 = \lim_{(x,y)\to(x_0,y_0)} \frac{|R_G(x,y)|}{|(x-x_0,y-y_0)|}. \tag{5.0.1}$$

Because (x_0,y_0) is a critical point we have $F(x_0,y_0) = G(x_0,y_0) = 0$. Therefore, we can rewrite the system as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \begin{pmatrix} R_F(x,y) \\ R_G(x,y) \end{pmatrix}.$$

However,

$$\frac{\left\|\binom{R_F(x,y)}{R_G(x,y)}\right\|}{\|(x,y)-(x_0,y_0)\|} = \sqrt{\left(\frac{|R_F(x,y)|}{\|(x-x_0,y-y_0)\|}\right)^2 + \left(\frac{|R_G(x,y)|}{\|(x-x_0,y-y_0)\|}\right)^2}$$

and so by 5.0.1 this goes to 0 as (x,y) tends to (x_0,y_0).

□

Examples

- We return to Duffing's equation from the previous section

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \ \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \frac{x^3}{6}.$$

1. We found that the critical points are $(0,0), (\pm\sqrt{6}, 0)$.
2. We obtain the linearization of the RHS of the system $\mathbf{F}(x,y) := \binom{y}{-x+\frac{x^3}{6}}$ around arbitrary critical point (x_0,y_0):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{F}(x,y) = \mathbf{F}(x_0,y_0) + \mathbf{J_F}(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + o(\|(x-x_0,y-y_0)\|) \\ &= \begin{pmatrix} 0 & 1 \\ -1+\frac{x^2}{2} & 0 \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + o(\|(x-x_0,y-y_0)\|)\end{aligned}$$

3. Next we study the stability behaviour around each of the critical points.
   - At the origin we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o(\|(x,y)\|)$$

   and so the eigenvalues are $\lambda = \pm i$. Therefore, the stability behaviour at the origin will be concentric circles.
   - At the $(\pm\sqrt{6}, 0)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \pm \sqrt{6} \\ y \end{pmatrix} + o(\left\|(x \pm \sqrt{6}, y)\right\|)$$

and so the eigenvalues are $\lambda = \pm\sqrt{2}$. Therefore, the stability behaviour at both $(\sqrt{6}, 0), (-\sqrt{6}, 0)$ will be unstable saddle nodes.

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + \varepsilon \sin(x), \ \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \varepsilon \sin(y),$$

for small $\varepsilon \in \mathbb{R}$. We will study the affect of the stability behaviour as $\varepsilon \to 0$.

1. First we find that the only critical point is the origin $(0, 0)$. We can deduce this by drawing the two curves $(x, -\varepsilon \sin(x))$, $(\varepsilon \sin(y), y)$ and see that they intersect only at the origin.

2. Next we linearize around the origin:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o(\|(x, y)\|).$$

Therefore, the eigenvalues are $\lambda_1 = \varepsilon + i$, $\lambda_1 = \varepsilon - i$.

(a) If $\varepsilon < 0$[1], the origin becomes a sink spiral point.

[1] It is generally a bad idea to use ϵ to denote a negative number as it has become a symbol of a generic small positive number. There are, in fact, jokes about this (see here for a discussion of this). Despite this, the authors hope there is no confusion.

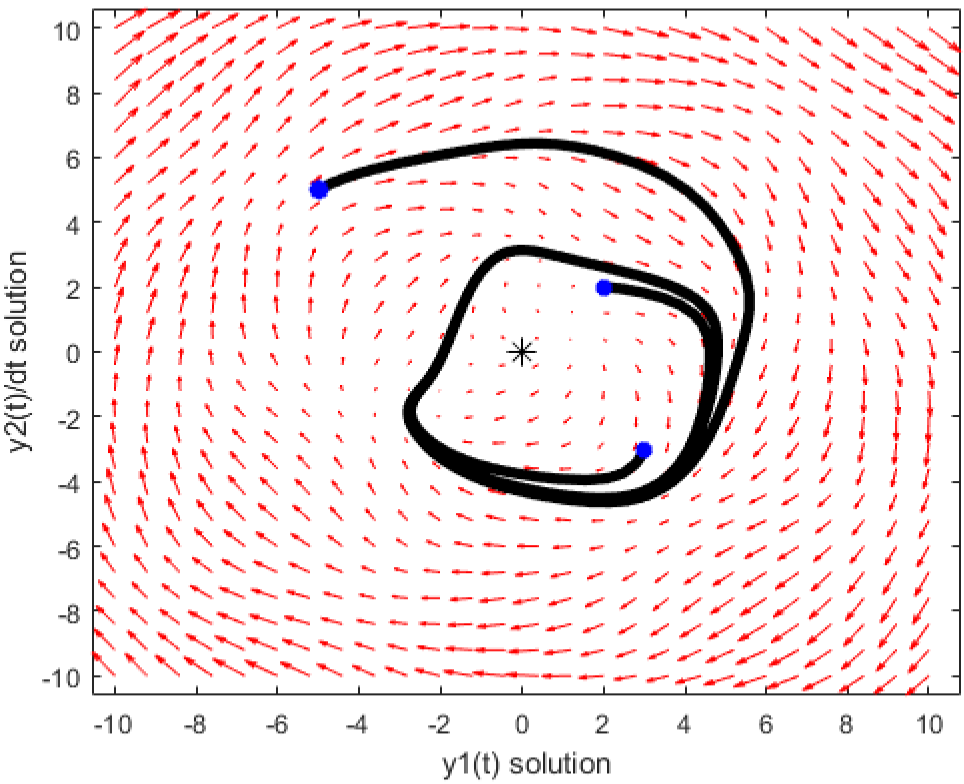


Figure 5.0.4: For negative perturbation we get sink spiral.

(b) If $\varepsilon > 0$, the origin becomes a source spiral point.

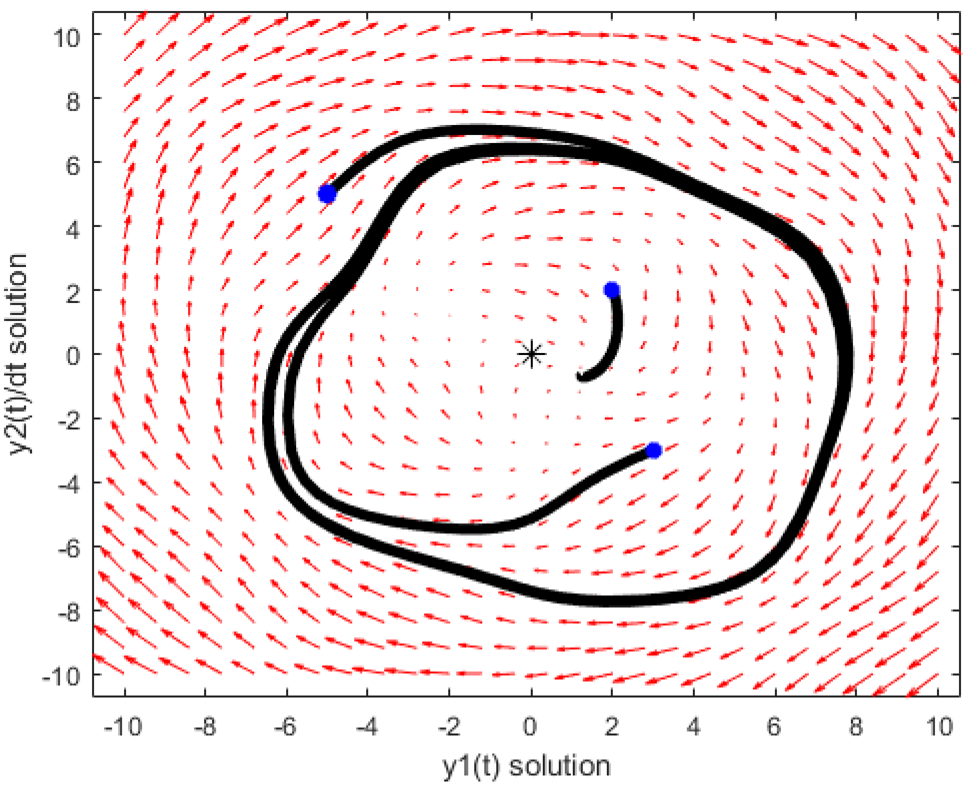


Figure 5.0.5: For positive perturbation we get source spiral.

(c) If $\varepsilon = 0$, the origin is a center of concentric circles.

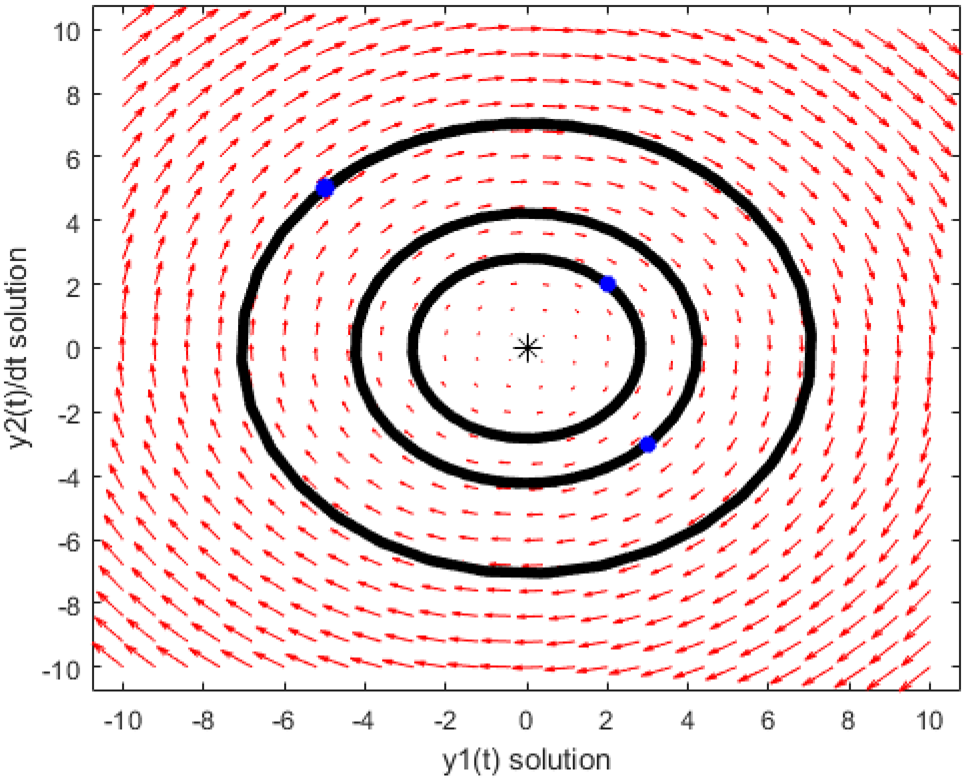


Figure 5.0.6: For zero perturbation we get circular behaviour.

If $\varepsilon = -0.1$ we get an almost circular picture:

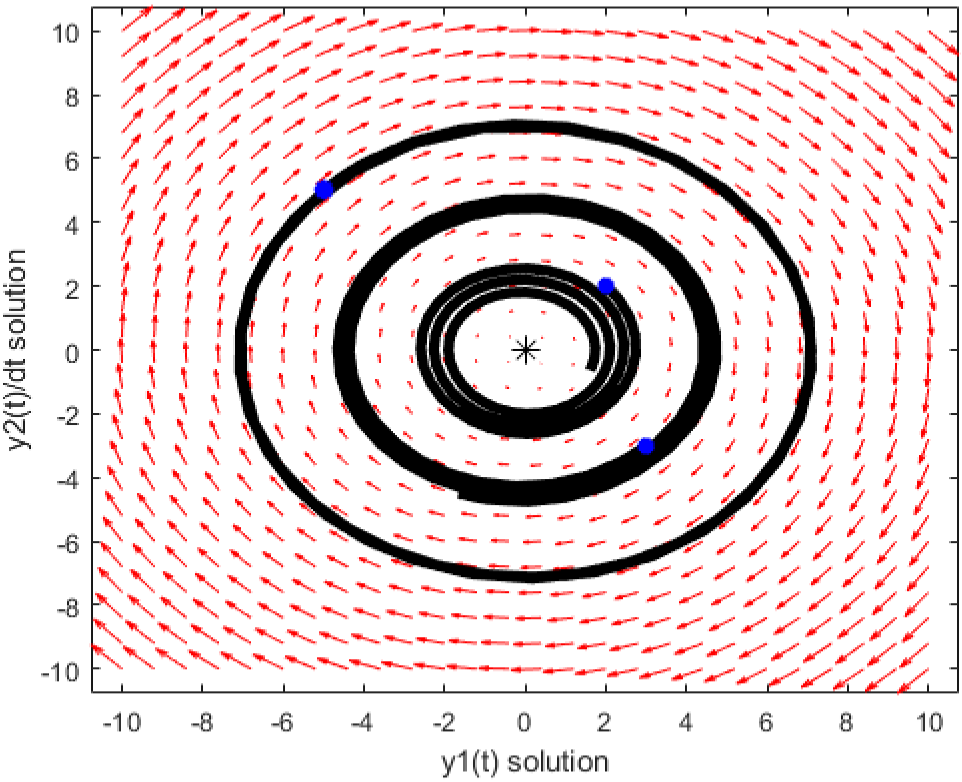


Figure 5.0.7: For small perturbation we get almost circular behaviour.

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + y^2, \frac{\mathrm{d}y}{\mathrm{d}t} = x + y.$$

1. First we find the critical points:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow$$
$$x = -y \text{ and } y(y-1) = 0 \Rightarrow (x,y) = (0,0), (-1,1).$$

2. We linearize around the origin:

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix} 1 & 2y \\ 1 & 1 \end{pmatrix}|_{(0,0)}\mathbf{x} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{x}\end{aligned}$$

3. The eigenpair is $(1, \binom{0}{1})$ and $\eta = k\binom{0}{1} + \binom{1}{0}$ and so the solution is:

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^t (t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

4. We linearize around the (-1,1):

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix}1 & 2y\\ 1 & 1\end{pmatrix}|_{(-1,1)}(\mathbf{x} - \begin{pmatrix}-1\\ 1\end{pmatrix})\\ &= \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1\\ 0\end{pmatrix}\end{aligned}$$

5. The eigenpairs are $(1+\sqrt{2}, \binom{\sqrt{2}}{1}), (1-\sqrt{2}, \binom{-\sqrt{2}}{1})$ and so the solution is:

$$\mathbf{x} = c_1 e^{(1+\sqrt{2})t}\begin{pmatrix}\sqrt{2}\\ 1\end{pmatrix} + c_2 e^{(1-\sqrt{2})t}\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix} + \begin{pmatrix}-1\\ 1\end{pmatrix}.$$

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - xy, \frac{\mathrm{d}y}{\mathrm{d}t} = x - y^3.$$

1. First we find the critical points:

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow\\ &1 = xy \text{ and } x = y^3 \Rightarrow (x, y) = (-1,-1), (1,1).\end{aligned}$$

2. We linearize around the critical point (-1,-1):

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix}-y & -x\\ 1 & 3y^2\end{pmatrix}|_{(-1,-1)}\left(\mathbf{x} - \begin{pmatrix}-1\\ -1\end{pmatrix}\right)\\ &= \begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\ 4\end{pmatrix}\end{aligned}$$

3. For solutions of the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{v}$ we have

$$\mathbf{x}_{nh} = \mathbf{A}^{-1}(-\mathbf{v}) = \begin{pmatrix}-1\\ -1\end{pmatrix}$$

4. The eigenpair is $(2+\sqrt{2}, \binom{-1+\sqrt{2}}{1}), (2-\sqrt{2}, \binom{-1-\sqrt{2}}{1})$ and so the general solution is:

$$\mathbf{x} = c_1 e^{(2+\sqrt{2})t}\begin{pmatrix}-1+\sqrt{2}\\ 1\end{pmatrix} + c_2 e^{(2-\sqrt{2})t}\begin{pmatrix}-1-\sqrt{2}\\ 1\end{pmatrix} + \begin{pmatrix}-1\\ -1\end{pmatrix}.$$

5. We linearize around the critical point (1,1):

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix}-y & -x\\ 1 & 3y^2\end{pmatrix}|_{(1,1)}\left(\mathbf{x} - \begin{pmatrix}1\\ 1\end{pmatrix}\right)\\ &= \begin{pmatrix}-1 & -1\\ 1 & 3\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\ -4\end{pmatrix}\end{aligned}$$

6. The eigenpair is $(1+\sqrt{3}, \binom{-2+\sqrt{3}}{1}), (1-\sqrt{3}, \binom{-2-\sqrt{3}}{1})$ and so the general solution is:
$$\mathbf{x} = c_1 e^{(1+\sqrt{3})t}\begin{pmatrix}-2+\sqrt{3}\\ 1\end{pmatrix} + c_2 e^{(1-\sqrt{3})t}\begin{pmatrix}-2-\sqrt{3}\\ 1\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}.$$

5.0.1 Applied Examples

Competing species

Suppose that in some closed environment there are two similar species competing for a limited food supply—for example, two species of fish in a pond that do not prey on each other but do compete for the available food. Let x and y be the populations of the two species at time t.

As discussed in Section 2.5, we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{dt}} &= x(\varepsilon_1 - \sigma_1 x)\\ \frac{\mathrm{d}y}{\mathrm{dt}} &= y(\varepsilon_2 - \sigma_2 y)\end{aligned}$$

However, when both species are present, each will tend to diminish the available food supply for the other. In effect, they reduce each other's growth rates and saturation populations:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{dt}} &= x(\varepsilon_1 - \sigma_1 x - \alpha_1 y)\\ \frac{\mathrm{d}y}{\mathrm{dt}} &= y(\varepsilon_2 - \sigma_2 y - \alpha_2 x)\end{aligned}$$

The α_1 is a measure of the degree to which species y interferes with species x and similarly for α_2. The values of the positive constants $\varepsilon_i, \sigma_i, \alpha_i$ depend on the particular species under consideration and, in general, must be determined from observation.

1. First we find the critical points

$$\begin{cases} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) = 0\\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) = 0\end{cases} \Longrightarrow (0,0),\ \left(\frac{\varepsilon_1}{\sigma_1}, 0\right),\ \left(0, \frac{\varepsilon_2}{\sigma_2}\right),\ \text{and}\ \left(\frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}\right).$$

For the last critical point to be a realistic steady state we require that both components be positive:

Case I: Both $\varepsilon_1\sigma_2 > \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 > \varepsilon_1\alpha_2$

which also imply $\sigma_1\sigma_2 > \alpha_1\alpha_2$. This happens if $\varepsilon_1 = \varepsilon_2 = 1$, $\sigma_1 = \sigma_2 = 2$, and $\alpha_1 = \alpha_2 = 1$.

Case II: Both $\varepsilon_1\sigma_2 < \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 < \varepsilon_1\alpha_2$

which also imply $\sigma_1\sigma_2 < \alpha_1\alpha_2$. This happens if $\varepsilon_1 = \varepsilon_2 = 1$, $\sigma_1 = \sigma_2 = 1$, and $\alpha_1 = \alpha_2 = 2$.

The *unrealistic* cases are:

Case III: $\varepsilon_1\sigma_2 > \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 < \varepsilon_1\alpha_2$

which also imply $\sigma_1\sigma_2 < \alpha_1\alpha_2$.This happens if $\varepsilon_1 = \varepsilon_2 = 1$, $\alpha_1 = \sigma_1 = 1$, and $\sigma_2 = \alpha_2 = 2$.

$$\text{Case IV: } \varepsilon_1\sigma_2 < \varepsilon_2\alpha_1 \text{ and } \varepsilon_2\sigma_1 > \varepsilon_1\alpha_2$$

which also imply $\sigma_1\sigma_2 < \alpha_1\alpha_2$. This happens if $\varepsilon_1 = \varepsilon_2 = 1$, $\sigma_1 = \alpha_1 = 2$, and $\alpha_2 = \sigma_2 = 1$.

2. We linearize the system by 2D-Taylor expanding

$$\mathbf{F}(x, y) = \begin{pmatrix} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{pmatrix}$$

around critical point (x_0, y_0):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{F}(x, y) = \mathbf{J}_{\mathbf{F}}(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(\|(x - x_0, y - y_0)\|)$$

$$= \begin{pmatrix} \varepsilon_1 - 2\sigma_1 x_0 - \alpha_1 y_0 & -\alpha_1 x_0 \\ & \\ -\alpha_2 y_0 & \varepsilon_2 - \alpha_2 x_0 - 2\sigma_2 y_0 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(\|(x - x_0, y - y_0)\|).$$

3. We determine the stability behaviour around each of the critical points.

   (a) At $(x_0, y_0) = (0, 0)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 \\ & \\ 0 & \varepsilon_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o(\|(x, y)\|).$$

   Therefore, the eigenvalues are $\lambda_1 = \varepsilon_1 > 0$, $\lambda_2 = \varepsilon_2 > 0$ and so the origin $(0, 0)$ is an unstable source node.

