Contents

1 Autonomous systems

As in the 1D case we will study the following system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x,y), \frac{\mathrm{d}y}{\mathrm{dt}} = G(x,y),$$

where F, G are continuously differentiable functions. Here again we might not be able to obtain explicit solutions, but we can provide a qualitative analysis.

Method formal steps

1. First, we find the critical points by setting

$$F(x,y) = 0 \text{ and } G(x,y) = 0.$$

2. For each critical point we carve out the regions of the phase portrait that converge to it, called *basin regions of attraction*. For example, as we will explain later in the competing species section, we obtain phase portraits of the form:

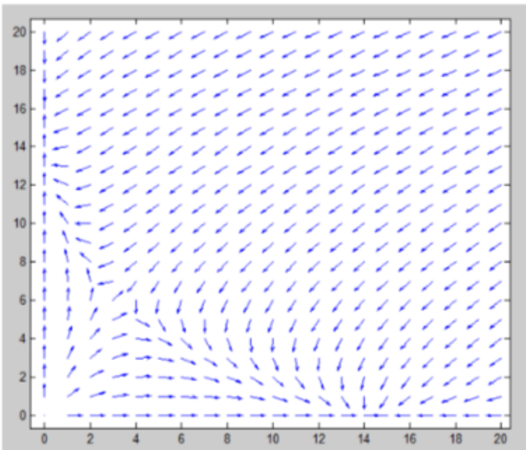


Figure 1.1: The critical points (14,0) and (0,14) have their own basin regions of attraction

Here the points (14,0) and (0,14) have their own basin regions of attraction (arrows pointing towards them) that are separated by curves called the *separatrix*.

3. Sometimes we can even solve such systems by taking their ratio and obtain a parametric solution:

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{\frac{\mathrm{d}y}{\mathrm{dt}}}{\frac{\mathrm{d}x}{\mathrm{dt}}} = \frac{G(x,y)}{F(x,y)}.$$

This ratio depends only on x and y (and not t), so methods from the first order section could be used.

4. Next we sketch the direction field by doing a nullcline analysis around the critical points. That is we study the signs of the pair $(\frac{\mathrm{d}x}{\mathrm{dt}}, \frac{\mathrm{d}y}{\mathrm{dt}})$.

5. Finally, we plot the parametric solution and check whether it agrees with the direction field from the above step.

Example-presenting the method

Consider the following oscillating pendulum: a mass m is attached to one end of a rigid, but weightless, rod of length L which hangs from the pivot point.

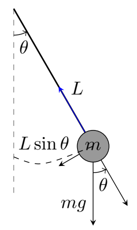


Figure 1.2: oscillating pendulum

The gravitational force mg acts downward and the damping force $c|\frac{d\theta}{dt}|$ is always opposite to the direction of motion. A rotational analog of Newton's second law of motion might be written in terms of torques:

$$mg \cdot Lsin(\theta) + \frac{\mathrm{d}\theta}{\mathrm{dt}} \cdot L + m\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}}L^2 = 0 \Rightarrow \frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \gamma\frac{\mathrm{d}\theta}{\mathrm{dt}} + \omega^2 sin(\theta) = 0.$$

This is a nonhomogeneous second order equation, but we can also view it as a system of equations by letting $x := \theta$ and $y := \frac{\mathrm{d}\theta}{\mathrm{dt}}$:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \omega^2 sin(x),$$

where γ is called the damping constant and as in the spring problem it is responsible for removing energy. This is an autonomous system.

1. First we find the critical points:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 0, \frac{\mathrm{d}y}{\mathrm{dt}} = 0 \Rightarrow y = 0, sin(x) = 0 \Rightarrow (k\pi, 0) \text{ for } k \in \mathbb{Z}.$$

2. Then we numerically draw the solutions

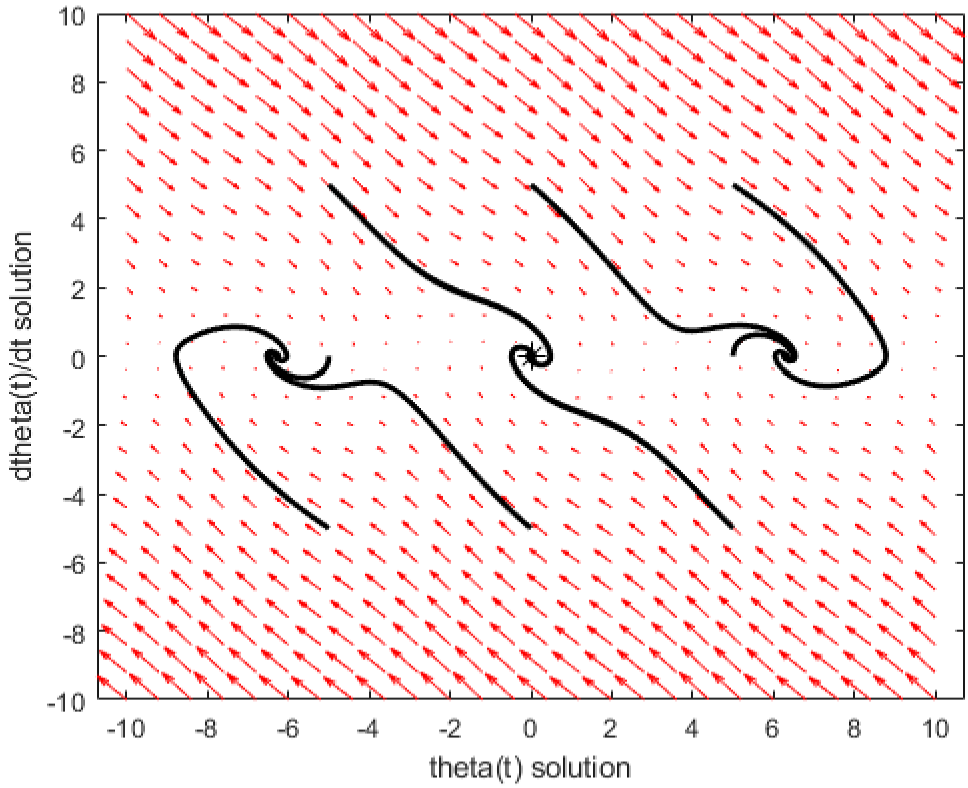


Figure 1.3: Phase portrait and solutions for oscillating pendulum.

