MAT244 ODEs **Name:** ______________

Summer 2018 **Student No:** ______________

Midterm 2 **Tutorial time** ______________
27/07/2018
Time Limit: 120 Minutes

This exam contains 22 pages (including this cover page) and 4 questions. No aids allowed. The back pages are for computations and rough work. If you want the grader to mark them, please make it clear.
Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	40	
Total:	100	

Formulas

- For repeated eigenvalues the ansatz is $x_2(t) = te^{\lambda t}\xi + \eta e^{\lambda t}$.
- The eigenvalue and eigenvector formulas for matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4 * det(A)}.$$

  - If $c \neq 0$ then $\xi_1 = \binom{\lambda_1 - d}{c}, \xi_2 = \binom{\lambda_2 - d}{c}$.
  - If $c = 0, b \neq 0$ then $\xi_1 = \binom{b}{\lambda_1 - a}, \xi_2 = \binom{b}{\lambda_2 - a}$.
  - If $b = c = 0$ then $\xi_1 = \binom{1}{0}$ for $\lambda_1 = a$,$\xi_2 = \binom{0}{1}$ for $\lambda_2 = d$.

1. (20 points) For some $a \in [-1, 2]$ consider the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \left(\begin{smallmatrix} -1 & -1 \\ -a & -1 \end{smallmatrix}\right)\mathbf{x}.$$

(a) (4 points) Find the eigenvalues in terms of a and determine the bifurcation values where the qualitative nature of the phase portrait for the system changes.
A: Using the formula from the sheet, the eigenvalues are

$$\lambda = -1 \pm \sqrt{a}.$$

So the bifurcation values are at $a = 1$ going between source and sink and at $a = 0$ going from real to repeated at $a = 0$ and then to spiral sink for negative a.

Common mistakes and advice

- Please use the eigenvalue and eigenvector formulas. Do not show us the middle steps in computing them. Do them at the back.
- Be careful with the eigenvalue calculation because it can mess up your entire answer.

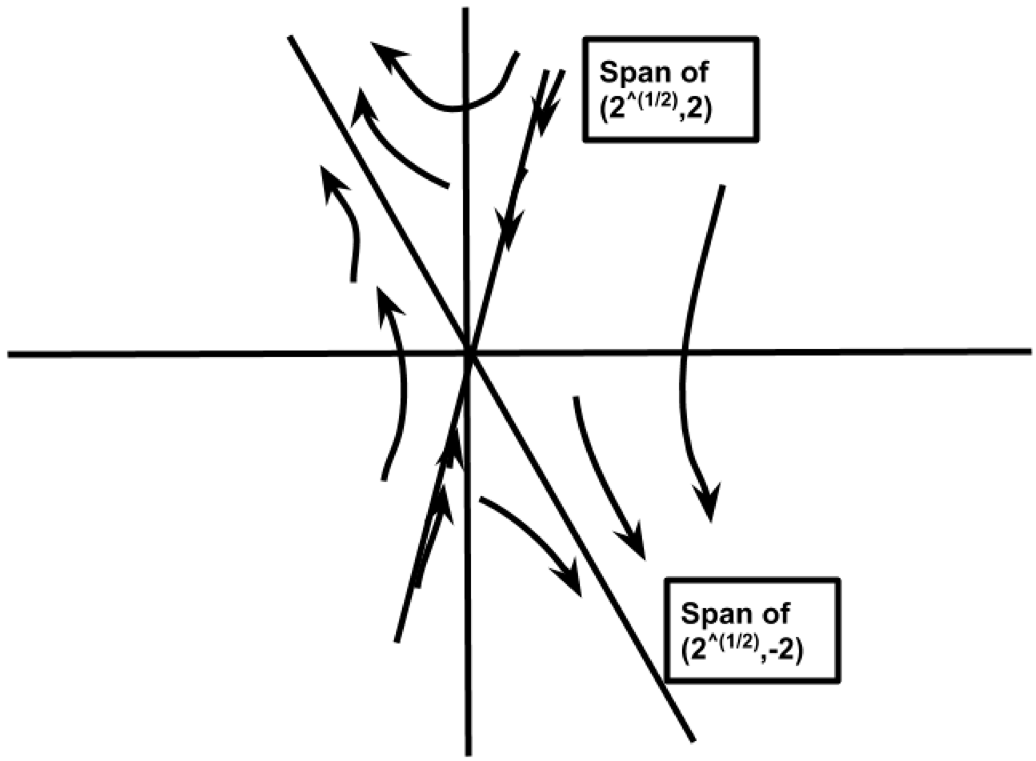
(b) (4 points) Find the general solution in terms of $a \in [-1, 0) \cup (0, 2]$.
A: the eigenpairs are $(1 + \sqrt{\alpha}, \left(\begin{smallmatrix} \sqrt{a} \\ -a \end{smallmatrix}\right))$ and $(1 + \sqrt{\alpha}, \left(\begin{smallmatrix} \sqrt{a} \\ a \end{smallmatrix}\right))$ and so

$$x = c_1 e^{(-1+\sqrt{a})t} \begin{pmatrix} \sqrt{a} \\ -a \end{pmatrix} + c_2 e^{(-1-\sqrt{a})t} \begin{pmatrix} \sqrt{a} \\ a \end{pmatrix}.$$