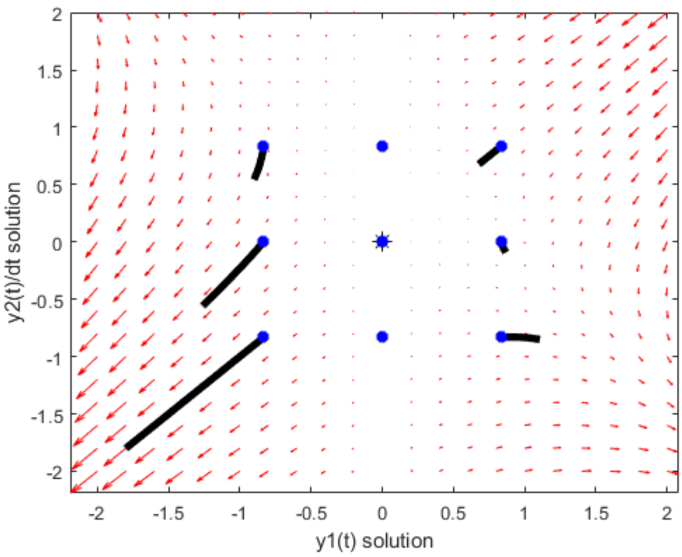


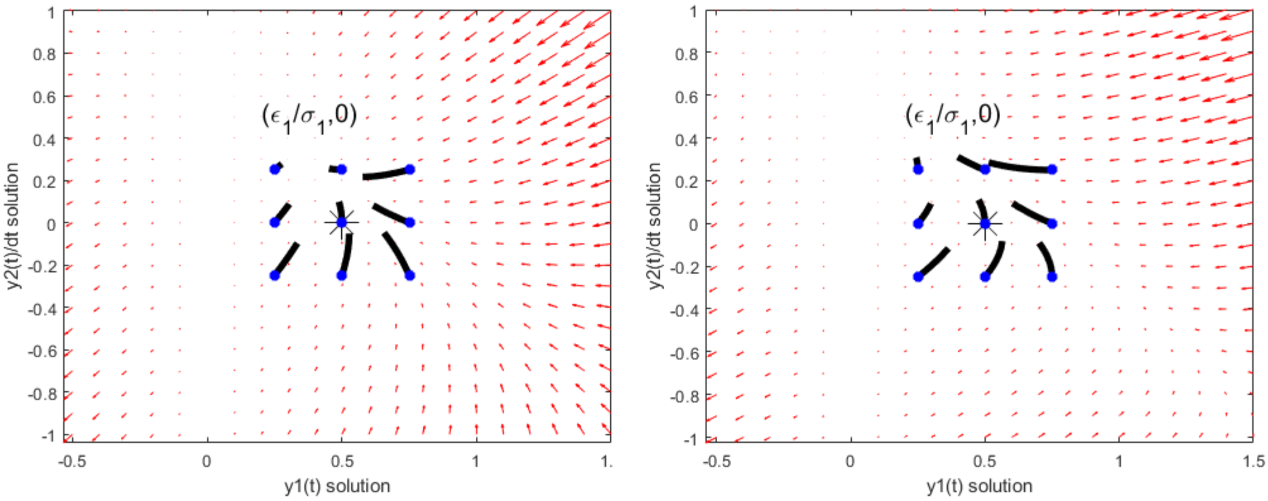
Figure 5.0.8: The origin is an unstable source node

(b) At $(x_0, y_0) = (\frac{\varepsilon_1}{\sigma_1}, 0)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\varepsilon_1 & \frac{-\alpha_1\varepsilon_1}{\sigma_1} \\ 0 & \varepsilon_2 - \frac{\alpha_2\varepsilon_1}{\sigma_1} \end{pmatrix}\begin{pmatrix} x - x_0 \\ y \end{pmatrix} + o(\|(x - x_0, y)\|).$$

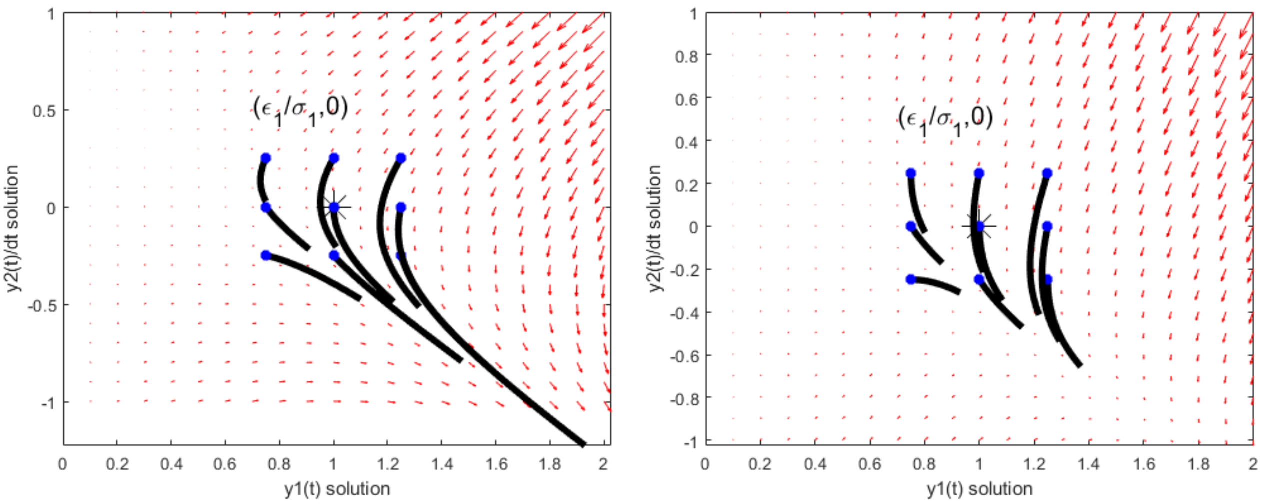
Therefore, the eigenvalues are $\lambda_1 = -\varepsilon_1 < 0$, $\lambda_2 = \frac{\sigma_1\varepsilon_2 - \alpha_2\varepsilon_1}{\sigma_1}$ and so

- in cases I and IV the point $(\frac{\varepsilon_1}{\sigma_1}, 0)$ is an unstable saddle node.



(a) Case I (b) Case IV

- in cases II and III the point $(\frac{\varepsilon_1}{\sigma_1}, 0)$ is an stable sink node.



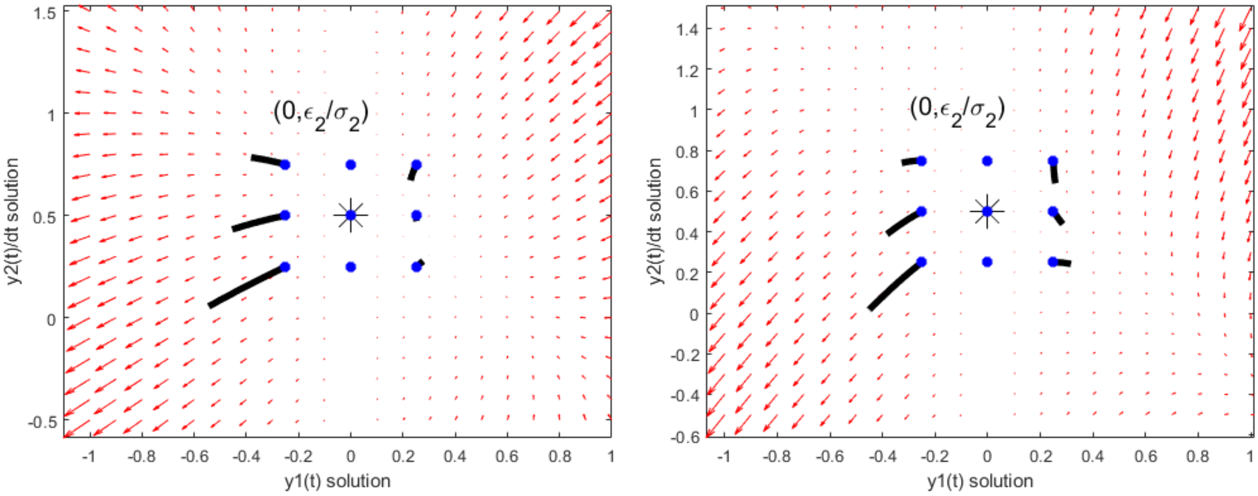
(a) Case II (b) Case III

(c) At $(x_0, y_0) = (0, \frac{\varepsilon_2}{\sigma_2})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1 - \frac{\alpha_1\varepsilon_2}{\sigma_2} & 0 \\ \frac{-\alpha_2\varepsilon_2}{\sigma_2} & -\varepsilon_2 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y \end{pmatrix} + o(\|(x, y - y_0)\|).$$

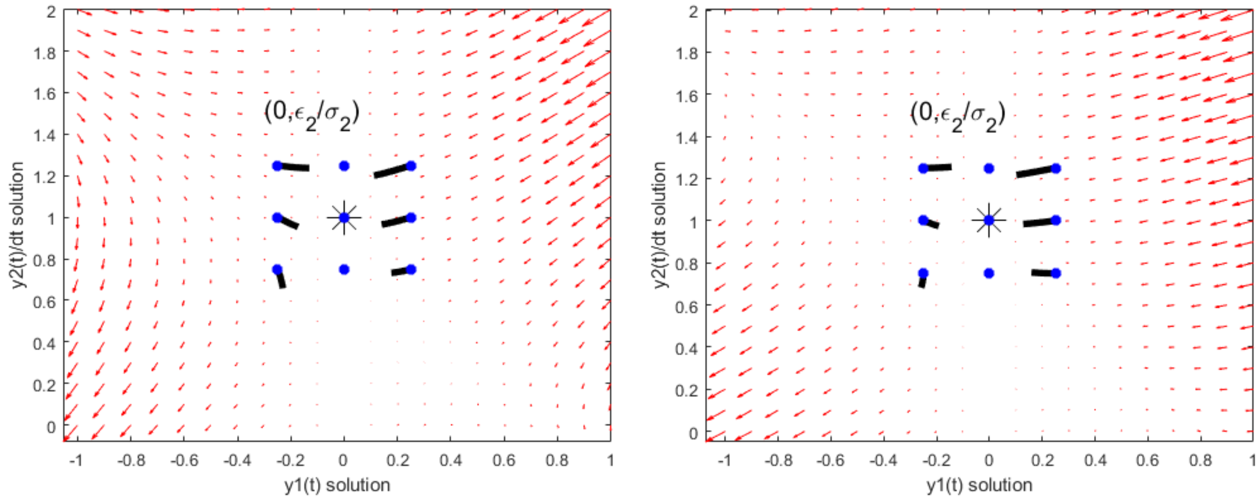
Therefore, the eigenvalues are $\lambda_1 = \frac{\varepsilon_1\sigma_2 - \alpha_1\varepsilon_2}{\sigma_2}$, $\lambda_2 = -\varepsilon_2 < 0$ and so

- in cases I and III the point $(0, \frac{\varepsilon_2}{\sigma_2})$ is an unstable saddle node.



(a) Case I (b) Case III

- in cases II and IV the point $(0, \frac{\varepsilon_2}{\sigma_2})$ is an stable sink node.



(a) Case II (b) Case IV

(d) At $(x_0, y_0) = (\frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_1 x_0 & -\alpha_1 x_0 \\ -\alpha_2 y_0 & \sigma_2 y_0 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y \end{pmatrix} + o(\|(x - x_0, y - y_0)\|).$$

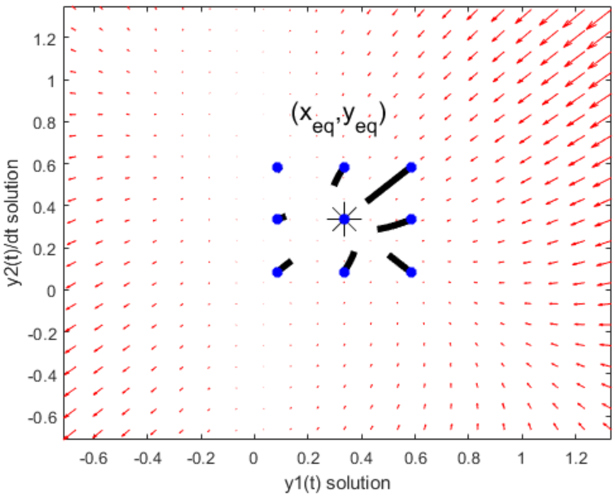
Therefore, the eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{-(\sigma_1 x_0 + \sigma_2 y_0) - \sqrt{(\sigma_1 x_0 + \sigma_2 y_0)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)x_0 y_0}}{2} \\ &= \frac{-(\sigma_1 x_0 + \sigma_2 y_0) - \sqrt{(\sigma_1 x_0 - \sigma_2 y_0)^2 + 4\alpha_1\alpha_2 x_0 y_0}}{2} \end{aligned}$$

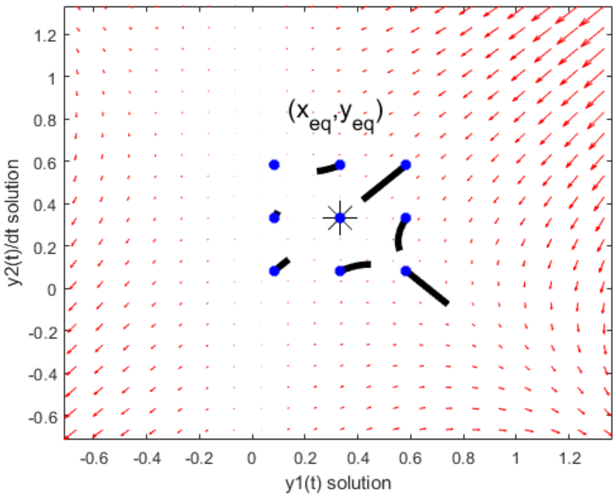
and

$$\lambda_2 = \frac{-(\sigma_1 x_0 + \sigma_2 y_0) + \sqrt{(\sigma_1 x_0 + \sigma_2 y_0)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)x_0 y_0}}{2}$$
$$= \frac{-(\sigma_1 x_0 + \sigma_2 y_0) + \sqrt{(\sigma_1 x_0 - \sigma_2 y_0)^2 + 4\alpha_1\alpha_2 x_0 y_0}}{2}.$$

- In cases I and II we have $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$. Thus, we observe that the radicand is positive and so the eigenvalues will always be real. Therefore, we get $\lambda_1 < 0$ and $\lambda_2 < 0$ and in turn (x_0, y_0) is stable sink node.

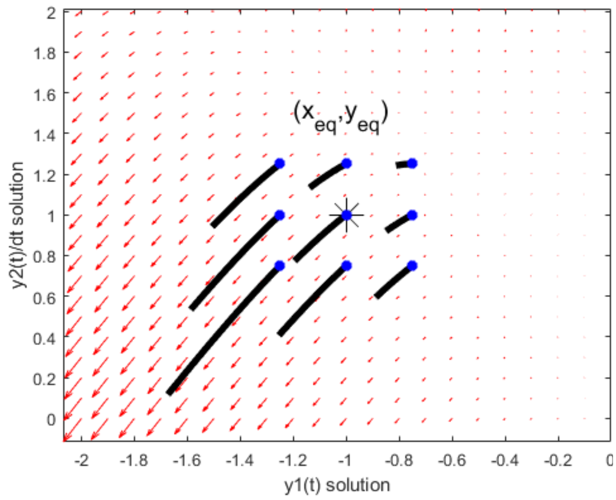


(a) Case I

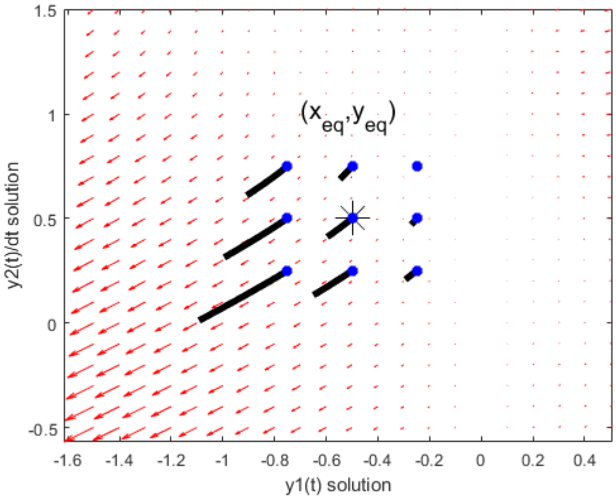


(b) Case II

- In cases III and IV the $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$ and so $\lambda_1 < 0, \lambda_2 > 0$. Thus, (x_0, y_0) is an unstable saddle node.



(a) Case III



(b) Case IV

4. We also do a nullcline analysis to predict how solutions will behave based on their initial data.

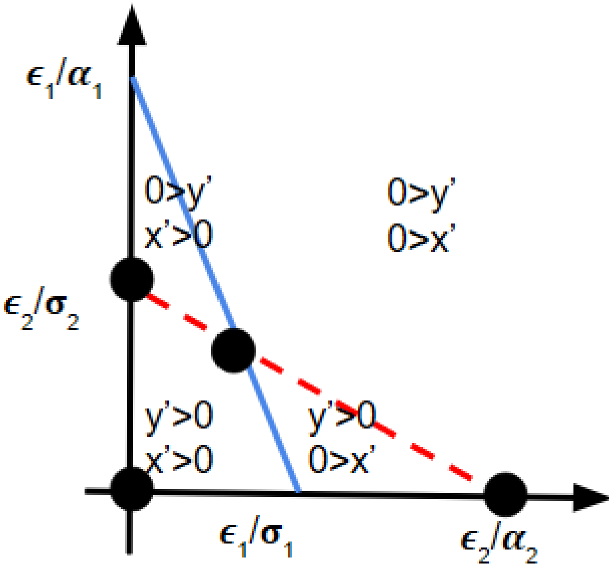


Figure 5.0.15: Case I

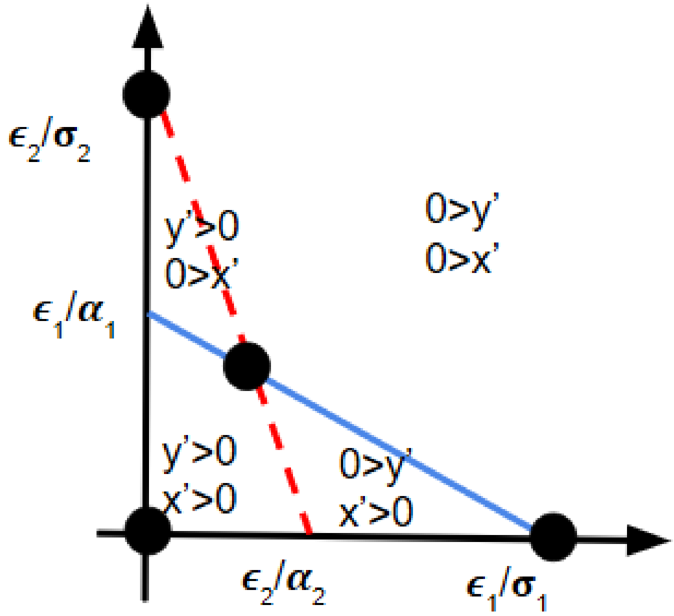


Figure 5.0.16: Case II

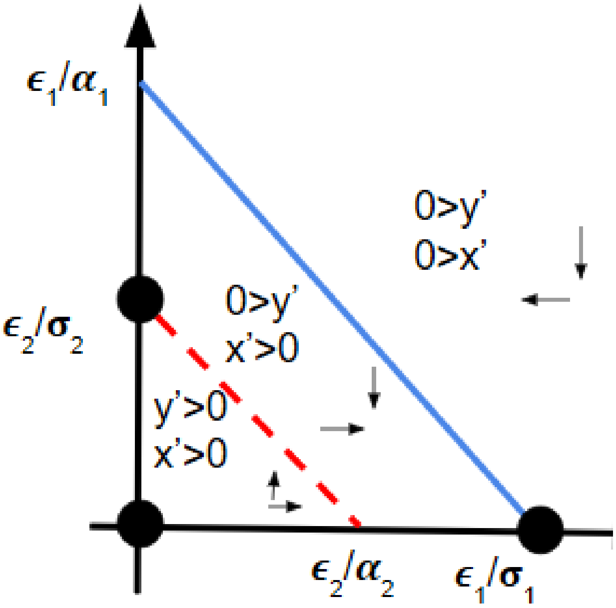


Figure 5.0.17: Case II

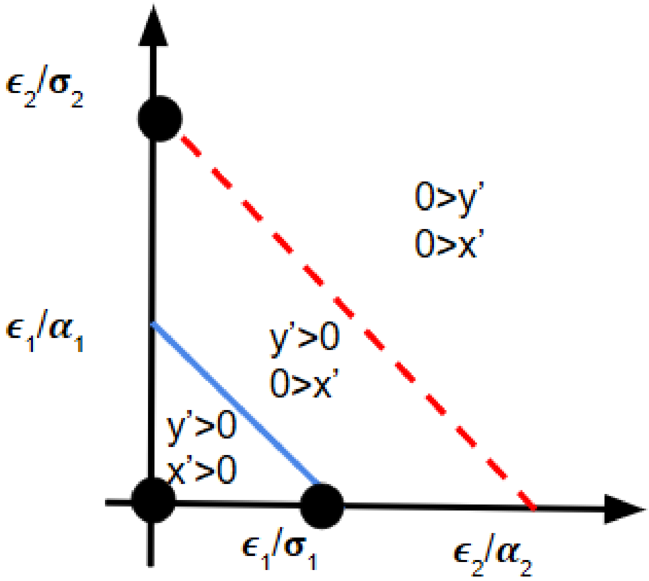


Figure 5.0.18: Case IV

(a) For simplicity lets start from case III and IV where the lines are well separated. First in case III we will show that all solutions tend to $(\frac{\varepsilon_1}{\sigma_1}, 0)$ (i.e. second species dies out).