We see that the basins of attractions for each critical point are regions separated by the black spiral curves.

3. The ratio is

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{\frac{\mathrm{d}y}{\mathrm{dt}}}{\frac{\mathrm{d}x}{\mathrm{dt}}} = \frac{-\gamma y - \omega^2 sin(x)}{y},$$

which is not amenable to known methods (eg. see Chini's equation).

4. However, if we set $\gamma = 0$ (undamped pendulum), we get a separable equation and in turn the implicit solution:

$$y^2 = 2(\omega^2 cos(x) + c) \Rightarrow \frac{y^2}{2} - \omega^2 cos(x) = constant.$$

5. Next we do a nullcline analysis around the origin.

- We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0, \frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff

$$y > 0, -\omega^2 sin(x) > 0 \Leftrightarrow y > 0, \frac{-\pi}{2} < x < 0.$$

- We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0, \frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff

$$y > 0, -\omega^2 sin(x) < 0 \Leftrightarrow y > 0, 0 < x < \frac{\pi}{2}.$$

- We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0, \frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff

$$y < 0, -\omega^2 sin(x) > 0 \Leftrightarrow y < 0, \frac{-\pi}{2} < x < 0.$$

- We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0, \frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff

$$y < 0, -\omega^2 sin(x) < 0 \Leftrightarrow y < 0, 0 < x < \frac{\pi}{2}.$$

6. Therefore, in summary around the origin we have the sketch:

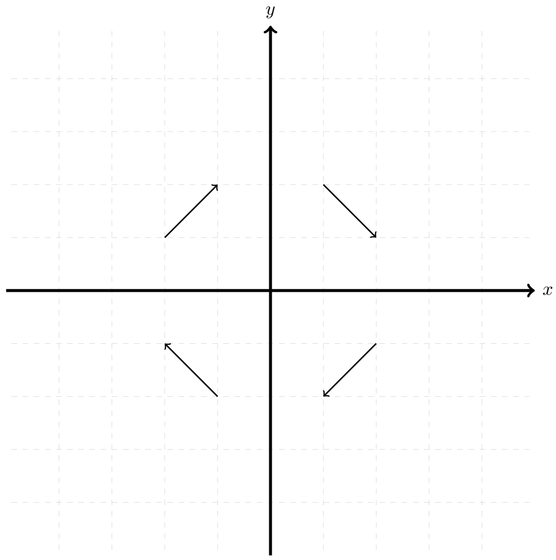


Figure 1.4: Phase portrait sketch for for undamped oscillating pendulum.

Indeed the parametric solution follows the circular behaviour of the above direction field.

7. The above sketch agrees with the numerically generated phase portrait:

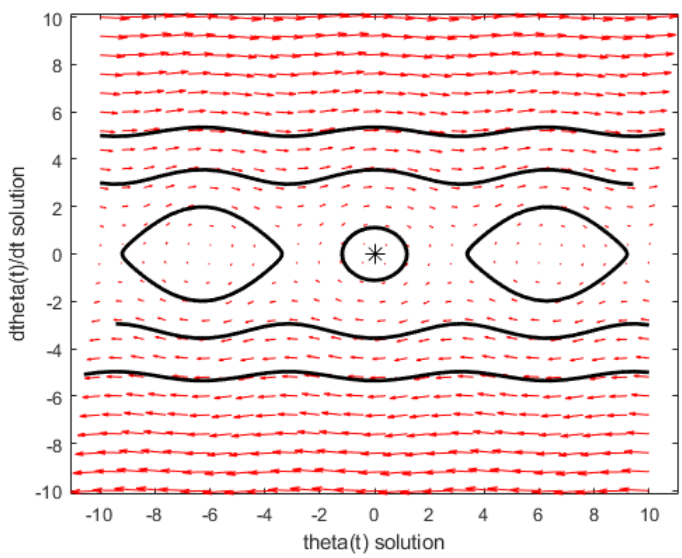


Figure 1.5: Phase portrait and solutions for undamped oscillating pendulum.

We see that the basins of attractions are separated by ellipses along the horizontal and they are separated from periodic behaviour along the vertical. Physically a closed curve around critical point represents the pendulum oscillating periodically since the velocity $y = \dot{\theta}$ oscillates periodically around that critical point. The wavy lines represent the pendulum spinning around the pivot point.