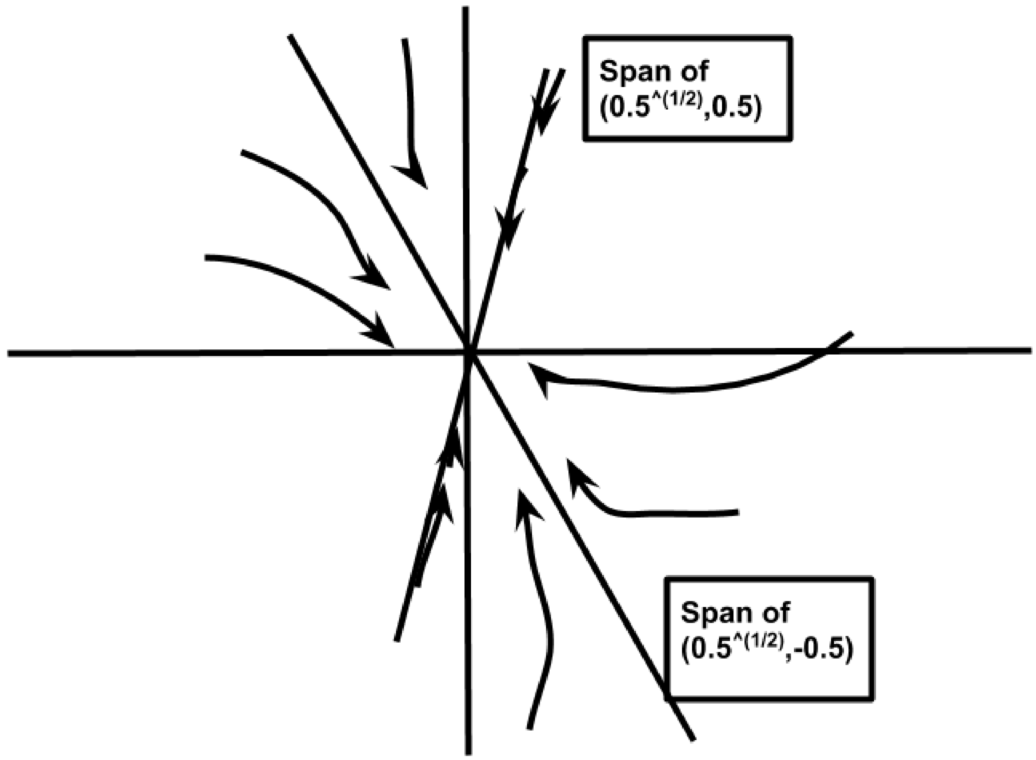
Common mistakes and advice

- Please use leave your answer as complex exponentials; I will mention that in the final as well.

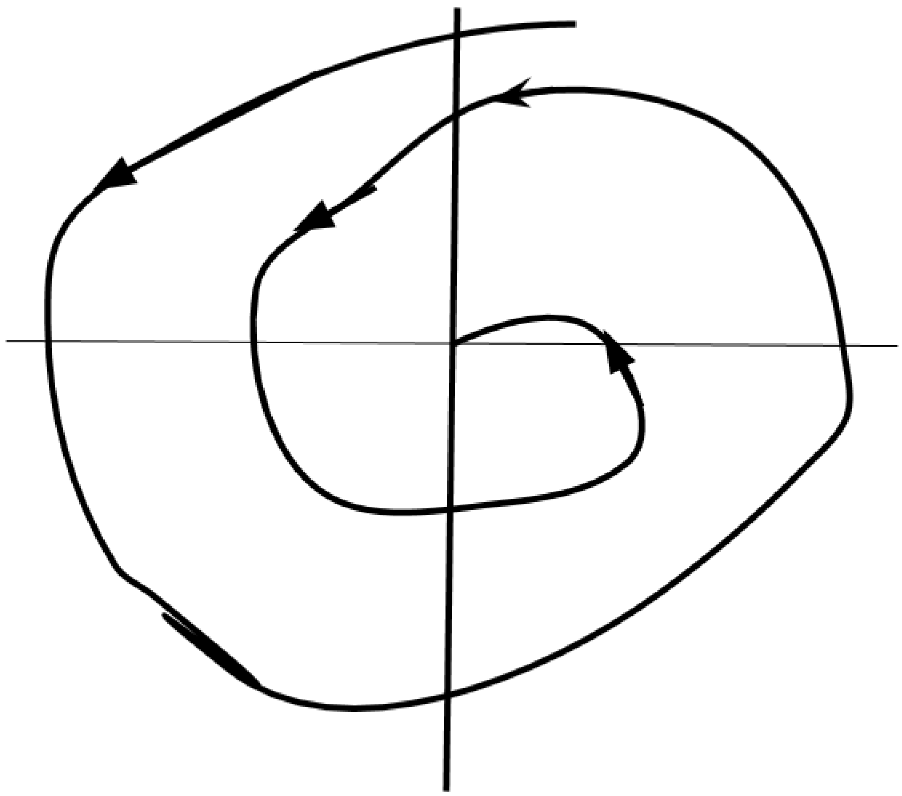
(c) (4 points) Determine and draw the asymptotic behaviour for $a = -1, \frac{1}{2}$ and 2.
A: at $a = 2$ it is an unstable saddle with $e^{(-1+\sqrt{2})t}\binom{\sqrt{2}}{-2}$ as the dominating term, so the solutions will converge to the span of $\binom{\sqrt{2}}{-2}$.



At $a = 1/2$ it is a stable sink with $e^{(-1+\sqrt{1/2})t}\binom{\sqrt{1/2}}{-1/2}$ still as the dominating term, so the solutions will converge to the origin along the span of $\binom{\sqrt{1/2}}{-1/2}$.



At $a = -1$, the eigenvalues are complex and the due to the negative real part we get a spiral sink centered at the origin.



Common mistakes and advice

- make sure you state what the dominating term is and exemplify that in your pictures.