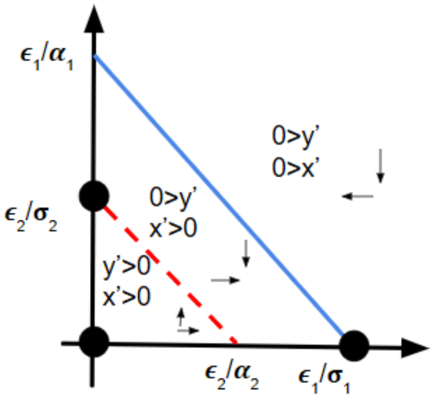


Figure 5.0.19: Case III

In the region where $0 > y', 0 > x'$ we will have the solutions flowing towards the left (left pointing arrow) and downwards (down pointing arrow). We similarly obtain the other arrows as shown in the figure. Therefore, solutions will escape the region where $y' > 0, x' > 0$, then once in the region $0 > y'x' > 0$ they will move south and rightwards till they hit the critical point $(\frac{\varepsilon_1}{\sigma_1}, 0)$.
The case IV similarly gives that $(0, \frac{\varepsilon_2}{\sigma_2})$ is the equilibrium point (i.e. first species dies out).

(b) Next we study case I. We will show that all solutions tend to the equilibrium point (both species coexist)

$$(x_0, y_0) = \Big(\frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}\Big).$$

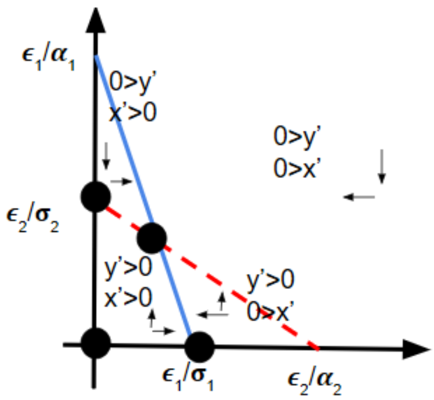


Figure 5.0.20: Case I

In the region where $0 > y'$, $0 > x'$ we will have the solutions flowing towards the left (left pointing arrow) and downwards (down pointing arrow). Solutions will escape the region where $y' > 0$, $x' > 0$, into either a) the region $0 > y'x' > 0$ in which they will move south and rightwards till they hit the equilibrium point or into b) the region $y' > 0$, $0 > x'$ in which they will move north and leftwards till they hit the equilibrium point. Case II similarly gives the same equilibrium point as the stable solution.

Price adjustment mechanism [SHS]

Consider the following system of differential equations:

$$p' = H_1(D_1(p, q) - S_1(p, q)), q' = H_2(D_2(p, q) - S_22(p, q)).$$

where p, q denote the prices of two different commodities with corresponding demand and supply D_i, S_i for $i = 1, 2$ and H_i are functions of one variable. Assume that $H_1(0) = H_2(0) = 0$ and that $H_1' > 0, H_2' > 0$.

Walras's law and the tâtonnement mechanism

Here, we consider the question of stability of a pure exchange, competitive equilibrium with an adjustment mechanism known as tâtonnement and directly inspired by the work of Léon Walras (1874), one of the founding fathers of mathematical economics.
The basic idea behind the tâtonnement mechanism is the same assumed in the rudimentary price adjustment mechanism models, namely that prices of commodities rise and fall in response to discrepancies between demand and supply (the so-called 'law of demand and supply'). In the present case, demand is determined by individual economic agents maximising a utility function subject to a budget constraint, given a certain initial distribution of stocks of commodities. The model can be described schematically as follows.

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{dt}} = \mathbf{f}(\mathbf{p}) = \begin{pmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \end{pmatrix},$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions with all their derivatives continuous as well.

1. A price point $\mathbf{p}_0$ is called an equilibrium if

$$\begin{aligned} &f_i(\mathbf{p}_0) \leq 0, p_i \geq 0,\text{and } p_j > 0\text{for some j} \\ &\text{or } f_i(\mathbf{p}_0) < 0, \mathbf{p}_0 = \mathbf{0}. \end{aligned}$$

   The first case makes economic sense (i.e. at least one price is nonzero) and so by *equilibrium point* we will mean the first case.

2. (Hypothesis **H**) The hypothesis that agents maximise utility is that the functions $f_i(\mathbf{p})$ are **homogeneous of degree zero**, namely $f_i(\lambda\mathbf{p}) = \lambda^0 f_i(\mathbf{p}) = f_i(\mathbf{p})$ for any $\lambda > 0$.

3. (**Walras's law**)Consider that the budget constraint for each individual k takes the form

$$\sum_{i=1}^{2} p_i f_i^k(\mathbf{p}) = p_1 f_1^k(\mathbf{p}) + p_2 f_2^k(\mathbf{p}) = 0,$$

   where f_i^k denotes the excess demand by the kth economic agent for the ith commodity, i.e., the difference between the agent's demand for, and the agent's initial endowment of, that commodity. In general for m commodities by summing over all N economic agents we have:

$$\sum_{k=1}^{N}\sum_{i=1}^{m} p_i f_i^k(\mathbf{p}) = \sum_i p_{i=1}^m f_i(\mathbf{p}) = 0.$$

   This law states that, in view of the budget constraints, for any set of semipositive prices p (not necessarily equilibrium prices), the value of aggregate excess demand, evaluated at those prices, must be zero.

4. The Jacobian matrix for $\mathbf{f}$ is

$$D\mathbf{f}(\mathbf{p}_0) = \begin{pmatrix} \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_2} \\ \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_2} \end{pmatrix}.$$

   We will need this matrix to be **Metzler**:

(a) Suppose that if the price of the ith commodity increases, while all the other prices remain constant, the excess demand for the ith commodity decreases (and vice versa). Suppose also that the effect of changes in the price of the ith commodity on its own excess demand is stronger than the combined effect of changes in the other prices (where the latter can be positive or negative). This can be formalised by assuming that

$$a_{ii} := \frac{\mathrm{d}f_i(\mathbf{p}_0)}{\mathrm{dp_i}} < 0$$

and the "strict diagonal dominance" (SDD) assumption that there exists a positive vector $d \in \mathbb{R}^m$ (in our case $m = 2$) s.t.

$$|a_{ii}|d_i > \sum_{j=1}^{3mm} j \neq i^m |a_{ij}|d_j.$$

(b) Moreover, we have the "gross substitutability" (GS) assumption that if we start from equilibrium and the price of a commodity increases (decreases) while the prices of all other commodities remain constant, then the excess demand of all of the other commodities increases (decreases):

$$a_{ij} := \frac{\mathrm{d}f_i(\mathbf{p}_0)}{\mathrm{dp_j}} > 0, i \neq j.$$

5. The eigenvalues for this system are:

$$\begin{aligned}\lambda &= \frac{a_{11}+a_{22}}{2} \pm \frac{1}{2}\sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \\ &= \frac{-|a_{11}+a_{22}|}{2} \pm \frac{1}{2}\sqrt{|(a_{11}-a_{22})^2 + 4a_{12}a_{21}|}\end{aligned}$$

5.1 Simulation code

Encoding the system If we start with the 2D system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = f_1(x,y,t) \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = f_2(x,y,t),$$

we first encode it as a function as follows:

```
f = @(t,y) [f_{1}(y(1),y(2),t);f_{2}(y(1),y(2),t)]; }
```

where $y(1) = x$ and $y(2) = y$. For example, for

$$\frac{\mathrm{d}x}{\mathrm{dt}} = x^2 + y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = \sin(y)$$

we write

```
f = @(t,y) [y(1)^2+y(2);sin(y(2))]; }
```

Direction field The following matlab-function will generate the direction field for the above function f:

```
function vectfield(func,y1val,y2val,t)
if nargin==3
  t=0;
end
```

```
n1=length(y1val);
n2=length(y2val);
yp1=zeros(n2,n1);
yp2=zeros(n2,n1);
for i=1:n1
  for j=1:n2
    ypv = feval(func,t,[y1val(i);y2val(j)]);
    yp1(j,i) = ypv(1);
    yp2(j,i) = ypv(2);
  end
end
quiver(y1val,y2val,yp1,yp2,'r','Autoscalefactor', 3)%'MaxHeadSize');
axis tight;
```

This is an example of calling it for the above example $\mathbf{F}(x, y) = (x^2 + y, \sin(y))$:

```
f = @(t,y) [y(1)^2+y(2);y(2)];
 vectfield(f,-1:0.1:1,-1:0.1:1);

```

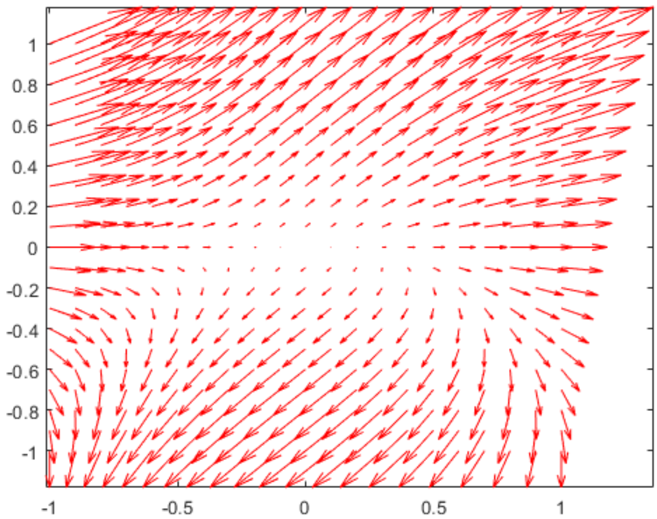


Figure 5.1.1: Direction field for system $\frac{dx}{dt} = x^2 + y$ and $\frac{dy}{dt} = \sin(y)$.

Solving the ODE system Given function $\mathbf{f}$ and initial data (x_0, y_0), the following ODE solver outputs a two dimensional solution run up to time T:

```
[ts,ys] = ode45(f,[0,T],[x0;y0]);
```

This is an example for the above function

```
x0=0.5;
y0=0.5;
T=4
f = @(t,y) [y(1)^2+y(2);y(2)];
vectfield(f,-1:0.1:1,-1:0.1:1);
hold on

plot(x0,y0,'o','MarkerFaceColor','b','MarkerSize',20)

hold on;
[ts,ys] = ode45(f,[0,T/10],[x0;y0]);
 plot(ys(:,1),ys(:,2),'k','Linewidth', 4)
xlabel('y1(t) solution')
    ylabel('y2(t)/dt solution')
```

that outputs:

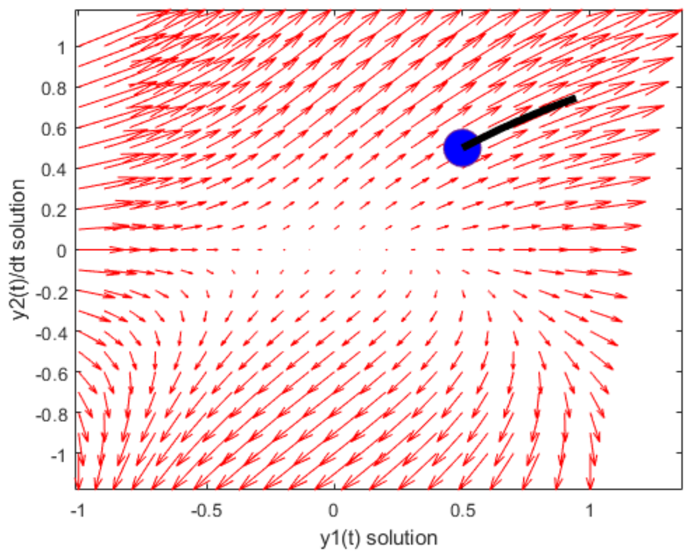


Figure 5.1.2: Solution curve for system $\frac{dx}{dt} = x^2 + y$ and $\frac{dy}{dt} = \sin(y)$.

Plotting implicit solutions If we obtain the implicit solutions $f(x, y) =$ constant then we can plot them with

```
fimplicit(@(x,y) f(x,y)=k,'LineWidth',1,'Color','k')
```

For example, for $x^2 + y^2 - \sin(y) =$ constant the following program

```
for k=1:10
f=@(x,y)x^2+y^2-sin(y)-k;
fimplicit(f,'LineWidth',1,'Color','k')
end
```

outputs

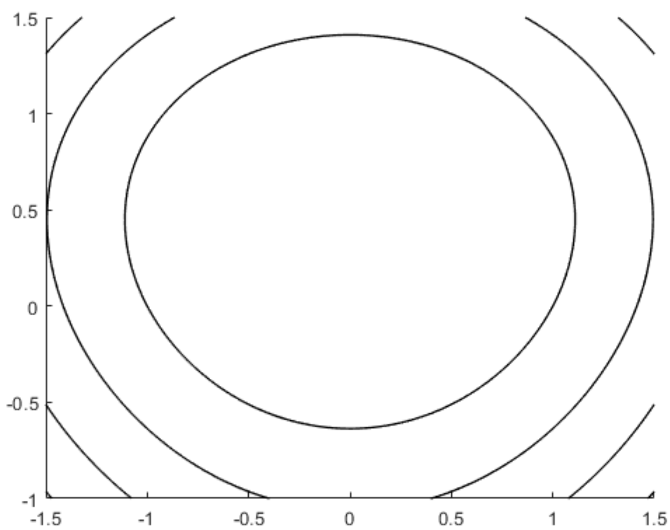


Figure 5.1.3: trajectories of implicit solutions.

Plotting eigenvectors for linear system Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

First we find the eigenvalues

```
[V,D]=eig(A);
```

The following code will generate linear spans for the eigenvectors and plot them:

```

A=[1 1; 0 2]
[V,D]=eig(A);

m1=1; %%% This two parameters scale the length of the line
m2=1;

xi11 = [-m1* V(1,1), m1*V(1,1)];
xi21 = [-m1*V(2,1), m1*V(2,1)];
pl1 = line(xi11,xi21,'Color','k','LineWidth',2);

x1 = V(1,1)/3;
y1 =V(2,1);
txt1 = ' \xi_1 ';
text(x1,y1,txt1,'FontSize',20)
hold on

xi12 = [-m2* V(1,2), m2*V(1,2)];
xi22 = [-m2*V(2,2), m2*V(2,2)];

pl2 = line(xi12,xi22,'Color','k','LineWidth',2);

x2 =  V(1,2)/3;
y2 =V(2,2)/3;
txt1 = '\xi_2 ';
text(x2,y2,txt1,'FontSize',20)
```

This is the full example with both the direction field and the eigenvector spans:

```
x0=0.5;
y0=0.5;
T=4

f = @(t,y) [y(1);2*y(2)];
vectfield(f,-1:0.1:1,-1:0.1:1);
hold on

plot(x0,y0,'o','MarkerFaceColor','b','MarkerSize',20)

hold on;
[ts,ys] = ode45(f,[0,T/10],[x0;y0]);
 plot(ys(:,1),ys(:,2),'k','Linewidth', 4)
hold on;

A=[1 1; 0 2]
[V,D]=eig(A);



m1=1;
m2=1;


xi11 = [-m1* V(1,1), m1*V(1,1)];
xi21 = [-m1*V(2,1), m1*V(2,1)];
pl1 = line(xi11,xi21,'Color','k','LineWidth',2);

x1 = V(1,1)/3;
y1 =V(2,1);
txt1 = ' \xi_1 ';
text(x1,y1,txt1,'FontSize',20)

```

```

hold on

xi12 = [−m2∗ V(1,2), m2∗V(1,2)];
xi22 = [−m2∗V(2,2), m2∗V(2,2)];

pl2 = line(xi12,xi22,'Color','k','LineWidth',2);

x2 =  V(1,2)/3;
y2 =V(2,2)/3;
txt1 = '\xi_2 ';
text(x2,y2,txt1,'FontSize',20)
```

It outputs the following figure:

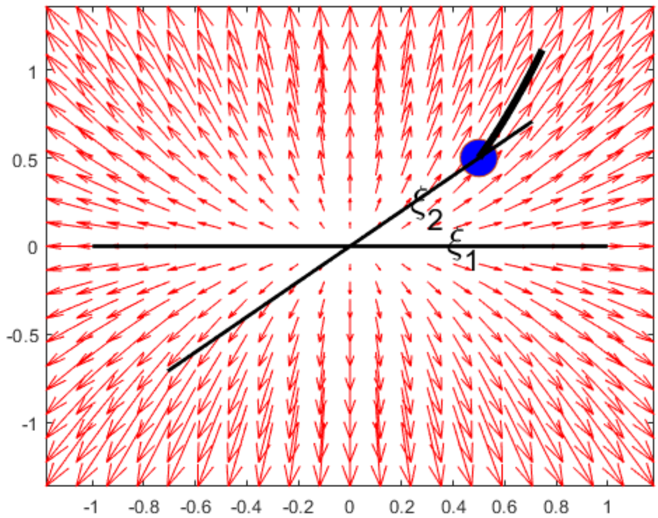


Figure 5.1.4: Eigenvectors spans and direction field

Chapter 6

Liapunov's Second Method

Consider the autonomous system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x,y) \text{ and } \frac{\mathrm{d}x}{\mathrm{dt}} = G(x,y).$$

We will obtain a criterion for concluding asymptotic stability and even determining the basin of attraction.

6.0.1 Example-presenting the method

We return to the damping-free pendulum

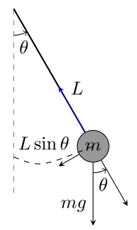


Figure 6.0.1: oscillating pendulum

whose angle θ satisfies the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \frac{mg}{L} sin(\theta) = 0.$$

This is a nonhomogeneous second order equation, but we can also view it as a system of equations by letting $x := \theta$ and $y := \frac{\mathrm{d}\theta}{\mathrm{dt}}$:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\frac{g}{L} sin(x).$$

1. Consider the total energy of the system:

$$\begin{aligned} E(x,y) &= \text{Potential } + \text{ Kinetic} \\ &= U(x,y) + K(x,y) \\ &:= mgL(1 - cos(x)) + \frac{1}{2} mL^2 y^2. \end{aligned}$$

2. Since the system is damping-free, the energy is conserved and so we should have:

$$\frac{\mathrm{d}V}{\mathrm{dt}} = 0.$$

Lets prove this:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{dt}} &= \frac{\mathrm{d}}{\mathrm{dt}}[mgL(1-cos(x)) + \frac{1}{2}mL^2y^2] \\ &= mgLsin(x)\frac{\mathrm{d}x}{\mathrm{dt}} + mL^2y\frac{\mathrm{d}y}{\mathrm{dt}}\end{aligned}$$

using the equations

$$\begin{aligned}&= mgLsin(x)y + mL^2y(-\frac{g}{L}sin(x)) \\ &= 0.\end{aligned}$$

3. Therefore, we obtain an implicit solution:

$$mgL(1-cos(x)) + \frac{1}{2}mL^2y^2 = constant.$$

4. Next we consider the case where there is damping i.e. θ satisfies the equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \gamma\frac{\mathrm{d}\theta}{\mathrm{dt}} + \omega^2 sin(\theta) = 0$$

and so the system is:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \frac{g}{L}sin(x).$$

5. By computing the time derivative of the total energy we obtain:

$$\frac{\mathrm{d}V}{\mathrm{dt}} = -mL^2\gamma y^2 = -mL^2\gamma(\dot{\theta})^2 \leq 0.$$

6. Physically this means that the energy will be decreasing over time as the damping force keeps slowing down the pendulum. Therefore, we expect that the system will be asymptotically stable towards the origin, where both the angle and the velocity are zero.