Examples

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 2y, \frac{\mathrm{d}y}{\mathrm{dt}} = -8x.$$

  1. The critical point is just (0,0).
  2. To determine the solutions we solve:

  $$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{-8x}{2y}.$$

  This equation is separable and so we easily obtain:

  $$y^2 = -4x^2 + c.$$

  3. Therefore, the solutions are ellipses $y^2 + 4x^2 = c$ centered at zero.
  4.
  5. Next we do a nullcline analysis around the origin.
     - We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0, \frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff $2y > 0, -8x > 0 \Leftrightarrow y > 0, x < 0$.
     - We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0, \frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff $y > 0, x > 0$.
     - We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0, \frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff $y < 0, x < 0$.
     - We have $\frac{\mathrm{d}x}{\mathrm{dt}} < 0, \frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff $y < 0, x > 0$.
  6. Therefore, in summary around the origin we have the sketch:

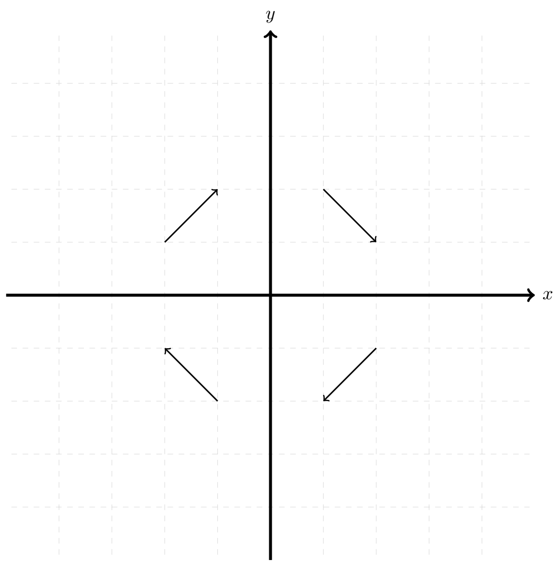


Figure 1.6: Phase portrait sketch.

  Indeed the parametric solution follows the circular behaviour of the above direction field.

7. The above sketch agrees with the numerically generated phase portrait:

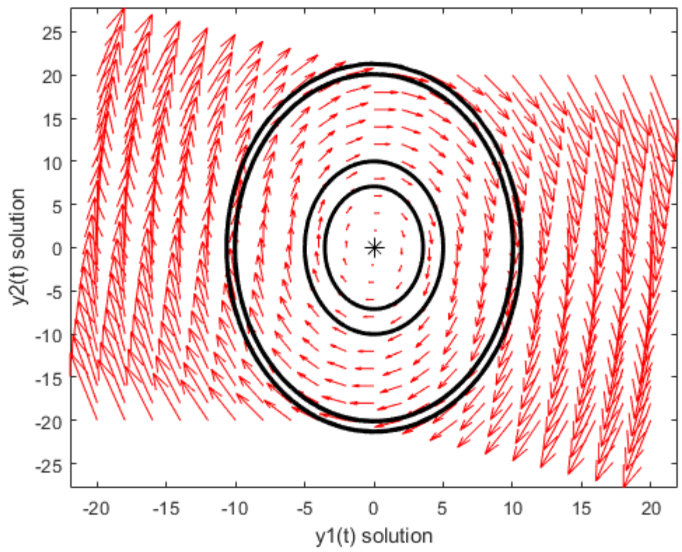


Figure 1.7: Phase portrait and solutions

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + y, \frac{\mathrm{d}y}{\mathrm{d}t} = -x - y.$$

1. First we find the critical point/s:

$$-x + y = 0 \text{ and } -x - y = 0 \Rightarrow (x, y) = (0, 0).$$

2. Next we find the parametric solution:

$$\frac{dy}{dx} = \frac{-x - y}{-x + y} = \frac{-1 - y/x}{-1 + y/x} = \frac{1 + y/x}{1 - y/x}.$$

We use the subtistution $v = y/x$ to obtain by chain rule

$$v' = \frac{y'}{x} + \frac{y}{-x^2} \Rightarrow y' = xv' + v.$$

Therefore,

$$xv' + v = \frac{1 + y/x}{1 - y/x} = \frac{1 + v}{1 - v} \Rightarrow$$

$$xv' = \frac{1 + v^2}{1 - v} \Rightarrow$$

this equation is separable and so we have:

$$\frac{1 - v}{1 + v^2} dv = \frac{1}{x} dx \Rightarrow$$

$$arctan(v) - \frac{1}{2}\log(1 + v^2) = \log(x) + c.$$

Undoing the change of variables we obtain the implicit solution:

$$arctan(\frac{y}{x}) - \frac{1}{2}\log(1 + (\frac{y}{x})^2) = \log(x) + c.$$

3. Next we do a nullcline analysis.

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0, \frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff $-x + y > 0, -x - y > 0 \Leftrightarrow y > x, y < -x$.
- We have $\frac{\mathrm{d}x}{\mathrm{d}t} > 0, \frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff $y > x, y > -x$.
- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0, \frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff $y < x, y < -x$.
- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0, \frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff $y < x, y > -x$.

4. Therefore, we have the following sketch:

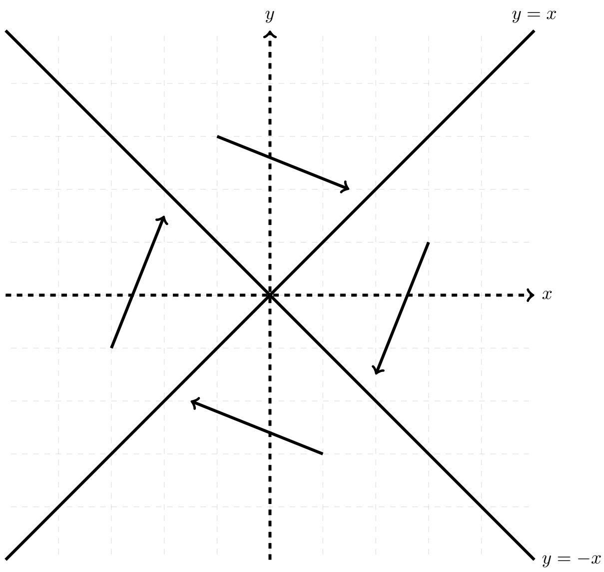


Figure 1.8: Phase portrait sketch.

