

1 Hamiltonian systems and Modeling of Mechanical systems

This project is taken from [NSS14, chapter 5, D Project]. The problems in this project explore the Hamiltonian formulation of the laws of motion of a system and its phase plane implications. This formulation replaces Newton's second law $F = ma = my''$ and is based on three mathematical manipulations:

- It is presumed that the force $F(t, y, y')$ depends only on y and $F(y) = -\frac{dV(y)}{dy}$, where $V(y)$ is called the potential.
- The velocity variable y' is replaced throughout by the momentum $y' = p/m$.
- The Hamiltonian of a system (conservative or not) is defined as

$$H(y, p) := \frac{p^2}{2m} + V(y),$$

i.e. the sum of the kinetic and potential energy.

With these in mind we express the equation $my'' = F(y) = -\frac{dV(y)}{dy}$ as the following system called *Hamilton's equations*:

$$\begin{aligned}\frac{dy}{dt} &= \frac{dH}{dp} = \frac{p}{m} \\ \frac{dp}{dt} &= -\frac{dH}{dy} = -\frac{dV(y)}{dy}.\end{aligned}$$

These equations imply by chain rule that

$$\frac{d}{dt}H(y, p) = \frac{dH}{dy} \frac{dy}{dt} + \frac{dH}{dp} \frac{dp}{dt} = \frac{dH}{dy} \frac{dH}{dp} - \frac{dH}{dp} \frac{dH}{dy} = 0.$$

Therefore, the Hamiltonian remains constant along solution curves $\gamma(t) = (y(t), p(t))$ of the above system. This is the equivalent formulation of the conservation of energy when the forces are conservative (i.e. do not change in time such as gravity).

Hamilton's formulation for mechanical systems and the conservation of energy principle imply that the phase plane trajectories of conservative systems lie on the curves where the Hamiltonian $H(y, p)$ is constant, and plotting these curves may be considerably easier than solving for the trajectories directly (which, in turn, is easier than solving the original system!).

A general system

$$\frac{dy}{dt} = f(y, p), \quad \frac{dp}{dt} = g(y, p)$$

has a Hamiltonian function H if

$$\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial p}.$$

Because then if we define H by

$$\frac{\partial H}{\partial p} = f(y, p), \quad \frac{\partial H}{\partial y} = -g(y, p)$$

then we have conservation of energy:

$$\frac{dH(y, p)}{dt} = H_y \frac{dy}{dt} + H_p \frac{dp}{dt} = -g(y, p)f(y, p) + g(y, p)f(y, p) = 0.$$

2 Project "Hamiltonian systems and Modeling of Mechanical systems" Problems

1. (30 points) For the mass-spring oscillator the spring force is given by $F(y) = -k \cdot y$ (where k is the spring constant).

- (a) (5 points) For a general Hamiltonian system

$$\frac{dy}{dt} = f(y, p), \quad \frac{dp}{dt} = g(y, p)$$

compute its linearization and plug in the Hamiltonian equations to conclude that a Hamiltonian system cannot have spiral sinks or sources critical point\`s.

- (b) (5 points) Show that the above system with spring force $F(y) = -k \cdot y$ is Hamiltonian and then find the Hamiltonian and express Hamilton's equations.
- (c) (10 points) As done for autonomous systems, take the ratio of the equations and show that the phase plane trajectories $H(y, p) = \text{constant}$ for this system are the ellipses given by $p^2/2m + ky^2/2 = \text{constant}$.
- (d) (10 points) Bonus: Plot the direction field and some of the ODE solutions using an ODE solver.
- (e) (10 points) Linearize the system on the critical point\`s and deduce the stability behaviour. Does that agree with part 1? Does it agree with the direction field (if you did the matlab part)?

2. (70 points) Stability and damping.

The damping force -by' considered is not conservative, of course. Physically speaking, we know that damping drains the energy from a system until it grinds to a halt at an equilibrium point. In the phase plane, we can qualitatively describe the trajectory as continuously migrating to successively lower constant-energy orbits; stable centers become asymptotically stable spiral points when damping is taken into consideration. The second Hamiltonian equation, which effectively states $p' = my'' = F$, has to be changed to

$$p' = -\frac{dH}{dy} - by' = -\frac{dH}{dy} - b\frac{p}{m},$$

when damping is present.

Periodic force: For a pendulum system with a periodic force given by

$$F = -l \cdot m \cdot g \cdot \sin(\theta) = -\frac{\partial}{\partial \theta}(-l \cdot m \cdot g \cdot \cos(\theta)) = -\frac{\partial}{\partial \theta}V(\theta),$$

where l is the length of the pendulum. With damping present and using that the angular momentum is $p = ml^2\theta'$, the Hamiltonian equations are:

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{dH}{dp} = \frac{p}{ml^2} \\ \frac{dp}{dt} &= -\frac{dH}{d\theta} - b\theta' = -\frac{dV(\theta)}{d\theta} - b\frac{p}{ml^2}. \end{aligned}$$

- (a) (5 points) Derive the Hamiltonian for the undamped system ($b=0$) and sketch the phase trajectories.
- (b) (5 points) Bonus: Plot the Hamiltonian trajectories using software.
- (c) (10 points) Obtain the linearization for the undamped system and the damped system around arbitrary point.
- (d) (30 points) Linearize the undamped system around each of the critical point\(-s and based on the eigenvalues identify the qualitative behaviour (saddle-unstable, unstable or stable node).
- (e) (5 points) Bonus: Plot the direction field and the ODE solutions using an ODE solver. Compare them to those of the Hamiltonian.
- (f) (30 points) Linearize the damped system around each of the critical point\(-s and based on the eigenvalues identify the qualitative behaviour (saddle-unstable, unstable or stable node). What happens to the stability behaviour as the damping b gets smaller?
- (g) (5 points) Bonus: Plot the direction field and the ODE solutions using an ODE solver. Compare them to those of the Hamiltonian as the damping b gets smaller.

References

- [NSS14] R Kent Nagle, Edward B Saff, and Arthur David Snider. *Fundamentals of differential equations*. Pearson Education, 2014.