6.0.2 Method formal steps

For the autonomous system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x,y) \text{ and } \frac{\mathrm{d}x}{\mathrm{dt}} = G(x,y)$$

let (x_0, y_0) denote a critical point. We will need the following definitions:

Definition 6.0.1. A point (x_0, y_0) is *Lyapunov-stable* if a solution that starts close to it, then it will stay close to that critical point for all future time. Given any desired $\varepsilon > 0$ we can find $\delta > 0$ s.t. if we start $\delta-$close

$$\|\mathbf{x}(0) - (x_0, y_0)\| \leq \delta,$$

then we stay $\varepsilon-$close:

$$\|\mathbf{x}(t) - (x_0, y_0)\| \leq \varepsilon, \forall t > 0.$$

A point (x_0, y_0) is *asymptotically stable* if a solution that starts close to it converges to that critical point:

$$\|\mathbf{x}(t) - (x_0, y_0)\| \to 0.$$

1. Find the "total energy" V of the autonomous system by solving:

$$\begin{aligned}\frac{\mathrm{d}V(x(t), y(t))}{\mathrm{ddt}} &= V_x \frac{\mathrm{d}x}{\mathrm{dt}} + V_y \frac{\mathrm{d}y}{\mathrm{dt}} \\ &= V_x F(x(t), y(t)) + V_y G(x(t), y(t)).\end{aligned}$$

2. If $\frac{\mathrm{d}V(x(t),y(t))}{\mathrm{ddt}} = 0$, then use to find the implicit solutions:

$$constant = V(x(t), y(t)).$$

3. If V satisfies the following conditions:

   - $V(x_0, y_0) = 0$,
   - $V(x, y) > 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0),
   - and it is nondecreasing in time $\frac{\mathrm{d}V}{\mathrm{dt}} \leq 0$ for all other $(x, y) \neq (x_0, y_0)$ in a disk around (x_0, y_0),

   then

   $$(x_0, y_0) \text{ is a Lyapunov-stable critical point.}$$

4. If instead we have

   - $V(x_0, y_0) = 0$,
   - $V(p) > 0$ for at least one point $p \neq (x_0, y_0)$ in a disk around (x_0, y_0),
   - and strictly decreasing energy $\frac{\mathrm{d}V}{\mathrm{dt}} < 0$ for all other $(x, y) \neq (x_0, y_0)$,

   then

   $$(x_0, y_0) \text{ is an asymptotically stable critical point.}$$

6.0.3 Finding the Lyapunov function

So clearly finding such a scalar function is the first serious obstacle. Here are some ideas and heuristics on guessing such a function:

Physical systems For physical systems the energy/Hamiltonian is a good guess. For example, if there is no new energy input, then the energy function will decay over time or remain constant.

Lur'e type systems Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -y - h_1(x) \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = h_2(x),$$

h_1 is differentiable and h_2 integrable. We will obtain a Lyapunov function for this type of system.By taking t-derivative of the first equation and using the second one we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}^2 \mathrm{t}} = -h_2(x) - h_1'(x) \frac{\mathrm{d}x}{\mathrm{dt}}$$

multiplying by $\frac{\mathrm{d}x}{\mathrm{dt}}$ we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}^2\mathrm{t}}\frac{\mathrm{d}x}{\mathrm{dt}} = -h_2(x)\frac{\mathrm{d}x}{\mathrm{dt}} - h_1'(x)(\frac{\mathrm{d}x}{\mathrm{dt}})^2 \Rightarrow$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(\frac{(\frac{\mathrm{d}x}{\mathrm{dt}})^2}{2} + \int_0^x h_2(s)ds\Big) = -h_1'(x)(\frac{\mathrm{d}x}{\mathrm{dt}})^2.$$

Thus, if we have $h_1'(x) > 0$ in a neighbourhood of x_0, then the function

$$V(x) := \frac{(\frac{\mathrm{d}x}{\mathrm{dt}})^2}{2} + \int_0^x h_2(s)ds$$

has a strictly negative derivative. Moreover, if $\int_0^x h_2(x)$

Cost functions Distance and cost functions with respect to the critical point (x_0, y_0) are also good guesses because close to the critical point the derivatives $\frac{\mathrm{d}x}{\mathrm{dt}}, \frac{\mathrm{d}y}{\mathrm{dt}}$ are decaying to zero and so we might indeed have:

$$V_x\frac{\mathrm{d}x}{\mathrm{dt}} + V_y\frac{\mathrm{d}y}{\mathrm{dt}} \leq 0.$$

Linear systems When the system is linear i.e. $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ then a candidate Lyapunov function is

$$V(\mathbf{x}) = \int_0^\infty \|e^{At}\mathbf{x}\|dt = \mathbf{x}^T\Big(\int_0^\infty e^{(A^T+A)t}dt\Big).$$

This is indeed a Lyapunov function for *exponentially stable systems* (see more details in the converse theorems 6.0.3)

Polynomial systems When f_1, f_2 are polynomials of highest degree m, then there are many algorithms for generating the corresponding Lyapunov functions of the form

$$V(x,y) = \sum_{j,k}^{m+1} c_{j,k}x^j y^k,$$

by optimizing over the coefficients (see [**giesl2015review**] for the sum-of-squares (SOS) theory).

6.0.4 General result:

Lyapunov's second method

Theorem 6.0.2. *Consider system*

$$\frac{\mathrm{d}x}{\mathrm{dt}} = f_1(x,y) \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = f_2(x,y),$$

and (x_0, y_0) a particular isolated equilibrium point. If we can find a continuously differentiable function $V : U_{(x_0,y_0)} \to \mathbb{R}$ around some neighbourhood $U_{(x_0,y_0)}$ of the critical point (x_0, y_0) with the following properties:

1. *$V(x_0, y_0) = 0$,*

2. *$V(x,y) > 0$ for $(x,y) \in U_{(x_0,y_0)} \setminus \{(x_0,y_0)\}$,*

3. and $\frac{\mathrm{d}V}{\mathrm{dt}} := \frac{\mathrm{d}V}{\mathrm{dx}} f_1 + \frac{\mathrm{d}V}{\mathrm{dy}} f_2 \leq 0$ *in punctured neighbourhood* $U_{(x_0,y_0)} \setminus \{(x_0, y_0)\}$

then the critical point (x_0, y_0) *is stable. In fact if we replace the last condition by strict inequality:*

$$\frac{\mathrm{d}V}{\mathrm{dx}} f_1 + \frac{\mathrm{d}V}{\mathrm{dy}} f_2 < 0$$

then critical point (x_0, y_0) *is asymptotically stable.*

Converse theorems: existence of Lyapunov function

Theorem 6.0.3. *Suppose that the linearization around an equilibrium point is*

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

s.t. $det(A) = ad - bc > 0$ *and* $a + d < 0$*. Then the function*

$$V(x, y) = Ax^2 + Bxy + Cy^2,$$

is a Lyapunov function for this system with $\dot{V} < 0$ *if*

$$\begin{aligned} A &= -\frac{c^2 + d^2 + det(A)}{2Tr(A)det(A)} \\ B &= \frac{bd + ac}{Tr(A)det(A)} \\ C &= -\frac{a^2 + b^2 + det(A)}{2Tr(A)det(A)} \end{aligned}$$

because then $\dot{V}(x, y) = -x^2 - y^2$ *and the matrix*

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

is positive definite.

6.0.5 Examples

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = -x - y.$$

  1. First we find that the equilibrium point is only the origin $(0, 0)$.
  2. We take the Euclidean distance function as a guess:

$$V(x, y) = \frac{1}{2}(x^2 + y^2).$$

  3. Next we check each of the properties
     - $V(0, 0) = 0$,

- $V(x, y) > 0$ for $(x, y) \neq 0$,
- and

$$\frac{\mathrm{d}V}{\mathrm{d}t} = V_x f_1 + V_y f_2 = (x)y + (y)(-x - y) = -y^2 < 0$$

for (x, y) in a punctured disk centered at the origin.

4. Therefore, $V(x, y)$ is a Lyapunov function and so the point (x_0, y_0) is asymptotically stable.

- Consider the linear harmonic oscillator

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = -kx,$$

for $k > 0$, with $V = Pot + Kin = \frac{1}{2}kx^2 + \frac{1}{2}y^2$ as its candidate Lyapunov function.

1. First, we check that we indeed have a Lyapunov function.
   - We indeed have $V(0, 0) = \frac{1}{2}k0 + \frac{1}{2}0 = 0$.
   - We have $V(x, y) = \frac{1}{2}kx^2 + \frac{1}{2}y^2 > 0$ for $(x, y) \neq (0, 0)$.
   - Finally, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = V_x\dot{x} + V_y\dot{y} = kxy - kxy = 0.$$

2. So the origin will be an asymptotically stable point (solutions that start close, remain close).
3. Indeed from linearization around the origin we obtain:

$$J_F = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix},$$

which has eigenvalues $\lambda = \pm i\sqrt{k}$ and so the phase portrait will be concentric circles centered at the origin. This agrees with our Lyapunov behaviour because solutions that start in a circle close to the origin, stay on that circle for all future time.

- Consider the linear harmonic oscillator with damping

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = -kx - \alpha y^3(1 + x^2),$$

for $k > 0$, with the same $V = Pot + Kin = \frac{1}{2}kx^2 + \frac{1}{2}y^2$ as its candidate Lyapunov function.

1. First, we check that we indeed have a Lyapunov function.
   - The first two conditions are the same.
   - Finally, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = V_x\dot{x} + V_y\dot{y} = kxy - kxy - \alpha y^4(1 + x^2) = -\alpha y^4(1 + x^2).$$

2. If $\alpha > 0$ (i.e. the is positive damping removing energy from the system), then $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$ and so the origin will be an asymptotically stable point (solutions that start close, remain close).
3. The linearization around the origin is:

$$J_F = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix},$$

which again has eigenvalues $\lambda = \pm i\sqrt{k}$ and so the phase portrait will be concentric circles centered at the origin.

4. For $\alpha > 0$ the behaviour is more complicated and we will explain it later.

- Consider the system
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 4y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = -x - y^3,$$
with $V = ax^2 + by^2$ as its candidate Lyapunov function.

  1. First, we check that we indeed have a Lyapunov function.
     - We indeed have $V(0,0) = 0$ and $V(x,y) > 0$ for $(x,y) \neq (0,0)$ if $a, b > 0$.
     - We have
$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= V_x\dot{x} + V_y\dot{y} \\ &= 2ax(-x+4y) + 2by(-x-y^3) \\ &= -2ax^2 + xy(8a-2b) - 2by^4,\end{aligned}$$
to make this strictly negative we set $a = 1, b = 4$ to get
$$= -2x^2 - 8y^4 < 0.$$
  2. So the origin will be a stable point (solutions converge to the origin).
  3. Indeed from linearization we obtain:
$$J_F = \begin{pmatrix} -1 & 4 \\ -1 & 0 \end{pmatrix},$$
which has repeated eigenvalue $\lambda = -1$ and so the solution will converge to the origin for any initial data.

- Consider the system:
$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\mathbf{x},$$
for Lyapunov function $V = \frac{1}{2}x^2 - \frac{2}{3}xy + \frac{7}{12}y^2$.

  1. At the origin we indeed have V(0,0)=0.
  2. Next we prove that $V > 0$. The goal is to complete the square. A quick formula for any monomial is
$$x^2 + bx + c = (x + \frac{1}{2}b)^2 + c - \frac{b^2}{4}.$$
So if we have $k > 0$ we are done. Indeed
$$c - \frac{b^2}{4} = y^2(\frac{7}{6} - \frac{4}{9}) > 0.$$
  3. Next we check the sign of $\dot{V}$. We have
$$\dot{V} = V_x\dot{x} + V_y\dot{y} = x^2 + y^2 > 0.$$
  4. So it says that the system is unstable. Indeed its eigenvalues are $1, 2$ and so the origin is a source (i.e. the initial data matters because for initial data it stays trapped in the origin).

- Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{x},$$

for Lyapunov function $V = \frac{1}{2}x^2 - xy + \frac{3}{2}y^2$.

1. As above in order to complete the square we find the sign of

$$c - \frac{b^2}{4} = y^2(\frac{1}{2} - \frac{1}{4}) = \frac{y^2}{4}.$$

2. Next we find the sign of $\dot{V}$:

$$\begin{aligned} \dot{V} &= V_x\dot{x} + V_y\dot{y} \\ &= (x-y)(x+2y) + (-x+3y)y \\ &= x^2 + xy(2-1-1) + y^2 \qquad\qquad = x^2 + y^2 > 0. \end{aligned}$$

3. So it says that the origin is unstable. Indeed the eigenvalues are 1,1 and so the origin is a source (i.e. for zero initial data it gets trapped whereas for nonzero initial data it moves to infinity and so the initial data is relevant).

- Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 2y + y^4, \frac{\mathrm{d}x}{\mathrm{d}t} = -y + x^4$$

for Lyapunov function $V = \frac{1}{2}x^2 + xy + \frac{3}{2}y^2$.

1. First we check the sign of V. In completing the square we find the sign of

$$c - \frac{b^4}{2} = y^2\frac{1}{4} > 0.$$

2. Next we check the sign of $\dot{V}$:

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = -x^2 - y^2 - (x+y)y^4 - (\frac{3x}{2} - 2y)x^4.$$

For (x, y) close to zero we have $x^4 << x^2, y^4 << y^2$ and so we indeed have $\dot{V} < 0$.
3. This agrees with the linearization since the eigenvalues will be the repeated -1, which makes the origin a source.

- Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 2y + y^4, \frac{\mathrm{d}x}{\mathrm{d}t} = 2y - 2x + x^4$$

for Lyapunov function $V = 4x^2 - 3xy + \frac{7}{4}y^2 = 4(x^2 - \frac{3}{4}xy + \frac{7}{16}y^2)$.

1. First we check the sign of V. In completing the square we find the sign of

$$c - \frac{b^4}{2} = y^2(\frac{7}{16} - \frac{9}{4 * 16}) > 0.$$

2. Next we check the sign of $\dot{V}$:

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = x^2 + y^2 - (8x - 3y)y^4 - (-3x + \frac{7}{2}y)x^4.$$

For (x, y) close to zero we have $x^4 << x^2, y^4 << y^2$ and so we indeed have $\dot{V} > 0$.

3. This agrees with the linearization since the eigenvalues will be the repeated $-1, 2$, which makes the origin an unstable saddle.

6.0.6 Applied examples

Walras's law and the *ttonnement* mechanism

Here, we consider the question of stability of a pure exchange, competitive equilibrium with an adjustment mechanism known as t^atonnement and directly inspired by the work of $LonWalras$ (1874), one of the founding fathers of mathematical economics.
The basic idea behind the t^atonnement mechanism is the same assumed in the rudimentary price adjustment mechanism models, namely that prices of commodities rise and fall in response to discrepancies between demand and supply (the so-called 'law of demand and supply').
In the present case, demand is determined by individual economic agents maximising a utility function subject to a budget constraint, given a certain initial distribution of stocks of commodities. The model can be described schematically as follows.

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{dt}} = \mathbf{f}(\mathbf{p}) = \begin{pmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \end{pmatrix},$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions with all their derivatives continuous as well.

1. A price point $\mathbf{p}_0$ is called an equilibrium if

$$\begin{aligned} &f_i(\mathbf{p}_0) \leq 0, p_i \geq 0, \text{and } p_j > 0 \text{for some j} \\ &\text{or } f_i(\mathbf{p}_0) < 0, \mathbf{p}_0 = \mathbf{0}. \end{aligned}$$

   The first case makes economic sense (i.e. at least one price is nonzero) and so by *equilibrium point* we will mean the first case.

2. (Hypothesis **H**)The hypothesis that agents maximise utility is that the functions $f_i(\mathbf{p})$ are **homogeneous of degree zero**, namely $f_i(\lambda\mathbf{p}) = \lambda^0 f_i(\mathbf{p}) = f_i(\mathbf{p})$ for any $\lambda > 0$.

3. (**Walras's law**)Consider that the budget constraint for each individual k takes the form

$$\sum_{i=1}^{2} p_i f_i^k(\mathbf{p}) = p_1 f_1^k(\mathbf{p}) + p_2 f_2^k(\mathbf{p}) = 0,$$

   where f_i^k denotes the excess demand by the kth economic agent for the ith commodity, i.e., the difference between the agent's demand for, and the agent's initial endowment of, that commodity. In general for m commodities by summing over all N economic agents we have:

$$\sum_{k=1}^{N} \sum_{i=1}^{m} p_i f_i^k(\mathbf{p}) = \sum_i p_{i=1}^m f_i(\mathbf{p}) = 0.$$

   This law states that, in view of the budget constraints, for any set of semipositive prices p (not necessarily equilibrium prices), the value of aggregate excess demand, evaluated at those prices, must be zero.

   The Jacobian matrix for $\mathbf{f}$ is

$$D\mathbf{f}(\mathbf{p}_0) = \begin{pmatrix} \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_2} \\ \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_2} \end{pmatrix}.$$

Chapter 7

Laplace transform

7.1 Laplace transform for 1D ODEs

The *Laplace transform* of continuous functions $f(t)$ with at most exponential growth, that is $|f(t)| \leq ce^{at}$ for $a \geq 0$ and $c \geq 0$, is defined as:

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t) \mathrm{d}t,$$

where $s > a$. In future, we denote continuous functions such that $|f(t)| \leq ce^{at}$ for $a \geq 0$, $c \geq 0$, and for all $t \geq 0$ as $C([0, \infty), e^{at})$, where if $a = 0$ we understand this as the space of bounded continuous functions on $[0, \infty)$. Note that we have dropped the constant c from the definition of $C([0, \infty), e^{at})$ since only a affects the region of definition of the Laplace transform. By integrating by parts we can easily check that we have:

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\} - f(0)$$

so for the second derivative we have, by iterating the previous observation,

$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\} - f'(0) = s^2\mathcal{L}\{f\} - f'(0) - sf(0).$$

Continuing with the above observation we see that

$$\mathcal{L}\{f^{(m)}\}(s) = s^m\mathcal{L}\{f\} - \sum_{j=0}^{m-1} s^j f^{m-j-1}(0)$$

where $f^{(m)}$ denotes the m^{th} derivative of f. Another useful property is that the Laplace transform is linear on continuous function of exponential growth. First observe that if $f, g \in C([0, \infty), e^{at})$ then linear combination of f, g also belong to $C([0, \infty), e^{at})$. Thus, the Laplace transform is defined on $f + g$, for $f, g \in C([0, \infty), e^{at})$, and satisfies:

$$\mathcal{L}\{f + g\}(s) = \mathcal{L}\{f\}(s) + \mathcal{L}\{g\}(s)$$

for $s > a$.

Method formal steps

1. Starting from the equation $ay''(t) + by'(t) + cy(t) = g(t)$ we compute the laplace transform of both sides, assuming an exponential growth condition on y and g, to obtain:

$$a\mathcal{L}\{y''\}(s) + b\mathcal{L}\{y'\}(s) + c\mathcal{L}\{y\}(s) = \mathcal{L}\{g\}(s)$$

we obtain from above:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2+bs+c}.$$

2. So by inverting the laplace transform (using linearity and known inversions) we can obtain solution $y(t)$ back. Note that inverting the laplce transform is permitted by Lerch's theorem (3) which says that if two functions, f_1 and f_2, have the same laplace transform then they are "essentially" equal.