5. The linearization around the origin is:

$$\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}.$$

   (a) The eigenpairs are $(-1 + i, \binom{i}{-1}), (-1 + i, \binom{-i}{-1})$. Therefore, the general solution is:

$$\mathbf{x}(t) = c_1 e^{(-1+i)t} \begin{pmatrix} i \\ -1 \end{pmatrix} + c_2 e^{(-1-i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

   (b)

6. The numerically generated phase portrait is:

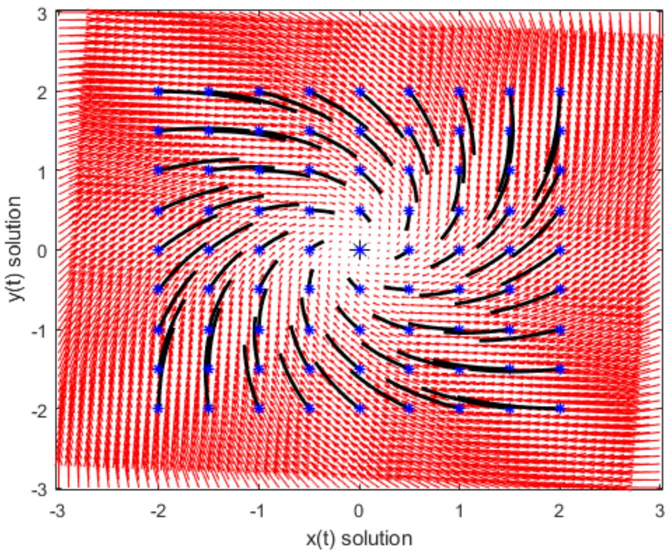


Figure 1.9: phase portrait

- Consider the system (Duffing's equation)

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -x + \frac{x^3}{6}.$$

  It describes the motion of a damped oscillator with a more complex potential than in simple harmonic motion; in physical terms, it models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law. The Duffing equation is an example of a dynamical system that exhibits chaotic behavior.

  1. The critical points are $(0,0), (\pm\sqrt{6}, 0)$.
  2. To determine the solutions we solve:

  $$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{-x + \frac{x^3}{6}}{y}.$$

     This equation is separable and so we easily obtain:

  $$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + \frac{x^4}{24} + c.$$

  3. Therefore, the solutions are the pairs of parabolas and ellipses $y^2 + \frac{1}{2}x^2 - \frac{x^4}{12} = c$ symmetric wrt to the x-axis.
  4. Next we do a nullcline analysis around the origin.
     - We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0, \frac{\mathrm{d}y}{\mathrm{dt}} > 0$ iff

  $$y > 0, -x + \frac{x^3}{6} > 0 \Leftrightarrow y > 0, x \in [-\sqrt{6}, 0] \cup [\sqrt{6}, \infty).$$

     - We have $\frac{\mathrm{d}x}{\mathrm{dt}} > 0, \frac{\mathrm{d}y}{\mathrm{dt}} < 0$ iff

  $$y > 0, -x + \frac{x^3}{6} < 0 \Leftrightarrow y > 0, x \in (-\infty, -\sqrt{6}] \cup [0, \sqrt{6}].$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0, \frac{\mathrm{d}y}{\mathrm{d}t} > 0$ iff

$$y < 0, -x + \frac{x^3}{6} > 0 \Leftrightarrow y > 0, x \in [-\sqrt{6}, 0] \cup [\sqrt{6}, \infty).$$

- We have $\frac{\mathrm{d}x}{\mathrm{d}t} < 0, \frac{\mathrm{d}y}{\mathrm{d}t} < 0$ iff

$$y < 0, -x + \frac{x^3}{6} < 0 \Leftrightarrow y > 0, x \in (-\infty, -\sqrt{6}] \cup [0, \sqrt{6}].$$

5. Therefore, in summary around the origin we have the sketch:

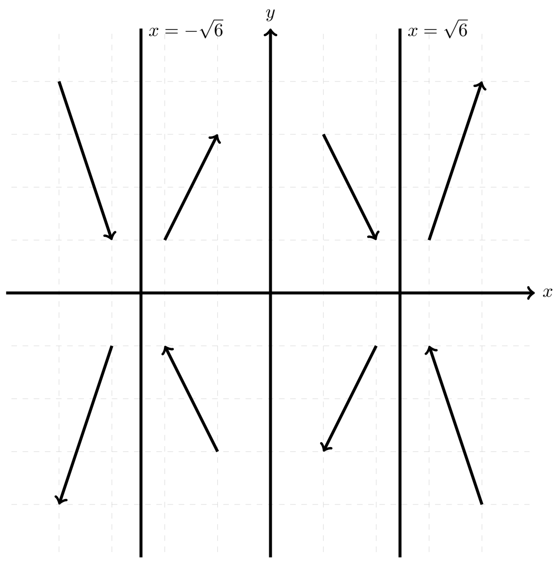


Figure 1.10: Phase portrait sketch.

Indeed the parametric solution follows the circular behaviour of the above direction field.