(d) (4 points) Find the general solution in terms of $a = 0$.
A:It is a repeated eigenvalue so we have to find η:

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \eta = k\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

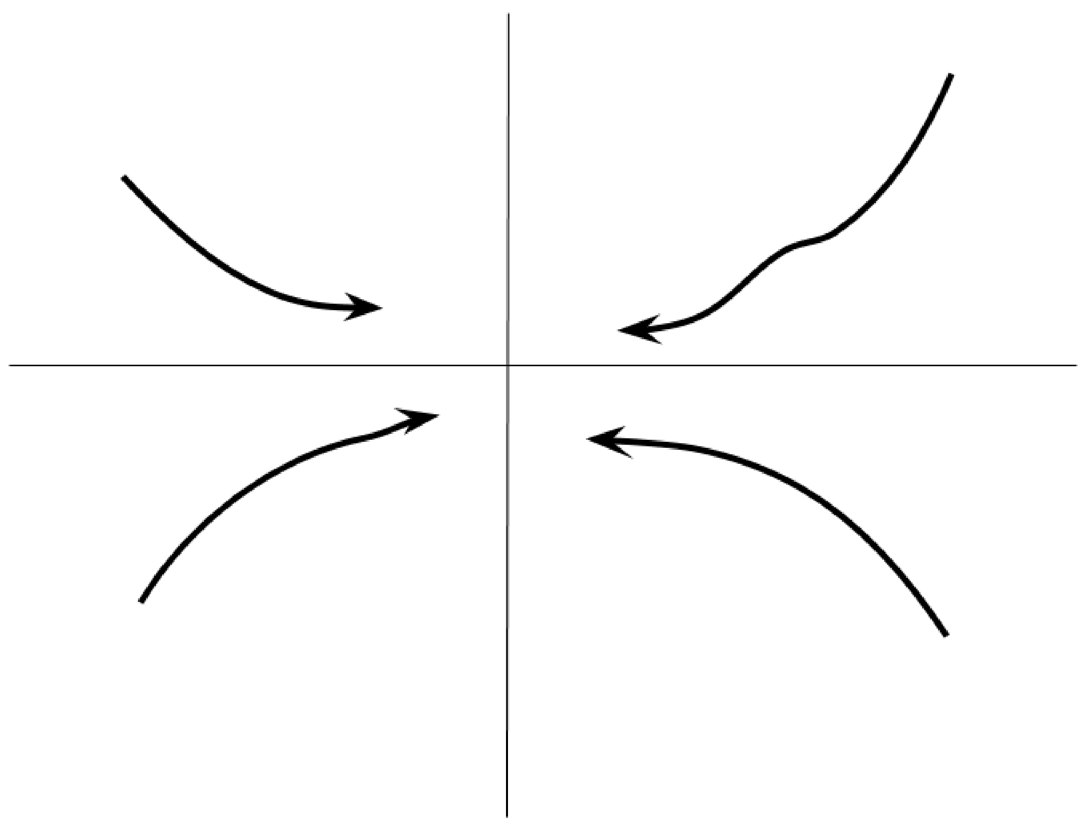
So the solution is

$$x = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(e^{-t}t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}).$$

Common mistakes and advice

- Many of you were unfamiliar with the repeated eigenvalue computations or what to do next.
- if you don't remember how to find η, simply plug in the ansatz from the formula sheet in the ode:
$$(\xi e^{\lambda t} t + \eta e^{\lambda t})' = A(\xi e^{\lambda t} t + \eta e^{\lambda t}) + g(t)$$
and do the cancelations to obtain:
$$(A - \lambda I)\eta = \xi.$$
- Many made small minus mistakes.

(e) (4 points) Determine and draw the asymptotic behaviour for $a = 0$.
A:The dominating term is $e^{-t}t\binom{1}{0}$ so the solutions converge to the origin along the span of $\binom{1}{0}$ (the x-axis).



2. (20 points) Consider the nonhomogeneous system

$$\mathbf{x}' = \left(\begin{smallmatrix} \alpha & 1 \\ -1 & \alpha \end{smallmatrix}\right)\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) (10 points) Find the general solution for the homogeneous system.

$$\mathbf{x}' = \left(\begin{smallmatrix} \alpha & 1 \\ -1 & \alpha \end{smallmatrix}\right)\mathbf{x}.$$

A: The eigenpairs are $(\alpha + i, \left(\begin{smallmatrix} i \\ -1 \end{smallmatrix}\right)), (\alpha - i, \left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right))$ and so:

$$x = c_1 e^{(\alpha+i)t}\begin{pmatrix} i \\ -1 \end{pmatrix} + c_2 e^{(\alpha-i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Common mistakes and advice

- Again be careful with eigenvalue,eigenvector computations. Do not show us the middle steps; only the final answer.

(b) (5 points) Find the general solution for the nonhomogeneous system

$$\mathbf{x}' = \left(\begin{smallmatrix} \alpha & 1 \\ -1 & \alpha \end{smallmatrix}\right)\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Assuming $x'_{nh} = 0$ we have

$$x_{nh} = \left(\begin{smallmatrix} \alpha & 1 \\ -1 & \alpha \end{smallmatrix}\right)^{-1}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{\alpha^2+1}\begin{pmatrix} 1-\alpha \\ -1-\alpha \end{pmatrix}.$$

So the general solution is:

$$x = c_1 e^{(\alpha+i)t}\begin{pmatrix} i \\ -1 \end{pmatrix} + c_2 e^{(\alpha-i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{1}{\alpha^2+1}\begin{pmatrix} 1-\alpha \\ -1-\alpha \end{pmatrix}.$$