3. The main computational aspect of this is splitting partial fractions to get the known relations. But Heaviside motivated by the same problem when computing the Laplace transform, came up with the cover-up method. In computing the coefficients below

$$\frac{p(s)}{(s-a_1)\cdots(s-a_n)} = \frac{A_1}{s-a_1} + \ldots + \frac{A_n}{s-a_n},$$

for polynomial $p(s)$, we see by rearranging that:

$$\frac{p(s)}{(s-a_1)\cdots(s-a_{i-1})(s-a_{i+1})\cdots(s-a_n)} = \frac{A_1(s-a_i)}{s-a_1} + \ldots + A_i + \ldots + \frac{A_n(s-a_i)}{s-a_n},$$

and by setting $s = a_i$ we obtain the ith coefficient A_i:

$$A_i = \frac{p(a_i)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}.$$

Here is a table of known Laplace transforms (see section 7.3 for proofs):

function $f(t) = \mathcal{L}^{-1}\{g\}(t)$	Laplace transform $g(s) = \mathcal{L}\{f\}(s)$	Region of definition
constant a	$\frac{a}{s}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2 + a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2 + a^2}$	$s > 0$
e^{at}	$\frac{1}{s - a}$	$s > a$
$\sin(bt)e^{at}$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$\cos(bt)e^{at}$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$f_{\text{step}}(t, a) := \begin{cases} 1, & 0 \le t \le a \\ 0, & t > a \end{cases}$	$\frac{1 - e^{-as}}{s}$	$s > 0$
$e^{a(t-b)} f_{\text{heavy}}(t, b) := e^{a(t-b)}\big(1 - f_{\text{step}}(t, b)\big)$	$\frac{e^{-bs}}{s - a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p, \ \ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

Examples

- Consider the equation

$$y''(t) - y'(t) - 6y(t) = 0, \ \ y(0) = 1, \ \ y'(0) = -1.$$

1. By taking the Laplace transform of both sides we obtain:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as + b)}{as^2 + bs + c}$$

for our equation we have

$$= \frac{y(0)(s-1) + y'(0)}{s^2 - s - 6}$$

for our IC we have

$$= \frac{(s-1) - 1}{s^2 - s - 6}$$
$$= \frac{s-2}{(s-3)(s+2)}$$

2. Next we split it into partial fractions

$$\begin{aligned}\mathcal{L}\{y\}(s) &= \frac{s-2}{(s-3)(s+2)} \\ &= \frac{1/5}{s-3} + \frac{4/5}{s+2}\end{aligned}$$

So we use $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \iff e^{at} = \mathcal{L}\{\frac{1}{s-a}\}^{-1}$

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\Big\{\frac{1/5}{s-3} + \frac{4/5}{s-(-2)}\Big\}(t) \\ &= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}.\end{aligned}$$

3. Indeed, using the method of characteristic equations for second order equations we obtain
$$y(t) = c_1 e^{3t} + c_2 e^{-2t}.$$
Therefore, by using the IC we have
$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ -1 = y'(0) = 3c_1 - 2c_2 \end{cases} \implies \begin{cases} c_1 = \frac{1}{5} \\ c_2 = \frac{4}{5} \end{cases}.$$

4. So for homogeneous equations it is clearly much faster and less error prone to use the method of characteristics.

- Consider the nonhomogeneous equation

$$y''(t) - 2y'(t) + 2y(t) = e^{-t}, \;\; y(0) = 0, \;\; y'(0) = 1$$

1. First we take the Laplace transform of both sides to obtain:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2 + bs + c}$$

for our equation it becomes

$$\begin{aligned} &= \frac{\frac{1}{s+1} + y'(0) + y(0)(s-2)}{s^2 - 2s + 2} \\ &= \frac{\frac{1}{s+1} + 1}{s^2 - 2s + 2} \\ &= \frac{s+2}{(s+1)(s^2 - 2s + 2)}\end{aligned}$$

by partial fractions we obtain

$$\begin{aligned} &= \frac{1}{5(s+1)} + \frac{8-s}{5(s^2 - 2s + 2)} \\ &= \frac{1}{5(s+1)} + \frac{8-s}{5(s-(1-i))(s-(1+i))}\end{aligned}$$

repeating partial fractions for the last term we have

$$= \frac{1}{5(s+1)} + \frac{\frac{7}{2}i - \frac{1}{2}}{5(s-(1-i))} + \frac{-\frac{7}{2}i - \frac{1}{2}}{5(s-(1+i))}.$$

So by inverting the Laplace transform we have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{5(s+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{7}{2}i - \frac{1}{2}}{5(s-(1-i))}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{7}{2}i - \frac{1}{2}}{5(s-(1+i))}\right\} \\ &= \frac{1}{5}e^{-t} + \left(\frac{7}{10}i - \frac{1}{10}\right)e^{(1+i)t} + \left(\frac{-7}{10}i - \frac{1}{10}\right)e^{(1-i)t}. \end{aligned}$$

2. Let's check this with the method of undetermined coefficients.
   (a) First we solve the homogeneous problem. The method of characteristics gives:

   $$x_h(t) = c_1 e^{(1+i)t} + c_2 e^{(1-i)t}.$$

   (b) We make the ansatz $x_{nh}(t) = ce^{-t}$ (here $s = 0$ because $r_* = -1$ is not a root). Plugging in we have

   $$ce^{-t} + 2ce^{-t} + 2ce^{-t} = e^{-t} \implies c + 2c + 2c = 1 \implies c = 1/5,$$

   which is the same nonhomogeneous solution as in the Laplace transform.
   (c) Next we evaluate the coefficients.

   $$\begin{cases} 0 = y(0) = c_1 + c_2 + \frac{1}{5} \\ 1 = y'(0) = (1+i)c_1 + (1-i)c_2 - \frac{1}{5} \end{cases} \implies \begin{cases} c_1 = \frac{7}{10}i - \frac{1}{10} \\ c_2 = -\frac{7}{10}i - \frac{1}{10} \end{cases}.$$

- Consider the equation

$$y''(t) + 4y(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$$

with initial data $y(0) = 1$, $y'(0) = 0$.

1. The Laplace transform of the above step function[1] is

$$\frac{1 - e^{-\pi s}}{s}.$$

2. We take the Laplace transform of both sides:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2 + bs + c}$$

for our equation the above becomes

$$\begin{aligned} \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2 + bs + c} &= \frac{\frac{1-e^{-\pi s}}{s} + y'(0) + sy(0)}{s^2+4} \\ &= \frac{1 - e^{-\pi s} + s^2}{s(s^2+4)} \\ &= \frac{s}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-\pi s}}{s(s^2+4)} \\ &= \frac{s}{s^2+4} + \left(\frac{1}{4s} - \frac{1}{8(s+2i)} - \frac{1}{8(s-2i)}\right)(1 - e^{-\pi s}). \end{aligned}$$

[1] Technically we have defined the Laplace transform only for continuous functions of controlled growth. However, the Laplace transform is extendable to Riemann integrable functions of controlled growth. In particular, for this example, one computes the Laplace transform by splitting the function into the two regions where it is understood. For more information see this exercises at the end of this section.

We will use the following Laplace transforms:

$$\begin{aligned}
\mathcal{L}\{\cos(2t)\}(s) &= \frac{s}{s^2+4} \\
\mathcal{L}\Big\{\frac{1}{4}\Big\}(s) &= \frac{1}{4s} \\
\mathcal{L}\Big\{\frac{1}{8}e^{-2it}\Big\}(s) &= \frac{1}{8(s+2i)} \\
\mathcal{L}\Big\{\frac{1}{8}e^{2it}\Big\}(s) &= \frac{1}{8(s-2i)} \\
\mathcal{L}\Big\{e^{-\pi s}\Big\}(s) &= \frac{1}{s-(-\pi)} = \frac{1}{s+\pi} \\
\mathcal{L}\Big\{e^{-a(t-b)}\big(1-f_{\text{step}}(t,b)\big)\Big\} &= \frac{e^{-bs}}{s+a}.
\end{aligned}$$

So we have

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} + \mathcal{L}^{-1}\Big\{\frac{1}{4s}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s+2i)}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s-2i)}\Big\} \\
&\quad + \mathcal{L}^{-1}\Big\{e^{-\pi s}\frac{1}{4s}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s+2i)}e^{-\pi s}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s-2i)}e^{-\pi s}\Big\} \\
&= \cos(2t) + \frac{1}{4} - \frac{1}{8}e^{-2it} - \frac{1}{8}e^{2it} \\
&\quad + \frac{1}{4}\big(1-f_{\text{step}}(t,\pi)\big) + \frac{1}{8}e^{-2i(t-\pi)}(1-f_{\text{step}}(t,\pi)) + \frac{1}{8}e^{2i(t-\pi)}(1-f_{\text{step}}(t,\pi))
\end{aligned}$$

7.2 Laplace transform for systems

Consider the Laplace transform of vectors $\mathcal{L}\{\mathbf{x}\}(s)$ defined componentwise[2]

$$\mathcal{L}\{\mathbf{x}\}(s) := \begin{pmatrix} \mathcal{L}\{x_1\}(s) \\ \vdots \\ \mathcal{L}\{x_n\}(s) \end{pmatrix}.$$

Therefore, as with the usual Laplace transform we obtain, by repeatedly using the scalar version of this identity, that:

$$\mathcal{L}\{\mathbf{x}'\}(s) = s\mathcal{L}\{\mathbf{x}\}(s) - \mathbf{x}(0).$$

7.2.1 Method formal steps

Consider the nonhomogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t).$$

[2]Recall that integration extends to vectors by integration componentwise. Since this transform is defined by integration it too extends to vectors by acting componentwise.

1. Taking the Laplace transform of each term in the above equation we have:

$$s\mathcal{L}\{\mathbf{x}\}(s) - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\{\mathbf{x}\}(s) + \mathcal{L}\{\mathbf{g}\}(s).$$

2. For simplicity we assume that $\mathbf{x}(0) = \mathbf{0}_{n\times 1}$.
3. We then obtain the system:

$$(s\mathbf{I}_n - \mathbf{A})\mathcal{L}\{\mathbf{x}\}(s) = \mathcal{L}\{\mathbf{g}\}(s).$$

4. By inverting the matrix, assuming s is not an eigenvalue of $\mathbf{A}$, we obtain:

$$\mathcal{L}\{\mathbf{x}\}(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\mathcal{L}\{\mathbf{g}\}(s).$$

5. Then we do inverse Laplace transform of each component using known Laplace transform relations.
6. The main computational aspect of this is splitting partial fractions to get the known relations. But Heaviside motivated by the same problem when computing the Laplace transform, came up with the cover-up method. In computing the coefficients below

$$\frac{p(s)}{(s-a_1)\cdots(s-a_n)} = \frac{A_1}{s-a_1} + ... + \frac{A_n}{s-a_n},$$

for polynomial $p(s)$, we see by rearranging that:

$$\frac{p(s)}{(s-a_1)\cdots(s-a_{i-1})(s-a_{i+1})\cdots(s-a_n)} = \frac{A_1(s-a_i)}{s-a_1} + ... + A_i + ... + \frac{A_n(s-a_i)}{s-a_n},$$

and by setting $s = a_i$ we obtain the ith coefficient A_i:

$$A_i = \frac{p(a_i)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}.$$

7.2.2 Examples

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

1. We take Laplace transform of both sides

$$(s\mathbf{I} - \mathbf{A})\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \begin{pmatrix} s-2 & 1 \\ 0 & s-1 \end{pmatrix}\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}.$$

2. We simplify

$$\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \begin{pmatrix} s-2 & 1 \\ 0 & s-1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$= \frac{1}{(s-2)(s-1)} \begin{pmatrix} s-1 & -1 \\ 0 & s-2 \end{pmatrix} \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{(s+1)(s-2)} - \frac{3}{(s-1)(s-2)s^2} \\ \frac{3}{(s-1)s^2} \end{pmatrix}.$$

By partial fractions we get

$$= \begin{pmatrix} \frac{2}{3(s-2)} - \frac{2}{3(s+1)} - \left(\frac{3}{2s^2} - \frac{3}{s-1} + \frac{9}{4}\frac{1}{s} + \frac{3}{4}\frac{1}{s-2} \right) \\ \frac{-3}{s^2} - \frac{3}{s} + \frac{3}{s-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{12(s-2)} - \frac{2}{3(s+1)} - \frac{3}{2s^2} + \frac{3}{s-1} - \frac{9}{4}\frac{1}{s} \\ \frac{-3}{s^2} - \frac{3}{s} + \frac{3}{s-1} \end{pmatrix}.$$

3. We use known Laplace transform relations to obtain $\mathbf{x}$ by inverting:

$$\mathbf{x}(t) = \begin{pmatrix} \mathcal{L}^{-1}\left\{ \frac{-1}{12(s-2)} - \frac{2}{3(s+1)} - \frac{3}{2s^2} + 3\frac{1}{s-1} - \frac{9}{4}\frac{1}{s} \right\}(t) \\ \mathcal{L}^{-1}\left\{ \frac{-3}{s^2} - \frac{3}{s} + \frac{3}{s-1} \right\}(t) \end{pmatrix}.$$

For the first component we have

$$\begin{aligned} x_1(t) &= \frac{-1}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) \\ &+ 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \frac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) \\ &= \frac{-1}{12}e^{2t} - \frac{2}{3}e^{-t} - \frac{3}{2}t + 3e^t - \frac{9}{4}. \end{aligned}$$

For the second component we have

$$\begin{aligned} x_2(t) &= -3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) \\ &= -3t - 3 + 3e^t. \end{aligned}$$

Therefore, together give

$$\mathbf{x}_{nh}(t) = \begin{pmatrix} \frac{-1}{12}e^{2t} - \frac{2}{3}e^{-t} - \frac{3}{2}t + 3e^t - \frac{9}{4} \\ -3t - 3 + 3e^t \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} \frac{-1}{12} \\ 0 \end{pmatrix} - \frac{2}{3}e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - 3\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} + 3e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4. Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} \frac{-1}{12} \\ 0 \end{pmatrix} - \frac{2}{3}e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - 3\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} + 3e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Consider the second order equation

$$w''(t) + w(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

with zero initial data. By setting $x = w$, $y = w'$ we obtain $x' = y$, $y' + x = f_{\text{step}}(t)$ or in system form

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0, & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f_{\text{step}}(t) \end{pmatrix}.$$

1. We take Laplace transform of both sides

$$(s\mathbf{I}_n - \mathbf{A})\mathcal{L}\{\mathbf{x}\}(s) = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}\mathcal{L}\{\mathbf{x}\}(s) = \begin{pmatrix} 0 \\ \mathcal{L}\{f_{\text{step}}\}(s) \end{pmatrix}.$$

2. The Laplace transform of the RHS is

$$\mathcal{L}\{f_{\text{step}}\}(s) = \int_0^\infty e^{-st} f_{\text{step}}(t)\mathrm{d}t = \int_0^1 e^{-st}\mathrm{d}t = \frac{1 - e^{-s}}{s}.$$

3. We simplify

$$\begin{aligned}
\mathcal{L}\{\mathbf{x}\}(s) &= \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1-e^{-s}}{s} \end{pmatrix} \\
&= \frac{1}{s^2+1}\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1-e^{-s}}{s} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-e^{-s}}{s(s^2+1)} \\ \frac{1-e^{-s}}{s^2+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-i}{2}\frac{1-e^{-s}}{s+i} - \frac{i}{2}\frac{1-e^{-s}}{s-i} + \frac{1-e^{-s}}{s} \\ \frac{-i}{2}\frac{1-e^{-s}}{s+i} + \frac{i}{2}\frac{1-e^{-s}}{s-i} \end{pmatrix}.
\end{aligned}$$

4. We use known Laplace transform relations to invert

   For the first component we have

$$\begin{aligned}
x_1(t) &= \mathcal{L}^{-1}\Big\{\frac{1-e^{-s}}{s}\Big\}^{-1}(t) + \mathcal{L}^{-1}\Big\{\frac{-i}{2}\frac{1-e^{-s}}{s+i} - \frac{i}{2}\frac{1-e^{-s}}{s-i}\Big\}^{-1}(t) \\
&= f_{step}(t,1) + \frac{-i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s+i}\Big\}(t) - \mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s+i}\Big\}(t)\Big] \\
&+ \frac{-i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s-i}\Big\}^{-1}(t) - \mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s-i}\Big\}^{-1}(t)\Big] \\
&= f_{step}(t,1) + \frac{-i}{2}\Big[e^{-it} - e^{-i(t-(-1))}(1 - f_{step}(t,1))\Big] \\
&+ \frac{i}{2}\Big[e^{it} - e^{i(t+1)}(1 - f_{step}(t,1))\Big].
\end{aligned}$$

For the second component we have

$$\begin{aligned}
x_2(t) &= \mathcal{L}^{-1}\Big\{\frac{-i}{2}\frac{1-e^{-s}}{s+i}+\frac{i}{2}\frac{1-e^{-s}}{s-i}\Big\}^{-1}(t)\\
&= \frac{-i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s+i}\Big\}(t)-\mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s+i}\Big\}(t)\Big]+\frac{i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s-i}\Big\}^{-1}(t)-\mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s-i}\Big\}^{-1}(t)\Big]\\
&= \frac{-i}{2}\Big[e^{-it}-e^{-i(t-(-1))}(1-f_{step}(t,1))\Big]+\frac{i}{2}\Big[e^{it}-e^{i(t+1)}(1-f_{step}(t,1))\Big].
\end{aligned}$$

Therefore, together we obtain

$$\mathbf{x}_{nh}(t) = f_{step}(t,1)\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\frac{-i}{2}\Big[e^{-it}-e^{-i(t-(-1))}f_{step}(t,1)\Big]+\begin{pmatrix}1\\-1\end{pmatrix}\frac{i}{2}\Big[e^{it}-e^{i(t+1)}f_{step}(t,1)\Big].$$

5. The general solution will be:

$$\begin{aligned}
\mathbf{x}(t) &= c_1e^{it}\begin{pmatrix}i\\1\end{pmatrix}+c_1e^{-it}\begin{pmatrix}-i\\1\end{pmatrix}\\
&+f_{step}(t,1)\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\frac{-i}{2}\Big[e^{-it}-e^{-i(t-(-1))}(1-f_{step}(t,1))\Big]\\
&+\begin{pmatrix}1\\-1\end{pmatrix}\frac{i}{2}\Big[e^{it}-e^{i(t+1)}(1-f_{step}(t,1))\Big].
\end{aligned}$$

6. For comparison we also compute the solution of the second order equation:

$$w''(t)+w(t)=\begin{cases}1, & 0\le t\le 1\\ 0, & t>1\end{cases}.$$

7. By taking the Laplace transform of both sides we obtain

$$s^2\mathcal{L}\Big\{w\Big\}-sw(0)-w'(0)+\mathcal{L}\Big\{w\Big\}=\mathcal{L}\Big\{f_{step}(\cdot,1)\Big\}(s)$$

using that $w(0)=w'(0)=0$ we obtain

$$\begin{aligned}
\mathcal{L}\Big\{w\Big\} &= \frac{\mathcal{L}\Big\{f_{step}(\cdot,1)\Big\}(s)}{s^2+1}\\
&= \frac{1-e^{-s}}{s(s^2+1)}\\
&= \frac{1-e^{-s}}{s(s^2+1)}\\
&= \frac{1-e^{-s}}{s}+\frac{-i}{2}\frac{1-e^{-s}}{s+i}-\frac{i}{2}\frac{1-e^{-s}}{s-i}.
\end{aligned}$$

Therefore, by inverting we obtain

$$w(t)=f_{step}(t,1)+\frac{-i}{2}\Big[e^{-it}-e^{-i(t+1)}(1-f_{step}(t,1))\Big]+\frac{i}{2}\Big[e^{it}-e^{i(t+1)}(1-f_{step}(t,1))\Big].$$

8. This is indeed the solution we obtained for the first component $x_1(t):=w(t)$.

- Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ e^{-(t-1)}(1 - f_{step}(t,1)) \end{pmatrix}.$$

1. First we take the Laplace transform forcing term:

$$\mathcal{L}\Big\{ \begin{pmatrix} 0 \\ e^{-(t-1)}(1 - f_{step}(t,1)) \end{pmatrix} \Big\} = \begin{pmatrix} 0 \\ \frac{e^{-s}}{s+1} \end{pmatrix}.$$

2. Therefore,

$$\begin{aligned} \mathcal{L}\Big\{\mathbf{x}\Big\} &= (s\mathbf{I} - \mathbf{A})^{-1} \begin{pmatrix} 0 \\ \frac{e^{-s}}{s+1} \end{pmatrix} \\ &= \frac{1}{(s-1)^2} \begin{pmatrix} s-2 & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{e^{-s}}{s+1} \\ &= \frac{1}{(s-1)^2} \begin{pmatrix} -1 \\ s \end{pmatrix} \frac{e^{-s}}{s+1}. \end{aligned}$$