6. The above sketch agrees with the numerically generated phase portrait:

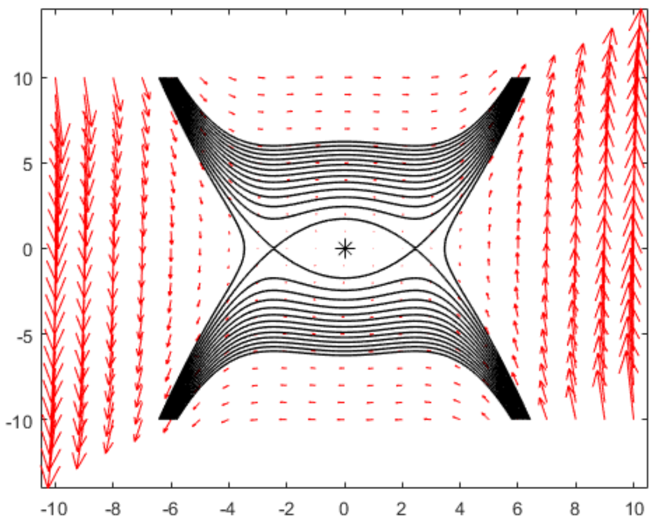


Figure 1.11: Phase portrait and solutions of duffing's system

2 Locally linear systems

We will study systems

$$\mathbf{x} = \mathbf{f}(\mathbf{x}),$$

where the components of $\mathbf{f}$ are C^1 functions so that we are able to Taylor expand them. The following system

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$$

is called *locally linear* around a critical point $\mathbf{x}_0$ if

$$\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0 \text{ and } as\mathbf{x} \to \mathbf{x}_0.$$

Example-presenting the method

We continue our study with the damped oscillating pendulum system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \omega^2 sin(x),$$

where γ is called the damping constant and as in the spring problem it is responsible for removing energy.

1. First we find the critical points. From the previous section we have:

$$(n \cdot \pi, 0) \text{ for any integer n.}$$

2. Second we Taylor expand the RHS of the system $F(x, y) := \binom{y}{-\gamma y - \omega^2 sin(x)}$ around arbitrary critical point (x_0, y_0):

$$\begin{aligned} F(x,y) &= F(x_0,y_0) + J_F(x_0,y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2) \\ &= \begin{pmatrix} 0 & 1 \\ -\omega^2 cos(x_0) & -\gamma \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2). \end{aligned}$$

Here $J_F(x_0, y_0)$ is the Jacobian matrix for function $F(x, y) = \binom{F_1(x,y)}{F_2(x,y)}$:

$$J_F(x_0, y_0) := \begin{pmatrix} \frac{d}{dx} F_1(x_0, y_0) & \frac{d}{dy} F_1(x_0, y_0) \\ \frac{d}{dx} F_2(x_0, y_0) & \frac{d}{dy} F_2(x_0, y_0) \end{pmatrix}$$

3. The linearization around $(x_0, y_0) = (n \cdot \pi, 0)$ for even integer n is:

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x - n \cdot \pi \\ y \end{pmatrix} + O(\|(x - n \cdot \pi, y)\|^2).$$

The eigenvalues of that matrix are:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}.$$

   (a) If $\gamma^2 - 4\omega^2 > 0$, then the eigenvalues are real,distinct and negative. Therefore, the critical points will be stable nodes.

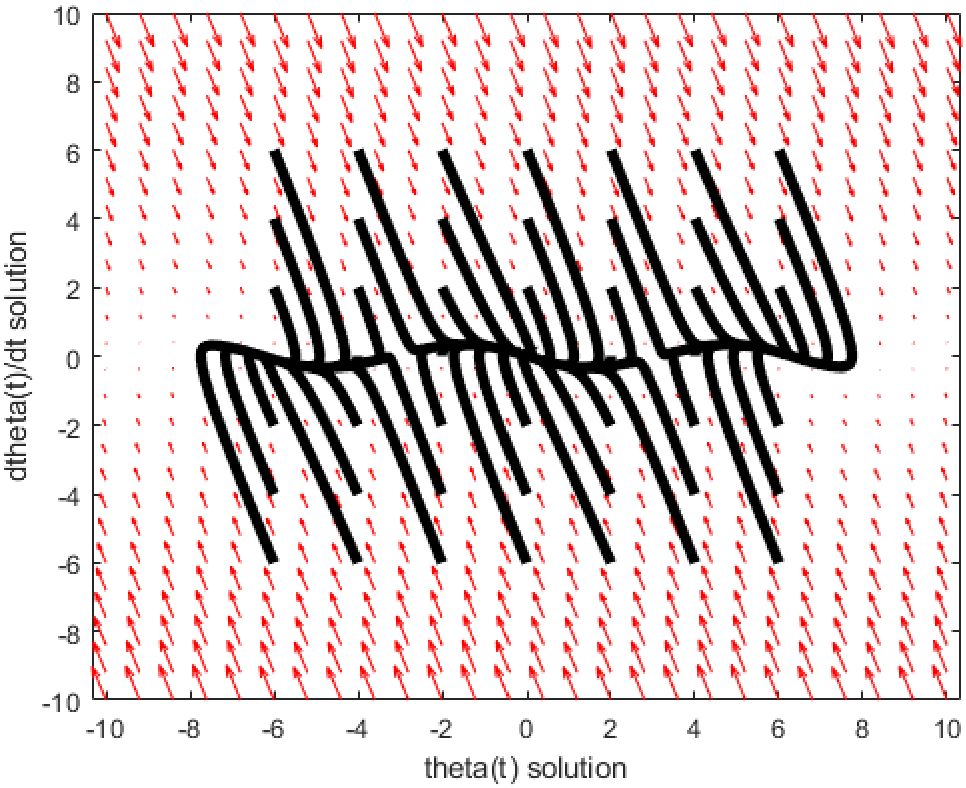


Figure 2.1: Stable nodes at even integer n critical points $(n\pi, 0)$ for n=0,2,-2.

We observe that the basins of attractions for each even-integer critical point are well-separated.

(b) If $\gamma^2 - 4\omega^2 = 0$, then the eigenvalues are repeated, real and negative. Therefore, the critical points will be stable nodes.