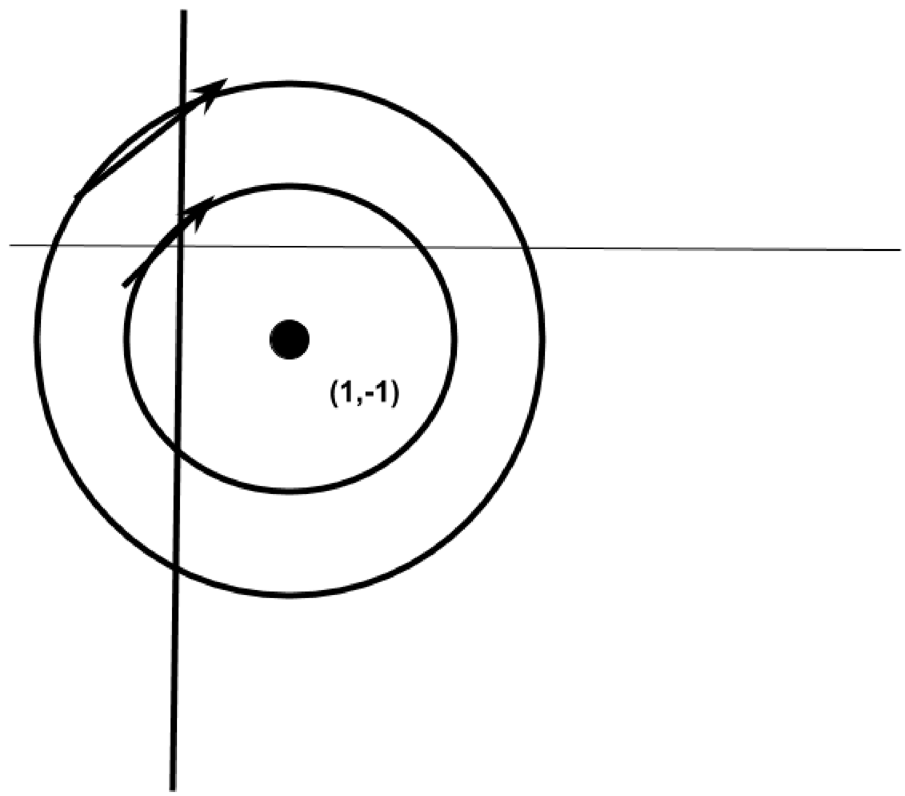
Common mistakes and advice

- It is incredible how many students forgot this method, even almost all the top students. I should have emphasized it more. In the final I will add the hint:"use $x' = 0$ approach".
- Again don't show us how you compute the inverse or multiply with vectors. Only the final answer.

(c) (5 points) Find the bifurcation value or values of α where the qualitative nature of the phase portrait for the system changes. Draw a phase portrait for a value of α slightly below, and for another value slightly above, each bifurcation value.

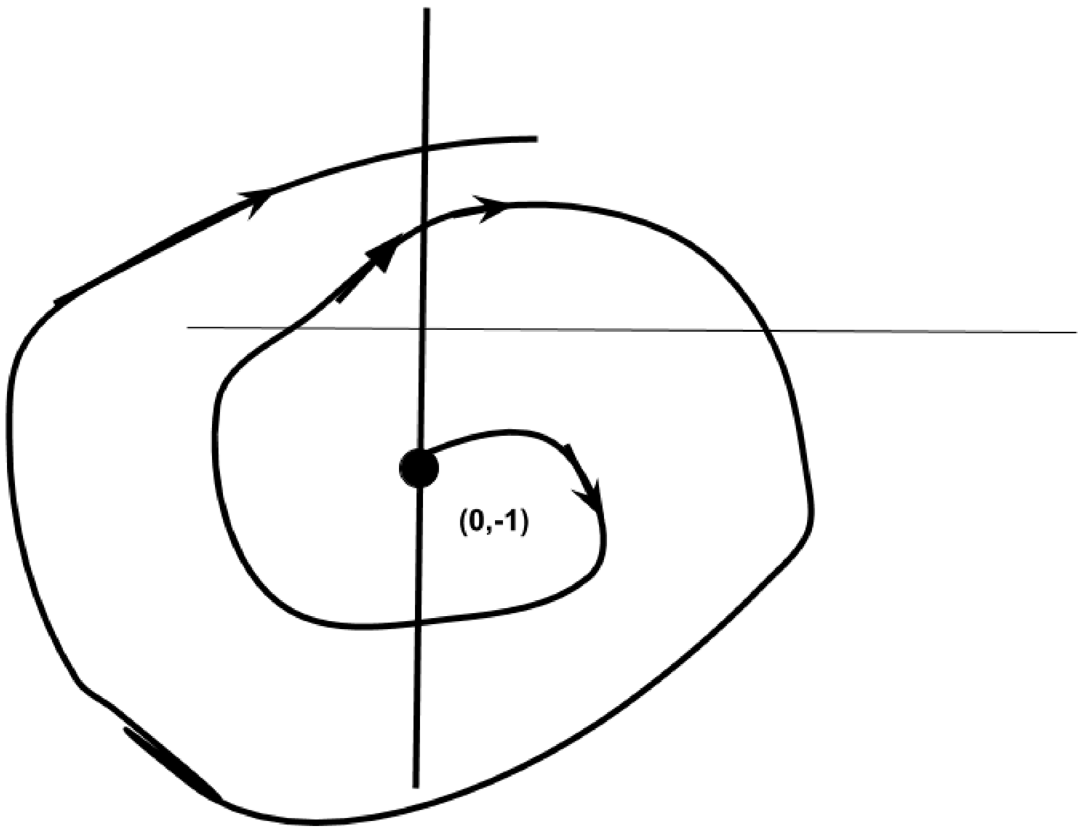
A: The bifurcation value is at $\alpha = 0$, where we have concentric circles centered at:

$$\frac{1}{\alpha^2+1}\begin{pmatrix} 1-\alpha \\ -1-\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$