Using the cover-up method we compute the partial fraction for the first component:

$$-e^{-s}\big(\frac{1}{2(s-1)^2} - \frac{1}{4(s-1)} + \frac{1}{4(s+1)}\big)$$

and for the second component

$$-e^{-s}\big(\frac{1}{2(s-1)^2} + \frac{1}{4(s-1)} - \frac{1}{4(s+1)}\big).$$

Next we invert the first component:

$$\begin{aligned} x &= \mathcal{L}^{-1}\Big\{ -e^{-s}\big(\frac{1}{2(s-1)^2} - \frac{1}{4(s-1)} + \frac{1}{4(s+1)}\big)\Big\} \\ &= -\frac{e^{t-1}(t-1)}{2} f_{heavy}(t,1) + \frac{e^{t-1}}{4} f_{heavy}(t,1) - \frac{e^{1-t}}{4} f_{heavy}(t,1) \end{aligned}$$

and the second component

$$y = -\frac{e^{t-1}(t-1)}{2} f_{heavy}(t,1) - \frac{e^{t-1}}{4} f_{heavy}(t,1) + \frac{e^{1-t}}{4} f_{heavy}(t,1).$$

3. So in vector notation we have the general solution:

$$\begin{aligned} \mathbf{x}_{gen} =& c_1 e^{-t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t}\big(t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\big) \\ &- \frac{e^{t-1}(t-1)}{2} f_{heavy}(t,1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &+ \Big[\frac{e^{t-1}}{4} f_{heavy}(t,1) - \frac{e^{1-t}}{4} f_{heavy}(t,1)\Big] \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

7.3 Properties of Laplace Transform

We begin by demonstrating the following commonly used identities for the Laplace transform:

Proposition 1. *The Laplace transform satisfies the following identities:*

1. *If $f, g \in C([0,\infty), e^{at})$ then for $b, c \in \mathbb{R}$ $\mathcal{L}\{bf + cg\}(s) = b\mathcal{L}\{f\}(s) + c\}(s)$ for $s > a$.*
2. *If $f \in C([0,\infty), e^{at})$ and $b > -a$ then $e^{bt} \cdot f \in C([0,\infty), e^{(a+b)t})$ and $\mathcal{L}\{e^{bt}f\}(s) = \mathcal{L}\{f\}(s-b)$ for $s > a+b$.*
3. *Suppose $f \in C([0,\infty), e^{at})$ for $a \neq 0$ and define $F : [0,\infty) \to \mathbb{R}$ by $F(t) = \int_0^t f(s)\mathrm{d}s$. Then $F \in C([0,\infty), e^{at})$ and for $s > a$ we have $\mathcal{L}\{F\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$.*
4. *Suppose $F : [0,\infty) \to \mathbb{R}$ is defined by $F(t) = \int_0^t f(s)\mathrm{d}s$ for $f \in C([0,\infty), e^{at})$ and we assume that $F \in C([0,\infty), e^{at})$ then for $s > a$ we have $\mathcal{L}\{F\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$.*
5. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{a\}(s) = \frac{a}{s}$ for $s > 0$.*
6. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2+a^2}$ for $s > 0$.*
7. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}$ for $s > 0$.*
8. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$ for $s > a$.*
9. *For $a, b \in \mathbb{R}$ we have $\mathcal{L}\{\sin(at)e^{bt}\}(s) = \frac{a}{(s-b)^2+a^2}$ for $s > b$.*
10. *For $a, b \in \mathbb{R}$ we have $\mathcal{L}\{\cos(at)e^{bt}\}(s) = \frac{s-b}{(s-b)^2+a^2}$ for $s > b$.*

Proof.

1. Suppose $f, g \in C([0,\infty), e^{at})$ and $b, c \in \mathbb{R}$. Observe that
$$|bf(t) + cg(t)| \leq |b| \cdot |f(t)| + |c| \cdot |g(t)| \leq |b| \cdot C_f e^{at} + |c|C_g e^{at} = (|b|C_f + |c|C_g)e^{at}$$
for $t \geq 0$ where C_f and C_g are non-negative constants. We conclude that $bf + cg \in C([0,\infty), e^{at})$. Thus, the Laplace transform of $bf + cg$, f, and g are all defined for $s > a$. Computing gives, for $s > a$, that:
$$\begin{aligned}\mathcal{L}\{bf + cg\}(s) &= \int_0^\infty e^{-st}(bf(s) + cg(s))\mathrm{d}s \\ &= b\int_0^\infty e^{-st}f(s)\mathrm{d}s + c\int_0^\infty e^{-st}g(s)\mathrm{d}s \\ &= b\mathcal{L}\{f\} + c\mathcal{L}\{g\}.\end{aligned}$$

2. Suppose $f \in C([0,\infty), e^{at})$ and $b > -a$. Then for $t \geq 0$ we have:
$$|f(t)e^{bt}| = e^{bt}|f(t)| \leq C_f e^{bt} \cdot e^{at} = C_f e^{(a+b)t}.$$
Thus, $e^{bt}f \in C([0,\infty), e^{(a+b)t})$ which means the Laplace transform of $e^{bt}f$ is defined for $s > a+b$. Observe that, for $s > a+b$
$$\mathcal{L}\{e^{bt}f\}(s) = \int_0^\infty e^{-st}e^{bt}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-b)t}f(t)\mathrm{d}t = \mathcal{L}\{f\}(s-b).$$

3. Suppose $f \in C([0,\infty), e^{at})$ and $F(t) = \int_0^t f(s)\mathrm{d}s$ for $t \geq 0$. Then for $t \geq 0$ we have, if $a \neq 0$

$$|F(t)| = \left|\int_0^t f(s)\mathrm{d}s\right| \leq \int_0^t |f(s)|\mathrm{d}s \leq C_f \int_0^t e^{as}\mathrm{d}s = C_f \cdot \frac{e^{e^{at}} - 1}{a} \leq \frac{C_f}{a} \cdot e^{at}.$$

Thus, for $a \neq 0$ the Laplace transform is defined for f for $s > a$. In particular, by integrating by parts, which is permitted since f is continuous, we get:

$$\mathcal{L}\{F\}(s) = \int_0^\infty e^{-st} F(t)\mathrm{d}t = \frac{-e^{-st}F(t)}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\mathrm{d}t = \frac{1}{s}\mathcal{L}\{f\}(s).$$

4. By assumption the Laplace transform of both f and F is defined for $s > a$. Thus, for $s > a$ we have, by integrating by parts.

$$\mathcal{L}\{F\}(s) = \int_0^\infty e^{-st} F(t)\mathrm{d}t = \frac{-e^{-st}F(t)}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\mathrm{d}t = \frac{1}{s}\mathcal{L}\{f\}(s).$$

5. Suppose $a \in \mathbb{R}$. Then the constant function defined by $f(t) = a$ for $t \geq 0$ is bounded and hence $f \in C([0,\infty), e^{0\cdot t})$. Thus, the Laplace transform is defined for $s > 0$. Computing this we obtain, for $s > 0$:

$$\mathcal{L}\{a\}(s) = \int_0^\infty e^{-st} a\mathrm{d}t = a\int_0^\infty e^{-st}\mathrm{d}t = \frac{a}{s}$$

6. For $a \in \mathbb{R}$ we have $\sin(at) \in C([0,\infty), e^{0\cdot t})$ since this function is bounded. Thus, the Laplace transform is defined for $s > 0$. Computing the transform we get, for $s > 0$:

$$\begin{aligned}\mathcal{L}\{\sin(at)\}(s) &= \int_0^\infty e^{-st}\sin(at)\mathrm{d}t = -\frac{e^{-st}\sin(at)}{s}\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\mathrm{d}t \\ &= -\frac{ae^{-st}\cos(at)}{s^2}\Big|_0^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin(at)\mathrm{d}t \\ &= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}(s).\end{aligned}$$

Thus, we obtain, for $s > 0$

$$\left(\frac{s^2 + a^2}{s^2}\right)\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2}$$

and so

$$\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2 + a^2}$$

7. Observe that, for $a \neq 0$, $\cos(at) = 1 - a\int_0^t \sin(as)\mathrm{d}s$ and that $\cos(at)$, $\sin(at)$, and the constant function -1 are all bounded functions. In particular, we see that

$$\int_0^t \sin(as)\mathrm{d}s = \frac{1 - \cos(at)}{a}$$

is bounded for $t \geq 0$. Thus, the Laplace transform of all functions involved is defined for $s > 0$. Applying properties 1, 4, and 6 we obtain for $s > 0$

$$\mathcal{L}\{\cos(at)\}(s) = \frac{1}{s} - \frac{a}{s}\cdot\frac{a}{s^2 + a^2} = \frac{1}{s}\cdot\frac{s^2 + a^2 - a^2}{s^2 + a^2} = \frac{s}{s^2 + a^2}.$$

8. By properties 2 and 5 we have
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$
for $s > a$ since 1 is bounded.

9. By properties 2 and 6 we have
$$\mathcal{L}\{\sin(at)e^{bt}\} = \frac{a}{(s-b)^2 + a^2}$$
for $s > b$ since $\sin(at)$ is bounded.

10. By properties 2 and 7 we have
$$\mathcal{L}\{\cos(at)e^{bt}\} = \frac{s-b}{(s-b)^2 + a^2}$$
for $s > b$ since $\sin(at)$ is bounded. □

The spaces $C([0,\infty), e^{at})$ for $a \geq 0$, while large enough to deal with simple functions we encounter in the wild, are not large enough to deal with some of the obstacles we may run into. In particular, these spaces are not suited to dealing with "modestly" growing functions like $x \mapsto x^n$ for $n \in \mathbb{N}$ which grows slower at infinity then any function of the form e^{at} for $a > 0$ but is not bounded. To get around this obstacle we define the spaces $L^p((0,\infty), e^{-at})$ consisting of functions, f, such that $\int_0^\infty e^{-at}|f(t)|^p \mathrm{d}t < \infty$ for $a \geq 0$ and $1 \leq p < \infty$. Observe that such functions have laplace transform defined for $s > a$ and if $a = 0$ then $L^p((0,\infty), e^{-at}) = L^p((0,\infty))$. We will, in particular, consider the case $p = 1$ as this allows an immediate extension to the Laplace transform. With this new definition we will demonstrate some properties of the extended Laplace transform. We will also show that the properties demonstrated in proposition 1 remain true for the extended Laplace transform.

Proposition 2. *The generalized Laplace transform satisfies the following identities:*

1. *Suppose $f \in C([0,\infty), e^{-at})$. Then $f \in L^1((0,\infty), e^{-at})$ and so the laplace transform is defined, by the same formula, for $s > a$.*

2. *If $p \geq 0$ then $f(s) = s^p$ is an element of $L^1((0,\infty))$ and satisfies $\mathcal{L}\{f\}(s) = \frac{\Gamma(p+1)}{s^{p+1}}$ for $s > 0$.*

Proof. 1. Observe that for $s > a$ we have

□

Theorem 3. *(Lerch's theorem) Suppose $f_1, f_2 \in L^p((0,\infty), e^{-at})$ and $\mathcal{L}\{f_1\}(s) = \mathcal{L}\{f_2\}(s)$ for all $s > a$. Then $f_1 = f_2$ almost everywhere on $(0,\infty)$.*

Chapter 8

Appendix

8.1 Inverse of a matrix

Given a 2×2 matrix $\mathbf{A}$ we will sometimes have to compute its inverse. To do this efficiently we provide an algorithm which gives the formula for the inverse of $\mathbf{A}$. We recall that if $\mathbf{v} = \binom{v_1}{v_2}$ then $\mathbf{v}^\perp = \binom{v_2}{-v_1}$ is its perpendicular vector.

8.1.1 Formal steps

1. We are given a 2×2 matrix $\mathbf{A}$ which takes the form

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\xi} & \boldsymbol{\eta} \end{bmatrix}$$

   where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are vectors in $\mathbb{R}^2$. We compute $\xi^\perp$ and $\eta^\perp$.

2. Next we compute $\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp$.

3. Finally, we form the inverse matrix

$$\mathbf{A}^{-1} = \frac{1}{\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp} \begin{bmatrix} (\boldsymbol{\eta}^\perp)^T \\ \\ -(\boldsymbol{\xi}^\perp)^T \end{bmatrix}.$$

8.1.2 Example of the method

1. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ \\ 2 & -7 \end{pmatrix}$$

   for which $\boldsymbol{\xi} = \binom{3}{2}$ and $\boldsymbol{\eta} = \binom{5}{-7}$. We compute that $\boldsymbol{\xi}^\perp = \binom{2}{-3}$ and $\boldsymbol{\eta}^\perp = \binom{-7}{-5}$.

2. $\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp = -31$.

3. Finally we obtain that

$$\mathbf{A}^{-1} = \frac{-1}{31}\begin{pmatrix} -7 & -5 \\ \\ -2 & 3 \end{pmatrix}.$$

We can check that this is correct by multipling $\mathbf{A}^{-1}$ and $\mathbf{A}$ to obtain $\mathbf{I}_2$.

8.1.3 Reasoning behind the method

This algorithm comes from the following reasoning. If we have a matrix $\mathbf{A}$ and we want to find its inverse $\mathbf{A}^{-1}$ then we require that

$$\begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix} = \mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} \boldsymbol{\alpha}^T \\ \\ \boldsymbol{\beta}^T \end{pmatrix}\begin{pmatrix} \boldsymbol{\xi} & \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}\cdot\boldsymbol{\xi} & \boldsymbol{\alpha}\cdot\boldsymbol{\eta} \\ \\ \boldsymbol{\beta}\cdot\boldsymbol{\xi} & \boldsymbol{\beta}\cdot\boldsymbol{\eta} \end{pmatrix}.$$

Comparing both sides we see that we must have $\boldsymbol{\alpha}\cdot\boldsymbol{\eta} = 0$ as well as $\boldsymbol{\beta}\cdot\boldsymbol{\xi} = 0$. The easiest way to achieve this is to choose $\boldsymbol{\alpha} = \boldsymbol{\eta}^T$ and $\boldsymbol{\beta} = \boldsymbol{\xi}^T$. This, however, ignores that we need $\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = 1$ and $\boldsymbol{\beta}\cdot\boldsymbol{\eta} = 1$. Fortunately, with the choices $\boldsymbol{\alpha} = \boldsymbol{\eta}^\perp$ and $\boldsymbol{\beta} = \boldsymbol{\xi}^\perp$ we have

$$\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = \boldsymbol{\eta}^\perp\cdot\boldsymbol{\xi} = \xi_1\eta_2 - \xi_2\eta_1$$

as well as

$$\boldsymbol{\beta}\cdot\boldsymbol{\eta} = \boldsymbol{\xi}^\perp\cdot\boldsymbol{\eta} = \xi_2\eta_1 - \xi_1\eta_2$$

which differ by a factor of -1. If we now choose $\boldsymbol{\beta} = -\boldsymbol{\xi}^\perp$ (essentially multiplying both sides of the last computation by -1) then we have $\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = \boldsymbol{\beta}\cdot\boldsymbol{\eta}$. Thus, the matrix $\mathbf{B}$ defined by

$$\mathbf{B} = \begin{pmatrix} (\boldsymbol{\eta}^\perp)^T \\ \\ -(\boldsymbol{\xi})^T \end{pmatrix}$$

satisfies

$$\mathbf{BA} = \begin{pmatrix} \boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp & 0 \\ \\ 0 & \boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp \end{pmatrix} = \boldsymbol{\xi}\cdot\boldsymbol{\eta}^\perp\begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix}.$$

Dividing both sides by $\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp$ we get that

$$\Big(\frac{1}{\boldsymbol{\xi} \cdot \boldsymbol{\eta}^\perp}\mathbf{B}\Big)\mathbf{A} = \begin{pmatrix} 1 & 0 \\ & \\ & \\ 0 & 1 \end{pmatrix}$$

which gives us the inverse matrix.

8.2 Exponential of matrix

8.2.1 Identities and formulas

Proposition 8.2.1. *We let $e^{t\mathbf{A}}$ denote the unique*[1] *matrix which solves*

$$\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{I}_n.$$

I claim that, in this case, $e^{t\mathbf{A}}$ satisfies:

1. $e^{\mathbf{0}} = I_n$
2. *The unique solution to the problem* $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{X}_0$ *is* $e^{t\mathbf{A}}\mathbf{X}_0$.
3. $\mathbf{A}e^{t\mathbf{A}} = e^{t\mathbf{A}}\mathbf{A}$
4. *The exponential of matrix is invertible and we have that* $(e^{t\mathbf{A}})^{-1} = e^{-t\mathbf{A}}$
5. *If* $\mathbf{AB} = \mathbf{BA}$ *then* $e^{t(\mathbf{A}+\mathbf{B})} = e^{t\mathbf{A}}e^{t\mathbf{B}}$
6. $(e^{t\mathbf{A}})^{\mathrm{T}} = e^{t\mathbf{A}^{\mathrm{T}}}$
7. *If* $\mathbf{T}$ *is an invertible matrix then* $e^{t\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T}$.
8. *If* $\mathbf{A}^2 = \mathbf{A}$ *then* $e^{t\mathbf{A}} = \mathbf{I}_n + (e^t - 1)\mathbf{A}$.
9. *Formulas of* $e^{t\mathbf{A}}$ *for* $n = 2$:
   (a) *If the eigenvalues are distinct then*
   $$exp\{t\mathbf{A}\} := e^{\lambda_1 t}\frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}_2) - e^{\lambda_2 t}\frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_1\mathbf{I}_2).$$
   (b) *If* $\lambda = \lambda_1 = \lambda_2$ *then*
   $$exp\{t\mathbf{A}\} := e^{\lambda t}\mathbf{I}_2 + e^{\lambda t}t(\mathbf{A} - \lambda\mathbf{I}_2).$$
   (c) *If* $\lambda_1 = a + ib, \lambda_2 = a - ib$ *then*
   $$exp\{t\mathbf{A}\} := \frac{e^{at}}{b}\{b\cos(bt)\mathbf{I}_2 + \sin(bt)(\mathbf{A} - a\mathbf{I}_2)\}.$$
10. *Suppose* $\mathbf{A}$ *is invertible. Then*
$$\int e^{t\mathbf{A}}\mathrm{d}t = \mathbf{A}^{-1}e^{t\mathbf{A}} + \mathbf{C}$$

[1]This is due to the uniqueness theorem for linear matrix ODEs.

Proof.

1. By definition we have
$$e^{\mathbf{0}} = e^{0\mathbf{A}} = \mathbf{X}(0) = \mathbf{I}_n.$$

2. Let $\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{X}_0$. Then,
$$\mathbf{Y}'(t) = \left(e^{t\mathbf{A}}\mathbf{X}_0\right)' = \mathbf{A}e^{t\mathbf{A}}\mathbf{X}_0 = \mathbf{A}\mathbf{Y}(t)$$
and
$$\mathbf{Y}(0) = e^{\mathbf{0}}\mathbf{X}_0 = \mathbf{I}_n\mathbf{X}_0 = \mathbf{X}_0.$$
By uniqueness of solutions of linear matrix IVPs we have that $\mathbf{Y}$ is the only solution to this matrix IVP.

3. Let $\mathbf{Y}(t) = \mathbf{A}e^{t\mathbf{A}}$. Observe that
$$\mathbf{Y}'(t) = \mathbf{A}\left(e^{t\mathbf{A}}\right)' = \mathbf{A}\left(\mathbf{A}e^{t\mathbf{A}}\right) = \mathbf{A}\mathbf{Y}(t)$$
and
$$\mathbf{Y}(0) = \mathbf{A}e^{\mathbf{0}} = \mathbf{A}\mathbf{I}_n = \mathbf{A}.$$
By 2 we must have
$$\mathbf{A}e^{t\mathbf{A}} = \mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{A}$$
for all t.