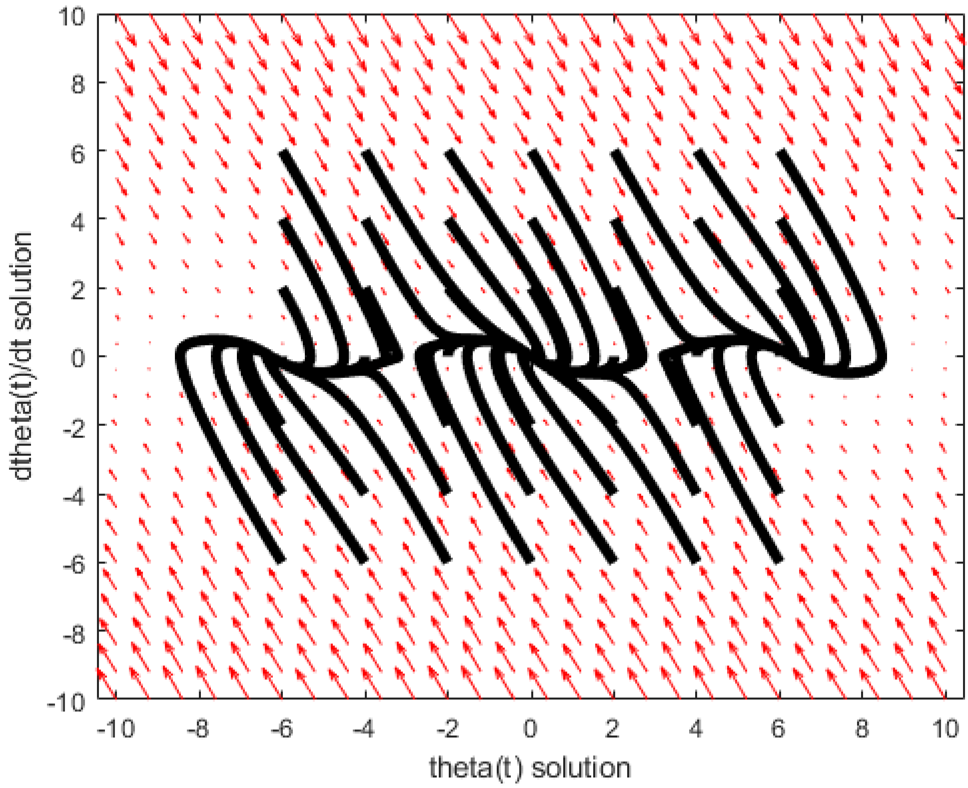


Figure 2.2: Stable nodes at even integer n critical points $(n\pi, 0)$.

(c) If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part. Therefore, the critical points will be stable spiral sinks.

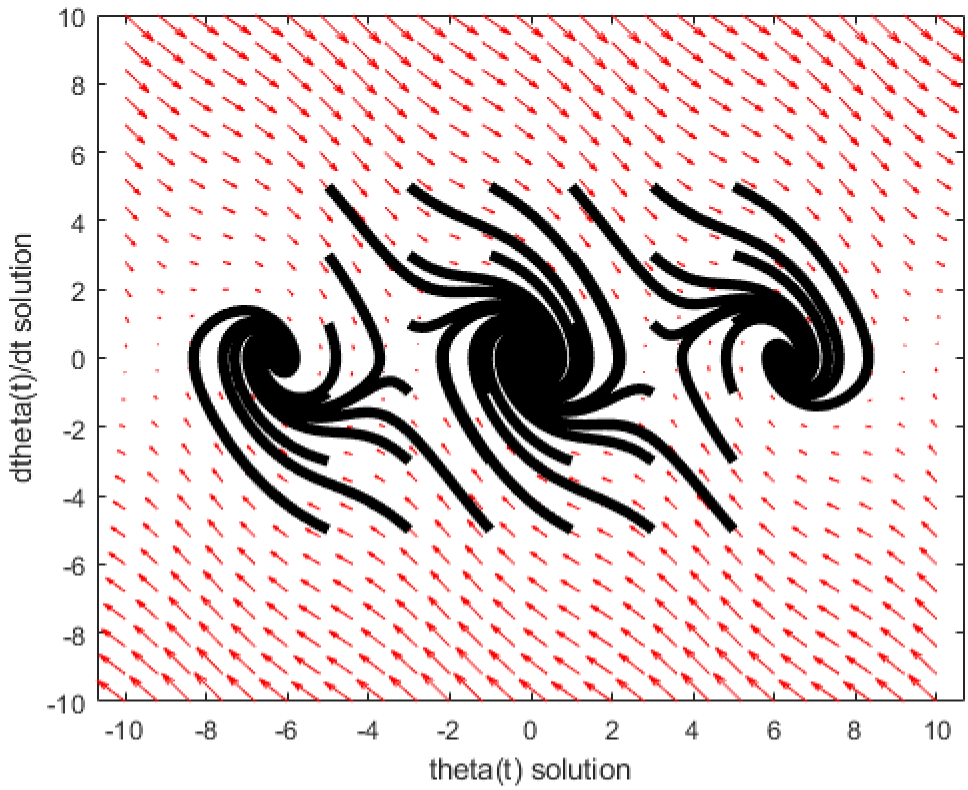


Figure 2.3: Stable spiral sinks at even integer n critical points $(n\pi, 0)$.

4. The linearization around $(x_0, y_0) = (n \cdot \pi, 0)$ for odd integer n is:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x - n \cdot \pi \\ y \end{pmatrix} + O(\|(x - n \cdot \pi, y)\|^2).$$

The eigenvalues of that matrix are:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}.$$

Therefore, it has one negative eigenvalue $\lambda_1 < 0$ and one positive eigenvalue $\lambda_2 > 0$, and so the critical points will be unstable saddle points.

Method formal steps

1. First we obtain the critical points for the system

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{x} = \mathbf{F}(\mathbf{x})$$

i.e. points (x_0, y_0) where $\mathbf{F}(x_0, y_0) = 0$.