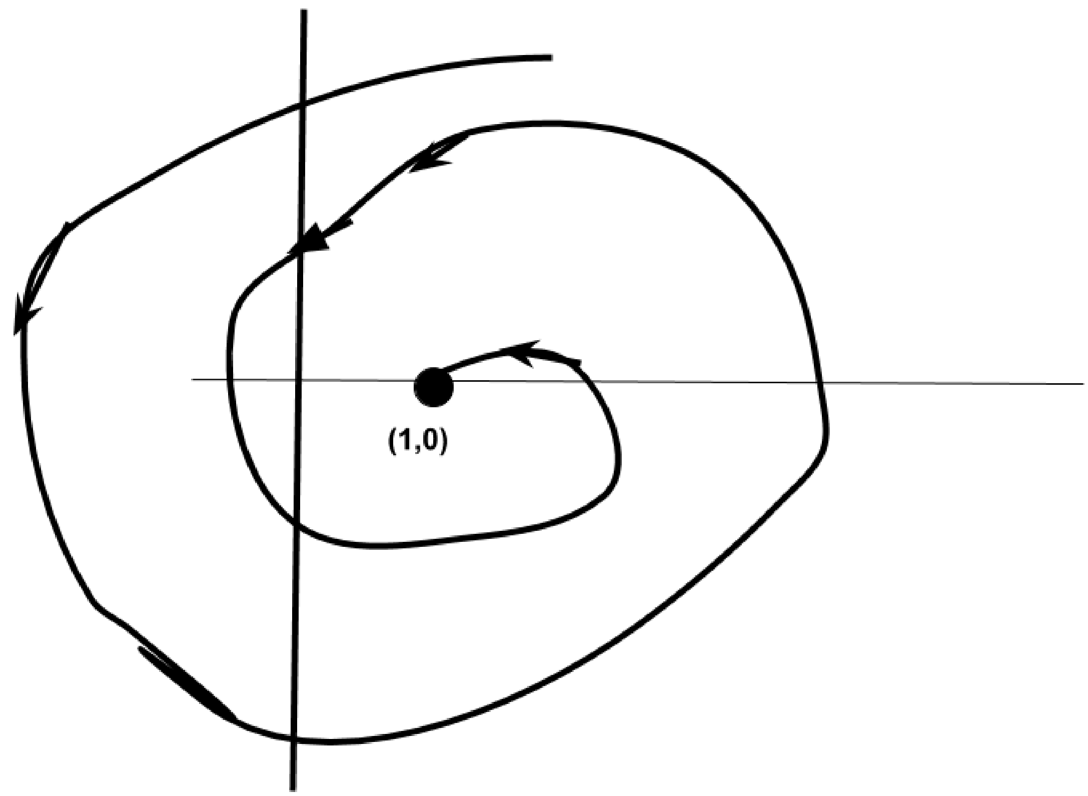
At $\alpha > 0$ we have spiral source. For example for $\alpha = 1$ we have spiral sources centered at:

$$\frac{1}{\alpha^2+1}\begin{pmatrix} 1-\alpha \\ -1-\alpha \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$



At $\alpha < 0$ we have spiral sink. For example for $\alpha = -1$ we have spiral sources centered at:

$$\frac{1}{\alpha^2+1}\begin{pmatrix} 1-\alpha \\ -1-\alpha \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$



Common mistakes and advice

- Many students didn't include the center of the spiral/circles. The vector $v = \frac{1}{\alpha^2+1}\binom{1-\alpha}{-1-\alpha}$ is the center because if $c_1 = c_2 = 0$ then the solution is trapped at v for all future time.

3. (20 points) Consider the autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + yx \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{y^2}{2}$$

(a) (2 points) Find the critical points.

A: We have $0 = -x + yx = x(y-1)$ and $0 = y - \frac{y^2}{2} = y(1 - y/2)$ implies $y = 1$ or $x = 0$ and $y = 0$ or $y = 2$. If $y = 1$, it contradicts the second set. So we just left with $(0,0)$ and $(0,2)$.

Common mistakes and advice

- Many students included the (0,1) point. The (0,1) is indeed a critical point for $x' = 0$, but we are not studying the x variable only. We are studying the system (x,y) and so we consider the points that are critical for both $x' = 0$ and $y' = 0$, so that we obtain sinks/sources for both x,y as a system.

(b) (3 points) Obtain the parametric solutions.
A:We take the ratio:

$$\frac{dy}{dx} = \frac{y - y^2/2}{-x + yx} \Rightarrow (-x + yx)dy - (y - y^2/2)dx = 0.$$

It is an exact equation and so the implicit solution is:

$$x((y-1)^2 - 1) = constant.$$

It can also be done by separation of variables.

Common mistakes and advice

- Many students didn't know what we meant for parametric solution even though we did this in the class notes multiple times. Here we try to obtain an equation only in terms of x and y (without any t). These will give the trajectories of x,y in the phase portrait.

(c) (5 points) Do a nullcline analysis around all the critical points.

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0, \frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff

$$-x + yx > 0, y - \frac{y^2}{2} > 0 \Leftrightarrow \{2 > y > 1, x > 0\} \cup \{1 > y > 0, x < 0\}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0, \frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff

$$-x + yx > 0, y - \frac{y^2}{2} < 0 \Leftrightarrow \{y > 2, x > 0\} \cup \{0 > y, x < 0\}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0, \frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff

$$-x + yx < 0, y - \frac{y^2}{2} > 0 \Leftrightarrow \{1 > y > 0, x > 0\} \cup \{2 > y > 1, x < 0\}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0, \frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff

$$-x + yx < 0, y - \frac{y^2}{2} < 0 \Leftrightarrow \{y < 0, x > 0\} \cup \{y > 2, x < 0\}.$$

(d) Therefore, in summary around the origin we have the sketch:

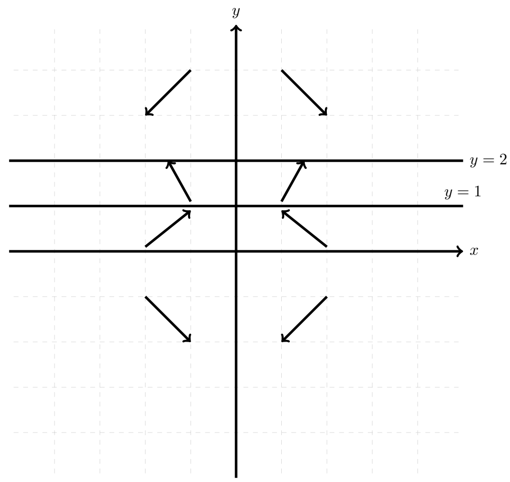


Figure 1: Phase portrait sketch.