4. Let $\mathcal{F}(t) = e^{t\mathbf{A}}e^{-t\mathbf{A}}$ for $t \in \mathbb{R}$. Observe that
$$\mathcal{F}'(t) = (e^{t\mathbf{A}})'e^{-t\mathbf{A}} + e^{t\mathbf{A}}(e^{-t\mathbf{A}})' = \mathbf{A}e^{t\mathbf{A}}e^{-t\mathbf{A}} - e^{t\mathbf{A}}\mathbf{A}e^{-t\mathbf{A}} = \mathbf{A}e^{t\mathbf{A}}e^{-t\mathbf{A}} - \mathbf{A}e^{t\mathbf{A}}e^{-t\mathbf{A}} = \mathbf{0}.$$
where I have used 3. Thus, $\mathcal{F}$ is constant. In particular, we have, by evaluating at $t = 0$
$$\mathcal{F}(0) = e^{\mathbf{0}}e^{\mathbf{0}} = \mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n.$$
By reversing the roles of $e^{t\mathbf{A}}$ and $e^{-t\mathbf{A}}$ we obtain the desired conclusion.

5. For this proof we let $\mathbf{X}(t) = e^{t\mathbf{A}}$, $\mathbf{Y}(t) = e^{t\mathbf{B}}$, and $\mathbf{Z}(t) = e^{t(\mathbf{A}+\mathbf{B})}$. Define $\mathcal{G}(t) = \mathbf{Z}(t) - \mathbf{X}(t)\mathbf{Y}(t)$. Differentiating we obtain
$$\begin{aligned}\mathcal{G}'(t) &= \mathbf{Z}'(t) - \mathbf{X}'(t)\mathbf{Y}(t) - \mathbf{X}(t)\mathbf{Y}'(t)\\ &= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - \mathbf{A}\mathbf{X}(t)\mathbf{Y}(t) - \mathbf{X}(t)\mathbf{B}\mathbf{Y}(t).\end{aligned}$$
Note that if we can show that $\mathbf{X}(t)\mathbf{B} = \mathbf{B}\mathbf{X}(t)$ then we get
$$\begin{aligned}&= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - \mathbf{A}\mathbf{X}(t)\mathbf{Y}(t) - \mathbf{X}(t)\mathbf{B}\mathbf{Y}(t)\\ &= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - \mathbf{A}\mathbf{X}(t)\mathbf{Y}(t) - \mathbf{B}\mathbf{X}(t)\mathbf{Y}(t)\\ &= (\mathbf{A}+\mathbf{B})\mathbf{Z}(t) - (\mathbf{A}+\mathbf{B})\mathbf{X}(t)\mathbf{Y}(t)\\ &= (\mathbf{A}+\mathbf{B})(\mathbf{Z}(t) - \mathbf{X}(t)\mathbf{Y}(t))\\ &= (\mathbf{A}+\mathbf{B})\mathcal{G}(t).\end{aligned}$$
We also have
$$\mathcal{G}(0) = \mathbf{Z}(0) - \mathbf{X}(0)\mathbf{Y}(0) = \mathbf{I}_n - \mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n - \mathbf{I}_n = \mathbf{0}.$$
By 2 we have
$$\mathcal{G}(t) = e^{t(\mathbf{A}+\mathbf{B})}\mathbf{0} = \mathbf{0}.$$
Thus,
$$e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}e^{t\mathbf{B}} = \mathbf{Z}(t) - \mathbf{X}(t)\mathbf{Y}(t) = \mathbf{0}$$

as we wanted. Now we show that $\mathbf{X}(t)\mathbf{B} = \mathbf{B}\mathbf{X}(t)$. Observe that

$$(\mathbf{B}\mathbf{X}(t))' = \mathbf{B}\mathbf{X}'(t) = \mathbf{B}\mathbf{A}\mathbf{X}(t) = \mathbf{A}\mathbf{B}\mathbf{X}(t) = \mathbf{A}(\mathbf{B}\mathbf{X}(t))$$

and

$$\mathbf{B}\mathbf{X}(0) = \mathbf{B}\mathbf{I}_n = \mathbf{B}.$$

By 2 we must have

$$\mathbf{B}\mathbf{X}(t) = \mathbf{X}(t)\mathbf{B}$$

for all $t \in \mathbb{R}$.

6. Observe that by 3 we have

$$(e^{t\mathbf{A}})' = \mathbf{A}e^{t\mathbf{A}} = e^{t\mathbf{A}}\mathbf{A}.$$

By transposing the previous equation we have

$$((e^{t\mathbf{A}})^T)' = \mathbf{A}^T(e^{t\mathbf{A}})^T.$$

Observe also that $(e^{0\mathbf{A}})^T = \mathbf{I}_n^T = \mathbf{I}_n$. Hence, by definition we have

$$e^{t\mathbf{A}^T} = (e^{t\mathbf{A}})^T.$$

7. Observe that

$$(\mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T})' = \mathbf{T}^{-1}(e^{t\mathbf{A}})'\mathbf{T} = \mathbf{T}^{-1}\mathbf{A}e^{t\mathbf{A}}\mathbf{T} = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})(\mathbf{T}^{-1}e^{t\mathbf{A}}T)$$

and $\mathbf{T}^{-1}e^{o\mathbf{A}}\mathbf{T} = \mathbf{T}^{-1}\mathbf{I}_n\mathbf{T} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}_n$. Thus, by definition we have

$$e^{t\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T}.$$

8. We have that

$$(e^{t\mathbf{A}})' = \mathbf{A}e^{t\mathbf{A}}$$

and so

$$(\mathbf{A}e^{t\mathbf{A}})' = \mathbf{A}(e^{t\mathbf{A}})' = \mathbf{A}^2e^{t\mathbf{A}} = \mathbf{A}e^{t\mathbf{A}}.$$

Also $\mathbf{A}e^{0\mathbf{A}} = \mathbf{A}$ so by 2 we have

$$\mathbf{A}e^{t\mathbf{A}} = e^t\mathbf{A}.$$

Observe that

$$(\mathbf{I}_n - \mathbf{A})\mathbf{A} = \mathbf{A} - \mathbf{A}^2 = \mathbf{0}.$$

This means that

$$((\mathbf{I}_n - \mathbf{A})e^{t\mathbf{A}})' = (\mathbf{I}_n - \mathbf{A})(e^{t\mathbf{A}})' = (\mathbf{I}_n - \mathbf{A})\mathbf{A}e^{t\mathbf{A}} = \mathbf{0}.$$

We conclude that $(I_n - \mathbf{A})e^{t\mathbf{A}}$ is constant and equal to $\mathbf{I}_n - \mathbf{A}$ at $t = 0$. Thus,

$$\mathbf{e^{tA}} = \mathbf{A}e^{t\mathbf{A}} + (\mathbf{I}_n - \mathbf{A})e^{t\mathbf{A}} = e^t\mathbf{A} + (\mathbf{I}_n - \mathbf{A}) = \mathbf{I}_n + (e^t - 1)\mathbf{A}.$$

9. (a)
   (b)
   (c)

10. Let $\mathbf{F}(t) = \mathbf{A}^{-1}e^{t\mathbf{A}}$. Observe that

$$\mathbf{F}'(t) = \mathbf{A}^{-1}\mathbf{A}e^{t\mathbf{A}} = e^{t\mathbf{A}}.$$

Thus,

$$\int e^{t\mathbf{A}}\mathrm{d}t = \mathbf{F}(t) + \mathbf{C}.$$

□

We observe that identity 7 allows for easier computation of the matrix exponential when the matrix $\mathbf{A}$ is diagonalizable. To see this, observe that in this case we can find an invertible matrix $\mathbf{T}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. Identity 7 gives

$$e^{t\mathbf{D}} = e^{t\cdot\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{t\mathbf{A}}\mathbf{T}$$

which means

$$e^{t\mathbf{A}} = \mathbf{T}e^{t\mathbf{D}}\mathbf{T}^{-1}.$$

Note that solving the vector ODE $\mathbf{x}' = \mathbf{D}\mathbf{x}$ is much simpler since $\mathbf{D}$ is a diagonal matrix.

8.2.2 Local lipschitz constant of exponential matrix*

We will demonstrate some results about the operator norm and Frobenius norm of the exponential matrix as well as the local lipschitz constant of the exponential matrix function. Before we begin we recall that if $\mathbf{A}$ is an $n \times n$ matrix then the operator norm of $\mathbf{A}$, denoted by $\|\mathbf{A}\|_{\mathrm{op}}$, is defined as

$$\|\mathbf{A}\|_{\mathrm{op}} = \sup_{\mathbf{x}\neq\mathbf{0}_{n\times 1}} \left\{ \frac{\|\mathbf{A}\mathbf{x}\|}{\mathbf{x}} \right\}.$$

We also define the Frobenius inner product of matrices $\mathbf{A}, \mathbf{B} \in M_{n\times n}(\mathbb{R})$ defined by

$$\langle \mathbf{A}, \mathbf{B} \rangle_F = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$$

as well as its associated norm, denoted by $\|\cdot\|_F$. Finally, recall the following result, which is a guided exercise in the section on first order equations, known as Grönwall's inequality:

Grönwall's inequality

Suppose $x : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), and satisfies $x'(t) \leq c(t)x(t) + b(t)$ on (a,b) where $c, b : [a,b] \to \mathbb{R}$ are continuous. Then

$$x(t) \leq x(a)e^{\int_a^t c(s)\mathrm{d}s} + e^{\int_a^t c(s)\mathrm{d}s}\int_a^t e^{\left[-\int_a^s c(s)\mathrm{d}s\right]} b(s)\mathrm{d}s$$

for $t \in [a,b]$.

In particular, we observe that if c is a constant function with value D then the inequality gives

$$x(t) \leq x(a)e^{D(t-a)} + e^{D(t-a)}\int_a^t e^{D(a-s)}b(s)\mathrm{d}s.$$

WIth these reminders in place we now prove the following

Proposition 8.2.2. *Let* $\mathbf{A}, \mathbf{B} \in M_{n\times n}(\mathbb{R})$. *Then we have:*

1. *if* $t \in [0,\infty)$ *then* $\|e^{t\mathbf{A}}\|_{op} \leq e^{t\|\mathbf{A}\|_{op}}$ *and, in particular,* $\|e^{\mathbf{A}}\|_{op} \leq e^{\|\mathbf{A}\|_{op}}$.
2. *If* $t \in [0,1]$ *then* $\|e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\|_{op} \leq \|\mathbf{B}\|_{op} \cdot e^{t\|\mathbf{A}\|_{op}}\left(\frac{e^{(\|\mathbf{A}+\mathbf{B}\|_{op}-\|\mathbf{A}\|_{op})t}-1}{\|\mathbf{A}+\mathbf{B}\|_{op}-\|\mathbf{A}\|_{op}}\right)$ *and, in particular,* $\|e^{(\mathbf{A}+\mathbf{B})} - e^{\mathbf{A}}\|_{op} \leq \|\mathbf{B}\|_{op} \cdot e^{\|\mathbf{A}\|_{op}}\left(\frac{e^{\|\mathbf{A}+\mathbf{B}\|_{op}-\|\mathbf{A}\|_{op}}-1}{\|\mathbf{A}+\mathbf{B}\|_{op}-\|\mathbf{A}\|_{op}}\right)$.

3. $\|e^{\mathbf{A}} - e^{\mathbf{B}}\|_{op} \leq \left(\frac{e^{\|\mathbf{A}\|_{op}} - e^{\|\mathbf{B}\|_{op}}}{\|\mathbf{A}\|_{op} - \|\mathbf{B}\|_{op}}\right)\|\mathbf{A}-\mathbf{B}\|_{op} \leq \|\mathbf{A}-\mathbf{B}\|_{op} \cdot e^{\max\{\|\mathbf{A}\|_{op}, \|\mathbf{B}\|_{op}\}}$

Proof.

1. Let $\mathbf{x}_0 \in \mathbb{R}^n$ be non-zero and $Z : \mathbb{R} \to \mathbb{R}$ be defined by

$$Z(t) = \|e^{t\mathbf{A}}\mathbf{x}_0\|^2.$$

Observe that by Cauchy-Schwarz as well as the definition of the operator norm we have

$$Z'(t) = 2\big(e^{t\mathbf{A}}\mathbf{x}_0\big) \cdot \big(\mathbf{A}e^{t\mathbf{A}}\mathbf{x}_0\big) \leq 2\|e^{t\mathbf{A}}\mathbf{x}_0\|\|\mathbf{A}\big(e^{t\mathbf{A}}\mathbf{x}_0\big)\| \leq 2\|\mathbf{A}\|_{\text{op}}\|e^{t\mathbf{A}}\mathbf{x}_0\|^2 = 2\|\mathbf{A}\|_{\text{op}}Z(t)$$

for $t \in \mathbb{R}$. Hence,

$$\big(e^{-2t\|\mathbf{A}\|_{\text{op}}}Z(t)\big)' \leq 0$$

for $t \in \mathbb{R}$. Thus, $w(t) = e^{-2t\|\mathbf{A}\|_{\text{op}}}Z(t)$ is non-increasing and hence $w(t) \leq w(0)$ for $t \geq 0$. This shows that for $t \geq 0$ we have

$$e^{-2t\|\mathbf{A}\|_{\text{op}}}\|e^{t\mathbf{A}}\mathbf{x}_0\|^2 \leq \|\mathbf{x}_0\|^2$$

and hence

$$\|e^{t\mathbf{A}}\mathbf{x}_0\| \leq \|\mathbf{x}_0\|e^{t\|\mathbf{A}\|_{\text{op}}}.$$

From this we conclude that

$$\frac{\|e^{t\mathbf{A}}\mathbf{x}_0\|}{\|\mathbf{x}_0\|} \leq e^{t\|\mathbf{A}\|_{\text{op}}}$$

for all $t \geq 0$. Since $\mathbf{x}_0$ was arbitrary we obtain

$$\|e^{t\mathbf{A}}\|_{\text{op}} \leq e^{t\|\mathbf{A}\|_{\text{op}}}$$

for $t \geq 0$. Choosing $t = 1$ gives the second part of the statement.

2. Let $\mathbf{x}_0 \in \mathbb{R}^n$ be non-zero and define $Z(t) = \|\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\|^2$ for $t \in \mathbb{R}$. Observe that by Cauchy-Schwarz and the definition of the operator norm we have, for $t \in (0,1)$

$$\begin{aligned}
Z'(t) &= 2\big[\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\big] \cdot \big[(\mathbf{A}+\mathbf{B})e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0 - \mathbf{A}e^{t\mathbf{A}}\mathbf{x}_0\big] \\
&= 2\big[\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\big] \cdot \big[\mathbf{A}\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\big] + 2\big[\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\big] \cdot \big[\mathbf{B}e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0\big] \\
&\leq 2\|\mathbf{A}\|_{\text{op}}\|\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\big)\|^2 + 2\|\big(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\big)\mathbf{x}_0\|\|\mathbf{B}e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0\| \\
&= 2\|\mathbf{A}\|_{\text{op}}Z(t) + 2\sqrt{Z(t)}\|\mathbf{B}e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0\|.
\end{aligned}$$

We can rewrite this as, assuming for now, to be removed later, that $Z(t) \neq 0$ for $t \in (0,1)$,

$$2\big(\sqrt{Z(t)}\big)' = \frac{Z'(t)}{\sqrt{Z(t)}} \leq 2\|\mathbf{A}\|_{\text{op}}\sqrt{Z(t)} + 2\|\mathbf{B}e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0\|$$

and so we get

$$\big(\sqrt{Z(t)}\big)' \leq \|\mathbf{A}\|_{\text{op}}\sqrt{Z(t)} + \|\mathbf{B}e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0\|$$

By 1 we can further estimate the last inequality as, for $t \geq 0$,

$$\begin{aligned}
&|\mathbf{A}\|_{\text{op}}\sqrt{Z(t)} + \|\mathbf{B}\|_{\text{op}}\|e^{t(\mathbf{A}+\mathbf{B})}\mathbf{x}_0\| \\
\leq&\|\mathbf{A}\|_{\text{op}}\sqrt{Z(t)} + \|\mathbf{B}\|_{\text{op}}\|e^{t(\mathbf{A}+\mathbf{B})}\|_{\text{op}}\|\mathbf{x}_0\| \\
\leq&\|\mathbf{A}\|_{\text{op}}\sqrt{Z(t)} + \|\mathbf{B}\|_{\text{op}}e^{t\|\mathbf{A}+\mathbf{B}\|_{\text{op}}}\|\mathbf{x}_0\|.
\end{aligned}$$

Altogether we conclude that

$$\left(\sqrt{Z(t)}\right)' \leq \|\mathbf{A}\|_{\mathrm{op}}\sqrt{Z(t)} + \|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\|\mathbf{x}_0\|$$

for for $t \in (0,1)$. By applying Grönwall's inequality we conclude that, for $t \in [0,1]$,

$$\sqrt{Z(t)} \leq \sqrt{Z(0)}e^{t\|\mathbf{A}\|_{\mathrm{op}}} + \|\mathbf{x}_0\|\|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}\|_{\mathrm{op}}}\int_0^t e^{-s\|\mathbf{A}\|_{\mathrm{op}}}e^{s\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\mathrm{d}s.$$

Note that $Z(0) = \|e^{\mathbf{0}_{n\times n}} - e^{\mathbf{0}_{n\times n}}\|^2 = \|0\|^2 = 0$. Thus, the inequality reduces to

$$\begin{aligned}\sqrt{Z(t)} &\leq \|\mathbf{x}_0\|\|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}\|_{\mathrm{op}}}\int_0^t e^{s(\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}})}\mathrm{d}s \\ &= \|\mathbf{x}_0\|\|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}\|_{\mathrm{op}}}\left(\frac{e^{t(\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}})}-1}{\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}}}\right).\end{aligned}$$

We deduce that, for $t \in [0,1]$,

$$\|\left(e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\right)\mathbf{x}_0\| \leq \|\mathbf{x}_0\|\|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}\|_{\mathrm{op}}}\left(\frac{e^{t(\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}})}-1}{\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}}}\right)$$

Dividing by $\|\mathbf{x}_0\|$ and noticing that $\mathbf{x}_0$ was an arbitrary non-zero vector gives

$$\|e^{t(\mathbf{A}+\mathbf{B})} - e^{t\mathbf{A}}\|_{\mathrm{op}} \leq \|\mathbf{B}\|_{\mathrm{op}} \cdot e^{t\|\mathbf{A}\|_{\mathrm{op}}}\left(\frac{e^{t(\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}})}-1}{\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}-\|\mathbf{A}\|_{\mathrm{op}}}\right).$$

for $t \in [0,1]$. Taking $t = 1$ gives the second part of the statement. Notice that the above was only derived for assuming $Z(t) \neq 0$ for $t \in (0,1)$. If this is not the case we consider $W(t) = Z(t) + \epsilon$. Then $W'(t) = Z'(t)$ and $W(t) > 0$ for $t \in (0,1)$. Thus, the previous analysis applied to W gives

$$\begin{aligned}W'(t) &\leq 2\|\mathbf{A}\|_{\mathrm{op}}\sqrt{Z(t)} + 2\|\mathbf{B}\|_{\mathrm{op}}\|\mathbf{x}_0\|_{\mathrm{op}}e^{t\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\sqrt{Z(t)} \\ &\leq 2\|\mathbf{A}\|_{\mathrm{op}}[\sqrt{Z(t)}+\epsilon] + 2\|\mathbf{B}\|_{\mathrm{op}}\|\mathbf{x}_0\|_{\mathrm{op}}e^{t\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\sqrt{[Z(t)+\epsilon]} \\ &= 2\|\mathbf{A}\|_{\mathrm{op}}W(t) + 2\|\mathbf{B}\|_{\mathrm{op}}\|\mathbf{x}_0\|_{\mathrm{op}}e^{t\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\sqrt{W(t)}\end{aligned}$$

and hence

$$\left(\sqrt{W(t)}\right)' \leq \|\mathbf{A}\|\sqrt{W(t)} + \|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\|\mathbf{x}_0\|.$$

Applying Grönwall's ineqality we get

$$\sqrt{Z(t)+\epsilon} = \sqrt{W(t)} \leq \sqrt{\epsilon}e^{t\|\mathbf{A}\|_{\mathrm{op}}} + \|\mathbf{x}_0\|\|\mathbf{B}\|_{\mathrm{op}}e^{t\|\mathbf{A}\|_{\mathrm{op}}}\int_0^t e^{-s\|\mathbf{A}\|_{\mathrm{op}}}e^{s\|\mathbf{A}+\mathbf{B}\|_{\mathrm{op}}}\mathrm{d}s.$$

Letting $\epsilon > 0$ tend to 0 we get the same inequality as before. Thus, we can repeat the same analysis to get the desired conclusion.

