Modeling of Economic growth with human capital

This project is taken from [MRW92]. You can find many online sources and analyses on the MRW model (Mankiw, Romer and Weil).
The predictions of the Solow model are, to a first approximation, consistent with the evidence. Examining recently available data for a large set of countries, we find that saving and population growth affect income in the directions that Solow predicted. Moreover, more than half of the cross-country variation in income per capita can be explained by these two variables alone.
Yet all is not right for the Solow model. Although the model correctly predicts the directions of the effects of saving and the data the effects of saving and population growth on income are too large. To understand the relation between saving, population growth, and income, one must go beyond the textbook Solow model.
We therefore augment the Solow model by including accumulation of human as well as physical capital. The exclusion of human capital from the textbook Solow model can potentially explain why the estimated influences of saving and population growth appear too large, for two reasons. First, for any given rate of human- capital accumulation, higher saving or lower population growth leads to a higher level of income and thus a higher level of human capital; hence, accumulation of physical capital and population growth have greater impacts on income when accumulation of human capital is taken into account. Second, human-capital accu- mulation may be correlated with saving rates and population growth rates; this would imply that omitting human-capital accu- mulation biases the estimated coefficients on saving and population growth.
Let us consider a closed economy, with a single productive sector, which uses physical capital (K_t), labor force (L_t) and human capital (H_t) as factors of production (Y_t).

- The economy is endowed with a technology defined by a Cobb-Douglas production function with constant returns to scale:

$$Y_t = K_t^{\alpha} H_t^{\beta} L_t^{1-\alpha-\beta}, \alpha, \beta, \alpha+\beta \in (0,1).$$

- The capital stock changes equal the gross investment $s_k Y_t$ minus the capital depreciation $\delta \cdot K_t$:

$$K_{t+\Delta t} - K_t = \Delta t \cdot (s_k Y_t - \delta K_t).$$

- The human capital stock changes equal the gross investment $s_h Y_t$ minus the capital depreciation $\delta \cdot H_t$:

$$H_{t+\Delta t} - H_t = \Delta t \cdot (s_h Y_t - \delta H_t).$$

- The model assumes that the population growths at a constant rate $n > 0$:

$$L_{t+\Delta t} = (1 + \Delta t \cdot n) L_t, L(0) > 0.$$

Project "Modeling of Economic growth with human capital" Problems

1. (70 points) If we define $k(t) := \frac{K}{L}$ as the physical capital per worker and $h(t) := \frac{H}{L}$ as the human capital per worker then derive the MRW system:

$$\frac{\mathrm{d}k}{\mathrm{dt}} = s_k k(t)^{\alpha} h(t)^{\beta} - (\delta + n) k(t)$$
$$\frac{\mathrm{d}h}{\mathrm{dt}} = s_h k(t)^{\alpha} h(t)^{\beta} - (\delta + n) h(t).$$

   Hint: start from

$$k' = \lim_{\Delta t \to 0} \frac{\frac{K_{t+\Delta t} - K_t}{\Delta t} L_t - K_t \frac{L_{t+\Delta t} - L_t}{\Delta t}}{L_t^2}$$

   and use the above model assumptions; work similarly for h.

(a) (10 points) Find the critical points for the MRW system.

(b) (20 points) Obtain the linearization around arbitrary point (section 2 in "nonlinear odes").

(c) (30 points) Linearize the system around each of the critical points and based on the eigenvalues of the Jacobian identify the qualitative behaviour (saddle-unstable, unstable or stable node). Sketch a picture (or just provide a plot using software).

(d) (10 points) How do the parameters α, β of the model affect the stable critical point (called the steady state)? In particular, based on the eigenvalues of the Jacobian, what happens when $\alpha, \beta, \alpha + \beta$ are not restricted to be in $(0, 1)$.

(e) (20 points) Bonus: plot the direction field and the ODE solutions using an ODE solver (eg.in MATLAB). Provide plots for both cases i.e. pick appropriate parameters so that a)$\alpha + \beta < 1$ and b)$\alpha + \beta > 1$.

2. (30 points) Comparing the rate of convergence with the Solow model.

(a) (10 points) For the Solow model we have $Y_t = K_t^\alpha L_t^{1-\alpha}$. Assuming the same capital stock change and population growth derive the equation:

$$\frac{\mathrm{d}k}{\mathrm{dt}} = s_k k^\alpha - (\delta + n)k.$$

(b) (10 points) Find the equilibrium point and linearize around that point. Find that the rate of convergence is $|(1 - \alpha)(n + \delta)|$.

(c) (10 points) In higher dimensions the rate of convergence is determined by the absolute value of largest eigenvalue of the Jacobian. Find the largest eigenvalue of the MRW model around the equilibrium point. Show that it is smaller that of Solow model.

(d) (5 points) Bonus: Put this result into context, that is, explain why introducing human capital accumulation implies slower rate of convergence for a given gap between current output per worker and steady-state output per worker.

(e) (5 points) Bonus: Plot solution trajectories of the Solow model and the MRW model using an ODE solver.

References

[MRW92] N Gregory Mankiw, David Romer, and David N Weil. "A contribution to the empirics of economic growth". In: *The quarterly journal of economics* 107.2 (1992), pp. 407–437.