Common mistakes and advice

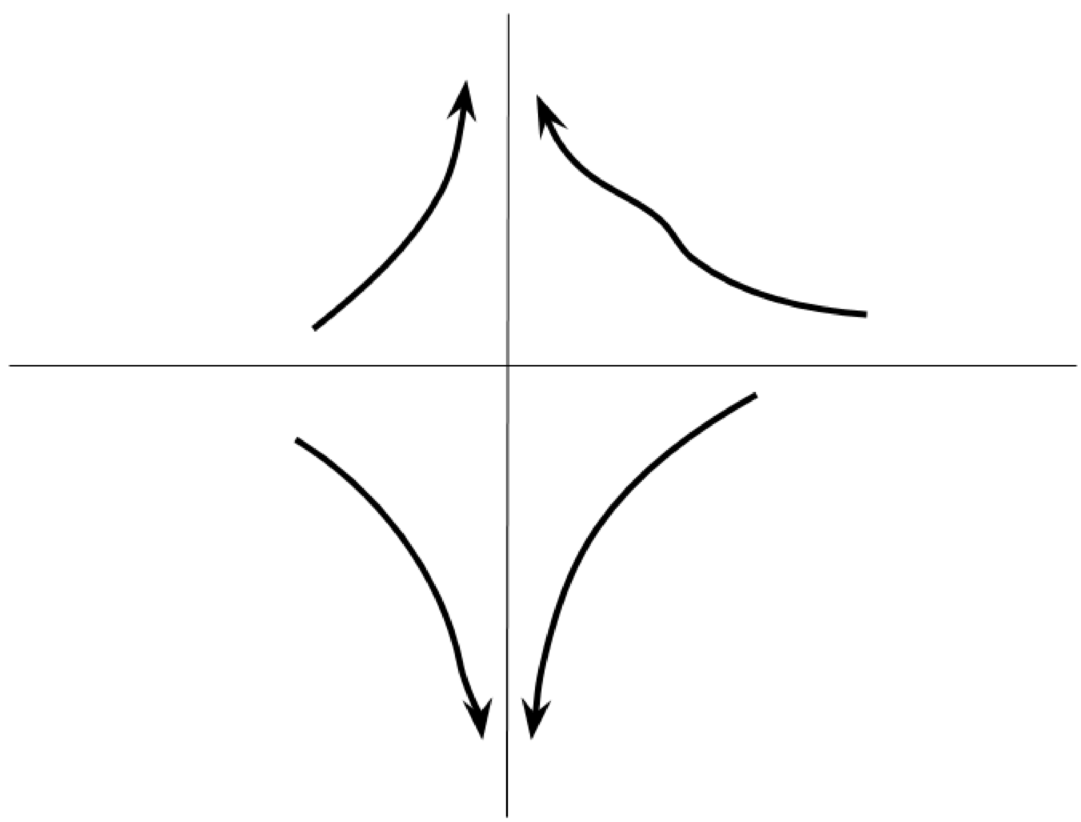
- Again many students were not familiar with what nullcline analysis and instead linearized the system. But the next questions should raised suspicion since I did the linearization for them.
- Quite a few students got their arrows mixed up.
- Please don't show us how you obtain the sets or the arrows. Just give the final clean answer as above.

(e) (5 points) If we linearize the previous autonomous system around the origin, we obtain the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Obtain the general solution and determine the asymptotic behaviour. Does it agree with your previous nullcline analysis?

A: the solution is $x = c_1 e^{-t}\binom{1}{0} + c_2 e^{t}\binom{0}{1}$. This is a saddle around the origin. The dominating term is $e^t\binom{0}{1}$ which indeed limits to the span of the y-axis. This picture agrees with our nullcline analysis.



Common mistakes and advice

- For all the questions the nullcline analysis and linearization have to agree. If they don't, then you have made a mistake.
- Many students got mixed with which eigenvalue corresponds to which eigenvector. If you are in doubt, simply use the formula $A\xi = \lambda\xi$, so in our case:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So the eigenvalue of $\binom{1}{0}$ is $\lambda = -1$.

(f) (5 points) If we linearize the previous autonomous system around the point (0,2), we obtain the system