3. Notice that $\mathbf{A}$ and $\mathbf{B}$ were arbitrary in 2 and so we placing $\mathbf{A}+\mathbf{B}$ with $\mathbf{X}$ and $\mathbf{A}$ with $\mathbf{Y}$ the inequality becomes, for $t = 1$,

$$\|e^{\mathbf{X}} - e^{\mathbf{Y}}\|_{\mathrm{op}} \leq \|\mathbf{X}-\mathbf{Y}\|_{\mathrm{op}} \cdot e^{\|\mathbf{Y}\|_{\mathrm{op}}}\left(\frac{e^{\|\mathbf{X}\|_{\mathrm{op}}-\|\mathbf{Y}\|_{\mathrm{op}}}-1}{\|\mathbf{X}\|_{\mathrm{op}}-\|\mathbf{Y}\|_{\mathrm{op}}}\right) = \|\mathbf{X}-\mathbf{Y}\|_{\mathrm{op}}\left(\frac{e^{\|\mathbf{X}\|_{\mathrm{op}}}-e^{\|\mathbf{Y}\|_{\mathrm{op}}}}{\|\mathbf{X}\|_{\mathrm{op}}-\|\mathbf{Y}\|_{\mathrm{op}}}\right)$$

which demonstrates one of the desired inequalities. Notice that since $\mathbf{A}$ and $\mathbf{B}$ can vary over all matrices and

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \\ \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

meaning that the linear map transforming $\mathbf{A}$ and $\mathbf{B}$ to $\mathbf{X}$ and $\mathbf{Y}$ is invertible then $\mathbf{X}$ and $\mathbf{Y}$ vary over all matrices. Now, consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(s) = e^s$. Observe that for any $s, t \in \mathbb{R}$ we have, for some $r \in (0, 1)$,

$$g(t) - g(s) = g'((1-r)s + rt)(t - s)$$

so

$$\frac{e^t - e^s}{t - s} = e^{(1-r)s+rt} \le e^{\max s,t}$$

where I have used that the exponential is an increasing function. From this we conclude that

$$\|e^{\mathbf{X}} - e^{\mathbf{Y}}\|_{\text{op}} \le \|\mathbf{X} - \mathbf{Y}\|_{\text{op}} \left(\frac{e^{\|\mathbf{X}\|_{\text{op}}} - e^{\|\mathbf{Y}\|_{\text{op}}}}{\|\mathbf{X}\|_{\text{op}} - \|\mathbf{Y}\|_{\text{op}}} \right) \le e^{\max\{\|\mathbf{X}\|_{\text{op}}, \|\mathbf{Y}\|_{\text{op}}\}} \|\mathbf{X} - \mathbf{Y}\|_{\text{op}}$$

which proves the desired inequality. □

It is worth remarking that the inequalities shown in 3 are in fact sharp in the sense that there is no constant $0 < c < 1$ such that either

$$\|e^{\mathbf{X}} - e^{\mathbf{Y}}\|_{\text{op}} \le c\|\mathbf{X} - \mathbf{Y}\|_{\text{op}} \left(\frac{e^{\|\mathbf{X}\|_{\text{op}}} - e^{\|\mathbf{Y}\|_{\text{op}}}}{\|\mathbf{X}\|_{\text{op}} - \|\mathbf{Y}\|_{\text{op}}} \right)$$

or

$$\|e^{\mathbf{X}} - e^{\mathbf{Y}}\|_{\text{op}} \le c e^{\max\{\|\mathbf{X}\|_{\text{op}}, \|\mathbf{Y}\|_{\text{op}}\}} \|\mathbf{X} - \mathbf{Y}\|_{\text{op}}$$

or

$$\|\mathbf{X} - \mathbf{Y}\|_{\text{op}} \left(\frac{e^{\|\mathbf{X}\|_{\text{op}}} - e^{\|\mathbf{Y}\|_{\text{op}}}}{\|\mathbf{X}\|_{\text{op}} - \|\mathbf{Y}\|_{\text{op}}} \right) \le c e^{\max\{\|\mathbf{X}\|_{\text{op}}, \|\mathbf{Y}\|_{\text{op}}\}} \|\mathbf{X} - \mathbf{Y}\|_{\text{op}}$$

can hold. To see this, consider the sequence $\mathbf{X}_m = m\mathbf{I}_n$ and $\mathbf{Y}_m = (m - \frac{1}{m})\mathbf{I}_n$. Then $\|\mathbf{X}_m\|_{\text{op}} = m$, $\|\mathbf{Y}_m\|_{\text{op}} = m - \frac{1}{m}$, $e^{\mathbf{X}_m} = e^m \mathbf{I}_n$, and $e^{\mathbf{Y}_m} = e^{m - \frac{1}{m}} \mathbf{I}_n$. Thus, $\|e^{\mathbf{X}_m} - e^{\mathbf{Y}_m}\|_{\text{op}} = e^m(1 - e^{-\frac{1}{m}})$ and $\|\mathbf{X}_m - \mathbf{Y}_m\|_{\text{op}} = \frac{1}{m}$. From this we conclude that

$$\|e^{\mathbf{X}_m} - e^{\mathbf{Y}_m}\|_{\text{op}} \approx \frac{1}{m} e^m$$

and

$$\|\mathbf{X}_m - \mathbf{Y}_m\|_{\text{op}} \left(\frac{e^{\|\mathbf{X}_m\|_{\text{op}}} - e^{\|\mathbf{Y}_m\|_{\text{op}}}}{\|\mathbf{X}_m\|_{\text{op}} - \|\mathbf{Y}_m\|_{\text{op}}} \right) \approx \frac{1}{m} e^m$$

and

$$e^{\max\{\|\mathbf{X}_m\|_{\text{op}}, \|\mathbf{Y}_m\|_{\text{op}}\}} \|\mathbf{X}_m - \mathbf{Y}_m\|_{\text{op}} \approx \frac{1}{m} e^m.$$

In fact, if we set $\mathbf{A} = a\mathbf{I}_m$ and $\mathbf{B} = b\mathbf{I}_m$, for $a > b > 0$, then we obtain $\|\mathbf{A}\|_{\text{op}} = a$, $\|\mathbf{B}\|_{\text{op}} = b$, $\|\mathbf{A} - \mathbf{B}\|_{\text{op}} = a - b$, and $\|e^{\mathbf{A}} - e^{\mathbf{B}}\|_{\text{op}} = e^a - e^b$. Hence,

$$\|e^{\mathbf{A}} - e^{\mathbf{B}}\|_{\text{op}} = e^a - e^b$$

$$\left(\frac{e^{\|\mathbf{A}\|_{\text{op}}} - e^{\|\mathbf{B}\|_{\text{op}}}}{\|\mathbf{A}\|_{\text{op}} - \|\mathbf{B}\|_{\text{op}}}\right)\|\mathbf{A}-\mathbf{B}\|_{\text{op}} = \left(\frac{e^a - e^b}{a-b}\right)(a-b) = e^a - e^b.$$

Thus, we actually have that strict equality is attained in the first inequality. Now we demonstrate a similar result for the Frobenius norm. The proof is fairly similar to the operator norm case so we will proceed quickly.

Proposition 8.2.3. *Let $\mathbf{A}, \mathbf{B} \in M_{n\times n}(\mathbb{R})$. Then we have:*

1. $\|e^{t\mathbf{A}}\|_{\text{F}} \leq e^{t\|\mathbf{A}\|_{\text{F}}}$ *for $t \geq 0$ and, in particular, for $t = 1$, $\|e^{\mathbf{A}}\|_{\text{F}} \leq e^{\|\mathbf{A}\|_{\text{F}}}$.*

2. $\|e^{\mathbf{A}} - e^{\mathbf{B}}\|_{\text{F}} \leq \|\mathbf{A}-\mathbf{B}\|_{\text{F}}\left(\frac{e^{\|\mathbf{A}\|_{\text{F}}} - e^{\|\mathbf{B}\|_{\text{F}}}}{\|\mathbf{A}\|_{\text{F}} - \|\mathbf{B}\|_{\text{F}}}\right) \leq e^{\max\{\|\mathbf{A}\|_{\text{F}}, \|\mathbf{B}\|_{\text{F}}\}}\|\mathbf{A}-\mathbf{B}\|_{\text{F}}$.

3. $\|\mathbf{A}\|_{op} \leq \|\mathbf{A}\|_{F}$.

Proof. 1. Observe that if $X(t) = \text{tr}((e^{t\mathbf{A}})^T e^{t\mathbf{A}})$ defined on $\mathbb{R}$ we have, by Cauchy-Schwarz,

$$\begin{aligned} X'(t) &= 2\text{tr}((e^{t\mathbf{A}})^T \mathbf{A} e^{t\mathbf{A}}) = 2\text{tr}((e^{t\mathbf{A}})^T e^{t\mathbf{A}}\mathbf{A}) \\ &\leq 2\|(e^{t\mathbf{A}})^T e^{t\mathbf{A}}\|_{\text{F}}\|\mathbf{A}\|_{\text{F}} =\leq 2\|e^{t\mathbf{A}}\|_{\text{F}}^2\|\mathbf{A}\|_{\text{F}}. \end{aligned}$$

Hence, by Grönwall's inequality we have, for $t \geq 0$,

$$X(t) \leq X(0)e^{2t\|\mathbf{A}\|_{\text{F}}} = e^{2t\|\mathbf{A}\|_{\text{F}}}$$

and hence

$$\|e^{t\mathbf{A}}\|_{\text{F}} \leq e^{t\|\mathbf{A}\|_{\text{F}}}$$

for $t \geq 0$, as we wanted.

2. Define, similar to the previous proof, $X : [0,1] \to \mathbb{R}$ by $X(t) = \|e^{t\mathbf{A}} - e^{t\mathbf{B}}\|_{\text{F}}^2$ and differentiate for $t \in (0,1)$ to get

$$\begin{aligned} X'(t) &= 2\langle e^{t\mathbf{A}} - e^{t\mathbf{B}}, \mathbf{A}e^{t\mathbf{A}} - \mathbf{B}e^{t\mathbf{B}}\rangle_{\text{F}} \\ &= 2\langle e^{t\mathbf{A}} - e^{t\mathbf{B}}, \mathbf{A}(e^{t\mathbf{A}} - e^{t\mathbf{B}})\rangle_{\text{F}} + 2\langle e^{t\mathbf{A}} - e^{t\mathbf{B}}, (\mathbf{A}-\mathbf{B})e^{t\mathbf{B}}\rangle_{\text{F}} \\ &\leq 2\|e^{t\mathbf{A}} - e^{t\mathbf{B}}\|_{\text{F}} \cdot \|\mathbf{A}(e^{t\mathbf{A}} - e^{t\mathbf{B}})\|_{\text{F}} + 2\|e^{t\mathbf{A}} - e^{t\mathbf{B}}\|_{\text{F}}\|(\mathbf{A}-\mathbf{B})e^{t\mathbf{B}}\|_{\text{F}} \\ &\leq 2\|\mathbf{A}\|_{\text{F}}X(t) + 2\sqrt{X(t)}\|\mathbf{A}-\mathbf{B}\|e^{t\|\mathbf{B}\|_{\text{F}}}. \end{aligned}$$

If $X(t) \neq 0$ for all $t \in (0,1)$, and maneuvering as in the proof of 2 if this is not the case, we obtain

$$\left(\sqrt{X}(t)\right)' \leq 2\|\mathbf{A}\|_{\text{F}}\sqrt{X(t)} + \|\mathbf{A}-\mathbf{B}\|_{\text{F}}e^{t\|\mathbf{B}\|_{\text{F}}}$$

which gives, by Grönwall's inequality and noting that $X(0) = 0$,

$$\|e^{t\mathbf{A}} - e^{t\mathbf{B}}\|_{\text{F}} = \sqrt{X(t)} \leq \|\mathbf{A}-\mathbf{B}\|_{\text{F}}\frac{e^{t\|\mathbf{B}\|_{\text{F}}} - e^{t\|\mathbf{A}\|_{\text{F}}}}{\|\mathbf{B}\|_{\text{F}} - \|\mathbf{A}\|_{\text{F}}}.$$

3. Note that for each non-zero $\mathbf{x}_0 \in \mathbb{R}^n$ we have

$$\begin{aligned} \|\mathbf{A}\mathbf{x}_0\|^2 &= (\mathbf{A}\mathbf{x}_0)^T(\mathbf{A}\mathbf{x}_0) = \mathbf{x}_0^T\mathbf{A}^T\mathbf{A}\mathbf{x}_0 = \text{tr}(\mathbf{x}_0^T\mathbf{A}^T\mathbf{A}\mathbf{x}_0) = \text{tr}(\mathbf{x}_0\mathbf{x}_0^T\mathbf{A}^T\mathbf{A}) \\ &\leq \sqrt{\text{tr}(\mathbf{x}_0\mathbf{x}_0^T\mathbf{x}_0\mathbf{x}_0^T)}\|\mathbf{A}^T\mathbf{A}\|_{\text{F}} \leq \|\mathbf{x}_0\|\sqrt{\text{tr}(\mathbf{x}_0\mathbf{x}_0^T)}\|\mathbf{A}\|_{\text{F}}^2 \\ &= \|\mathbf{x}_0\|\sqrt{\text{tr}(\mathbf{x}_0^T\mathbf{x}_0)}\|\mathbf{A}\|_{\text{F}}^2 = \|\mathbf{x}_0\|^2\|\mathbf{A}\|_{\text{F}}^2. \end{aligned}$$

We conclude that for all non-zero $\mathbf{x}_0$ that

$$\frac{\|\mathbf{A}\mathbf{x}_0\|}{\|\mathbf{x}_0\|} \leq \|\mathbf{A}\|_{\mathrm{F}}$$

and so

$$\|\mathbf{A}\|_{\mathrm{op}} \leq \|\mathbf{A}\|_{\mathrm{F}}.$$

□

8.2.3 Liouville's Formula*

Proposition 8.2.4. *Suppose $\mathbf{A} : \mathbb{R} \to M_{n\times n}(\mathbb{R})$ is a matrix-valued function. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A}(t))) = \sum_{i=1}^{n} \det(\mathbf{A}^i(t))$$

where

$$\mathbf{A}^i(t) = \begin{pmatrix} A_{1,1}(t) & A_{1,2}(t) & \cdots & A_{1,n}(t) \\ A_{2,1}(t) & A_{2,2}(t) & \cdots & A_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A'_{i,1}(t) & A'_{i,2}(t) & \cdots & A'_{i,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}(t) & A_{n,2}(t) & \cdots & A_{n,n}(t) \end{pmatrix}$$

for $i = 1, \ldots, n$.

Proof. We proceed by induction. For $n = 1$, since $\mathbf{A}(t)$ will be a 1×1 matrix then $\det(\mathbf{A}(t)) = A_{1,1}(t)$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A}(t))) = A'_{1,1}(t) = \sum_{i=1}^{1} \det(A^i(t))$$

where the last equality follows from the fact that there is only one row to take the derivative of and

$$\det(A^1(t)) = \det(A'_{1,1}(t)) = A'_{1,1}(t).$$

Now we presume this formula holds for $n-1$, $n \geq 2$ and we show it holds for n. Observe that

$$\det(\mathbf{A}(t)) = \sum_{i=1}^{n} (-1)^{1+i} A_{1,i}(t) \det(\tilde{\mathbf{A}}_{1,i}(t))$$

where $\tilde{\mathbf{A}}_{1,i}(t)$ denotes the matrix, of size $(n-1)\times(n-1)$ obtained from $\mathbf{A}(t)$ which has row 1 and column i removed. Differentiating and using the induction hypothesis we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A}(t))) = \sum_{i=1}^{n} (-1)^{1+i} A'_{1,i}(t) \det(\mathbf{A}_{1,i}\tilde{}(t)) + \sum_{i=1}^{n} (-1)^{1+i} A_{1,i}(t) \frac{\mathrm{d}}{\mathrm{d}t}(\det(\tilde{\mathbf{A}}_{1,i}(t)))$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (-1)^{1+i} A'_{1,i}(t) \det(\tilde{\mathbf{A}_{1,i}}(t)) + \sum_{i=1}^{n} (-1)^{1+i} A_{1,i}(t) \sum_{j=1}^{n-1} \det(\tilde{\mathbf{A}}^{j}_{1,i}(t)) \\
&= \det(A^1(t)) + \sum_{i=1}^{n} (-1)^{1+i} A_{1,i}(t) \sum_{j=1}^{n-1} \det(\tilde{\mathbf{A}}^{j}_{1,i}(t)) \\
&= \det(A^1(t)) + \sum_{j=1}^{n-1} \sum_{i=1}^{n} (-1)^{1+i} A_{1,i}(t) \det(\tilde{\mathbf{A}}^{j}_{1,i}(t)) \\
&= \det(A^1(t)) + \sum_{j=1}^{n-1} \det(\mathbf{A}^{j+1}(t)) \\
&= \det(A^1(t)) + \sum_{j=2}^{n} \det(\mathbf{A}^{j}(t)) \\
&= \sum_{j=1}^{n} \det(\mathbf{A}^{j}(t))
\end{aligned}
$$

□

Proposition 8.2.5. *Let $\mathbf{X}(t)$ denote the matrix of fundamental solutions to the problem*

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

where $\mathbf{x}(t)$ is an n-vector. Then

$$\det(\mathbf{X}(t)) = \det(\mathbf{X}(0)) e^{t \cdot \mathrm{tr}(\mathbf{A})}.$$

As a consequence we have

$$\det(e^{t\mathbf{A}}) = e^{t \cdot \mathrm{tr}(\mathbf{A})}.$$

Proof. We first notice that by proposition 2.2 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{X}(t)) = \sum_{i=1}^{n} \det(\mathbf{X}^i(t)).$$

Observe that

$$
\mathbf{X}^i(t) = \begin{pmatrix}
X_{1,1}(t) & X_{1,2}(t) & \cdots & X_{1,n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
X'_{i,1}(t) & X'_{i,2}(t) & \cdots & X'_{i,n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
X_{n,1}(t) & X_{n,2}(t) & \cdots & X_{n,n}(t)
\end{pmatrix}
$$

$$= \begin{pmatrix} X_{1,1}(t) & X_{1,2}(t) & \cdots & X_{1,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} A_{i,j} X_{j,1}(t) & \sum_{j=1}^{n} A_{i,j} X_{j,2}(t) & \cdots & \sum_{j=1}^{n} A_{i,j} X_{j,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ X_{n,1}(t) & X_{n,2}(t) & \cdots & X_{n,n}(t) \end{pmatrix}.$$

Recall that subtracting multiples of one row from another does not change the value of the determinant. So subtracting $A_{i,1}$ times row 1 of $\mathbf{X}(t)$ from row i and then subtracting $A_{2,i}$ times row 2 of $\mathbf{X}(t)$ from row i and so on does not change the value of the determinant but leads us to

$$\det(\mathbf{X}^i(t)) = \det \begin{pmatrix} X_{1,1}(t) & X_{1,2}(t) & \cdots & X_{1,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{i,i} X_{i,1}(t) & A_{i,i} X_{i,2}(t) & \cdots & A_{i,i} X_{i,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ X_{n,1}(t) & X_{n,2}(t) & \cdots & X_{n,n}(t) \end{pmatrix} = A_{i,i} \det(\mathbf{X}(t)).$$

The above conclusions are true for each $i = 1, \ldots, n$. We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{X}(t))) = \sum_{i=1}^{n} A_{i,i} \det(\mathbf{X}(t)) = \mathrm{tr}(\mathbf{A}) \det(\mathbf{X}(\mathrm{t})).$$

We conclude that

$$\det(\mathbf{X}(t)) = Ce^{t \cdot \mathrm{tr}(\mathbf{A})}.$$

Evaluating at $t = 0$ gives

$$C = \det(\mathbf{X}(0)).$$

To obtain the second identity notice that the first identity can be written as

$$\det(\mathbf{X}(t)(\mathbf{X}(0))^{-1}) = e^{t \cdot \mathrm{tr}(\mathbf{A})}$$

and notice that $\mathbf{X}(t)(\mathbf{X}(0))^{-1}$ solves

$$\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{I}_n.$$

□

8.2.4 Remarks

The construction of the matrix exponential given in section 8.2 is not the standard development. Generally, one defines this matrix through the use of infinite series of matrices. It is then a theorem that the matrix exponential solves the matrix ODE $\mathbf{X}' = \mathbf{A}\mathbf{X}$ with $\mathbf{X}(0) = \mathbf{I}_n$. The construction given in section 8.2 is probably not new though the authors of these notes have no citations for this technique. It is, perhaps, worth noting that many of the standard identities involving the exponential matrix can be obtained from 8.2.1 and 8.2.3 by simply setting $t = 1$.