2. We Taylor expand $\mathbf{F}$ in higher dimensions around an arbitrary critical point:

$$\begin{aligned} F(x, y) &= F(x_0, y_0) + J_F(x_0, y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2) \\ &= J_F(x_0, y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2). \end{aligned}$$

Here $J_F(x_0, y_0)$ is the Jacobian matrix for function $F(x, y) = \binom{F_1(x,y)}{F_2(x,y)}$:

$$J_F(x_0, y_0) := \begin{pmatrix} \frac{d}{dx}F_1(x_0, y_0) & \frac{d}{dy}F_1(x_0, y_0) \\ \frac{d}{dx}F_2(x_0, y_0) & \frac{d}{dy}F_2(x_0, y_0) \end{pmatrix}$$

3. Around each critical point we determine the eigenvalues and identify the type of qualitative behaviour.

General result:

The system

$$x' = F(x, y), y' = G(x, y)$$

is locally linear around a critical point (x_0, y_0) if $F, G \in C^2$ around it.

Proof. First we Taylor expand them

$$\begin{aligned} F(x, y) &= F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + R_F(x, y), \\ G(x, y) &= G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) + R_G(x, y), \end{aligned}$$

where by Taylor's thm the residues satisfy

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R_F(x,y)}{|(x_0,y_0)|} = 0 = \lim_{(x,y)\to(x_0,y_0)} \frac{R_G(x,y)}{|(x_0,y_0)|}.$$

Because (x_0, y_0) is a critical point we have $F(x_0, y_0) = G(x_0, y_0) = 0$. Therefore, we can rewrite the system as:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} R_F(x,y) \\ R_G(x,y) \end{pmatrix}.$$

However,

□

Examples

- We return to Duffing's equation from the previous section

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -x + \frac{x^3}{6}.$$

  1. We found that the critical points are $(0,0), (\pm 6, 0)$.
  2. We obtain the linearization of the RHS of the system $F(x,y) := \binom{y}{-x+\frac{x^3}{6}}$ around arbitrary critical point (x_0, y_0):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} &= F(x,y) = F(x_0,y_0) + J_F(x_0,y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2) \\ &= \begin{pmatrix} 0 & 1 \\ -1 + \frac{x^2}{2} & 0 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2)\end{aligned}$$

  3. Next we study the stability behaviour around each of the critical points.
     - At the origin we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(\|(x,y)\|^2)$$

       and so the eigenvalues are $\lambda = \pm i$. Therefore, the stability behaviour at the origin will be concentric circles.
     - At the $(\pm 6, 0)$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 17 & 0 \end{pmatrix}\begin{pmatrix} x \pm 6 \\ y \end{pmatrix} + O(\|(x \pm 6, y)\|^2)$$

       and so the eigenvalues are $\lambda = \pm\sqrt{17}$. Therefore, the stability behaviour at both $(6,0), (-6,0)$ will be unstable saddle nodes.

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y + \varepsilon sin(x), \frac{\mathrm{d}y}{\mathrm{dt}} = -x - \varepsilon cos(y),$$

  for small $\varepsilon \in \mathbb{R}$. We will study the affect of the stability behaviour as $\varepsilon \to 0$.

1. First we find that the only critical point is the origin (0,0). We can deduce this by drawing the two curves $(x, \varepsilon sin(x))$, $(\varepsilon cos(y), y)$ and see that they intersect only at the origin.
2. Next we linearize around the origin:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(\|(x,y)\|^2).$$

   Therefore, the eigenvalues are $\lambda_1 = \varepsilon + i, \lambda_1 = \varepsilon - i$.

   (a) If $\varepsilon < 0$, the origin becomes a sink spiral point.

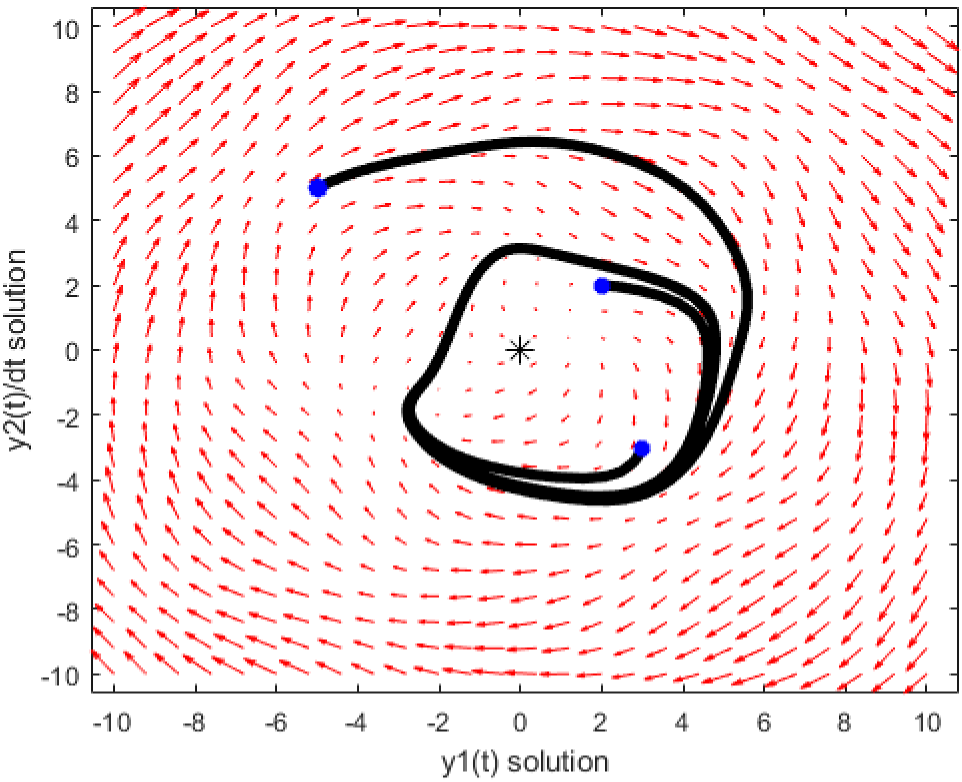


Figure 2.4: For negative pertubation we get sink spiral.