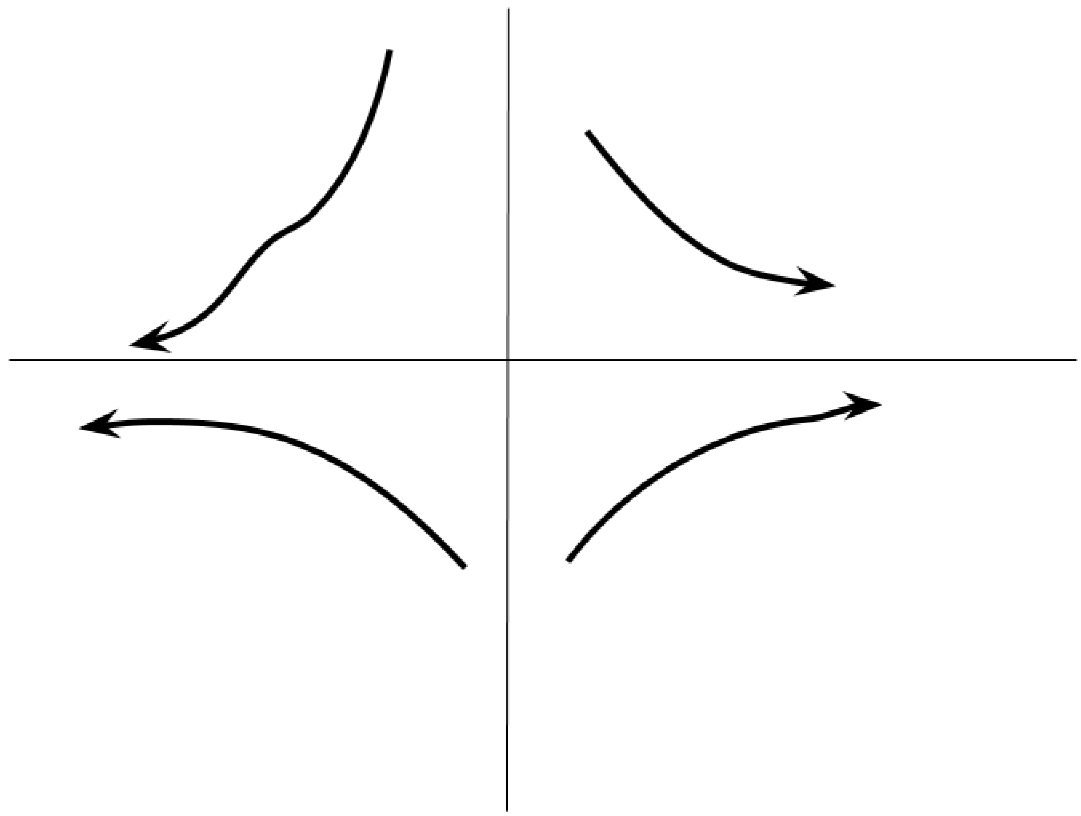
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Obtain the general solution and determine the asymptotic behaviour. Does it agree with your previous nullcline analysis?

A: The nonhomogeneous part can be found by the $x' = 0$ approach:

$$x_{nh} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

So the solution is $x = c_1 e^t \binom{1}{0} + c_2 e^{-t} \binom{0}{1} + \binom{0}{2}$, which indeed limits to the span of the x-axis when centered at the point $\binom{0}{2}$. This agrees with our nullcline analysis.



Common mistakes and advice

- Again many students forgot the $x' = 0$ approach and wasted too much time using the diagonalization method.
- Some students used that this system is diagonal to obtain the decoupled first order odes $x' = x$ and $y' = -y + 2 \Rightarrow x = c_1e^t, y = c_2e^t + 2$. This is an even faster approach.

4. (40 points) Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \frac{x^2}{2}.$$

(a) (2 points) Find the critical points.
A:The critical points are: (0,0) and $(2, 0)$.

(b) (3 points) Obtain the parametric solutions.
A: the equation is separable and so we have:

$$y^2 = -x^2 + \frac{x^3}{6} + c.$$

(c) (5 points) Do a nullcline analysis around all the critical points.
A:

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff

$$y > 0, -x + \frac{x^2}{2} > 0 \Leftrightarrow \{y > 0, x > 2\} \cup \{y > 0, x < 0\}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff

$$y > 0, -x + \frac{x^2}{2} < 0 \Leftrightarrow \{y > 0, 0 < x < 2\}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff

$$y < 0, -x + \frac{x^2}{2} > 0 \Leftrightarrow \{y < 0, x > 2\} \cup \{y < 0, x < 0\}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff

$$y < 0, -x + \frac{x^2}{2} > 0 \Leftrightarrow \{y < 0, 0 < x < 2\}.$$

(d) Therefore, in summary around the origin we have the sketch:

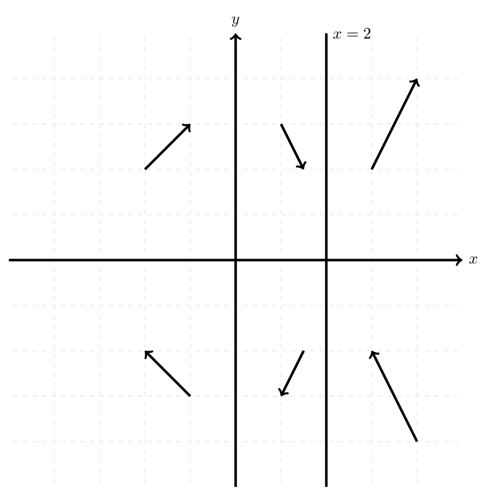


Figure 2: Phase portrait sketch.

Common mistakes and advice

- Quite a few students got their arrows mixed up.
- Please don't show us how you obtain the sets or the arrows. Just give the final clean answer as above.

(e) (15 points) If we linearize the previous system around the origin and assume exponential error, we get the nonhomogeneous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the general solution for this nonhomogeneous system. What type of phase portrait do you get (no need to sketch)? Does it agree with your nullcline analysis?

1. The eigenpairs are $(-i, \binom{-i}{1}), (i, \binom{i}{1})$ and so T is

$$T = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}.$$

2. The $T^{-1}g(t) = \frac{e^{-t}}{2i}\binom{i-1}{i+1}$.
3. So we obtain the solutions:

$$\begin{aligned} y_1 &= e^{-it}(\int e^{is}(s)e^{-s}\frac{i-1}{2i}ds + c_2) \\ &= e^{-it}\frac{i-1}{2i}(\frac{e^{(i-1)t}}{i-1}) \\ &= e^{-t}\frac{1}{2i} \\ y_2 &= e^{it}(\int e^{-is}(s)e^{-s}ds + c_2) \\ &= e^{it}\frac{i+1}{2i}\frac{e^{-(i+1)t}}{-(i+1)} \\ &= e^{-t}\frac{1}{-2i} \end{aligned}$$

4. Therefore, $x_{nh} = Ty$ gives

$$x = Ty = \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t}.$$

5. So the general solution is

$$x = c_1 e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

6. it is concentric circles because $Re(\lambda) = 0$. It agrees with the nullcline analysis because we also get circles there.

Common mistakes and advice

- Many students got mixed up which eigenvalue corresponds to which eigenvector.
- Please don't show us how you compute the inverse of T or how you multiply $T^{-1}g(t)$ or how you multiply Ty.
- To save you time with matrix vector multiplication always factor out common terms: instead of multiplying

$$Ty = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \frac{e^{-t}}{2i} \\ -\frac{e^{-t}}{2i} \end{pmatrix}$$

factor out $\frac{e^{-t}}{2i}$ and multiply

$$\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2i \\ 0 \end{pmatrix}$$

and then multiply by $\frac{e^{-t}}{2i}$ to get

$$Ty = e^{-t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Trust me it will save you time and also numerous transfer computation mistakes.

(f) (10 points) If we linearize the previous system around the critical point (2,0) and assume exponential decaying error, we get the nonhomogeneous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the general solution for this system.(Hint: first find the solution for $x' = Ax + \binom{0}{-2}$ and $x' = Ax + e^{-t}\binom{1}{1}$ and then add them).

A:

1. For the system $x' = Ax + \binom{0}{-2}$ we use the $x' = 0$ approach to quickly get:

$$x_{nh,1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

2. The eigenpairs are $(-1, \binom{-1}{1}), (1, \binom{1}{1})$ and so T is

$$T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

3. The $T^{-1}g(t) = \binom{0}{e^{-t}}$.
4. So we obtain the system:

$$\begin{aligned} y_1 &= 0 \\ y_2 &= e^t(\int e^{-s}(s)e^{-s}ds + c_2) \\ &= e^t e^{-2t}\frac{1}{-2} \\ &= e^{-t}\frac{1}{-2} \end{aligned}$$

5. Therefore, $x_{nh} = Ty$ gives

$$x = Ty = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}\frac{1}{-2}.$$

6. The general solution is

$$x = c_1 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{e^{-t}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Common mistakes and advice

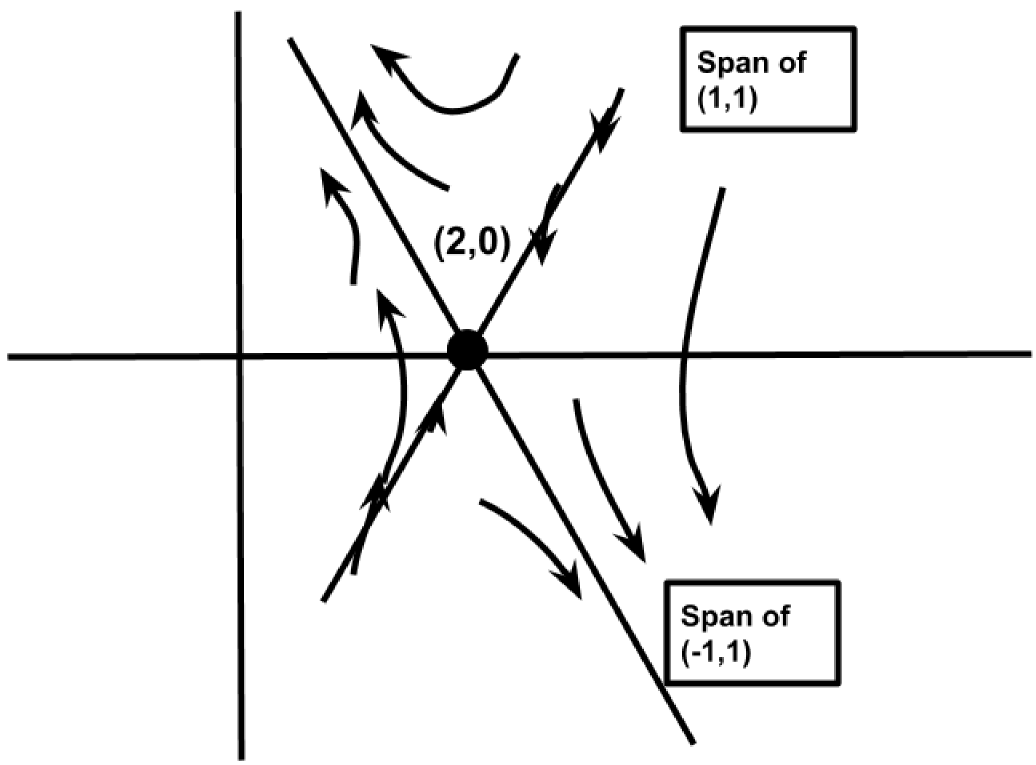
- Please make sure that you are consistent with your choice of first and second eigenvector. Because the first column of T must be the first eigenvector and so on and so forth. But more importantly y_1 has to correspond to the first eigenvalue.

(g) (5 points) Sketch the phase portrait for this solution. Where is it centered? Does it agree with your nullcline analysis?

A: The solution

$$x = c_1 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}1\\1\end{pmatrix} - \frac{e^{-t}}{2}\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\0\end{pmatrix}$$

has a saddle centered "close" to the critical point $\binom{2}{0}$.



In fact the saddle is centered at the place where $c_1 = c_2 = 0$:

$$x(0) = -\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}1.5\\-0.5\end{pmatrix}.$$

More space