   (b) If $\varepsilon > 0$, the origin becomes a source spiral point.

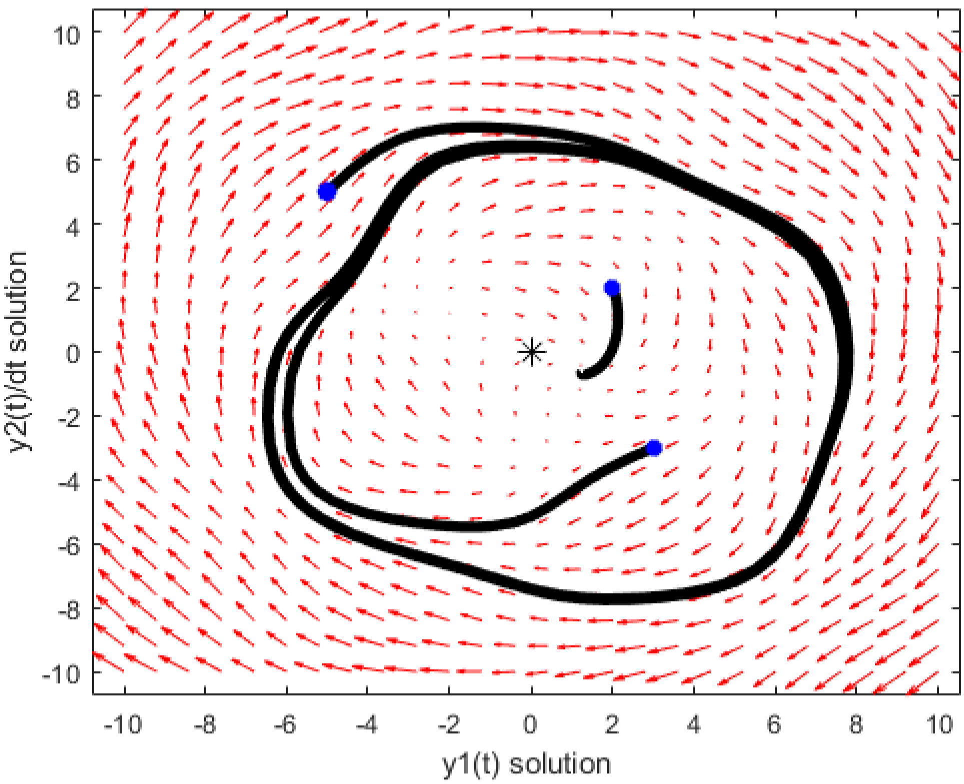


Figure 2.5: For positive pertubation we get source spiral.

(c) If $\varepsilon = 0$, the origin is a center of concentric circles.

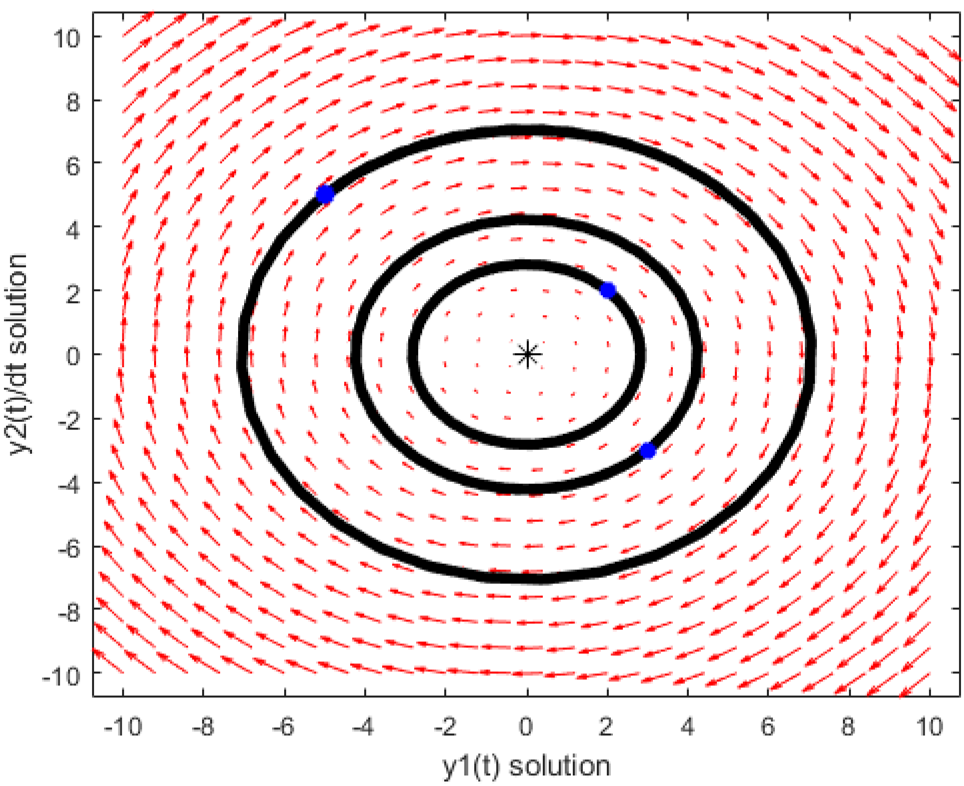


Figure 2.6: For zero pertubation we get circular behaviour.

If $\varepsilon = -0.1$ we get an almost circular picture:

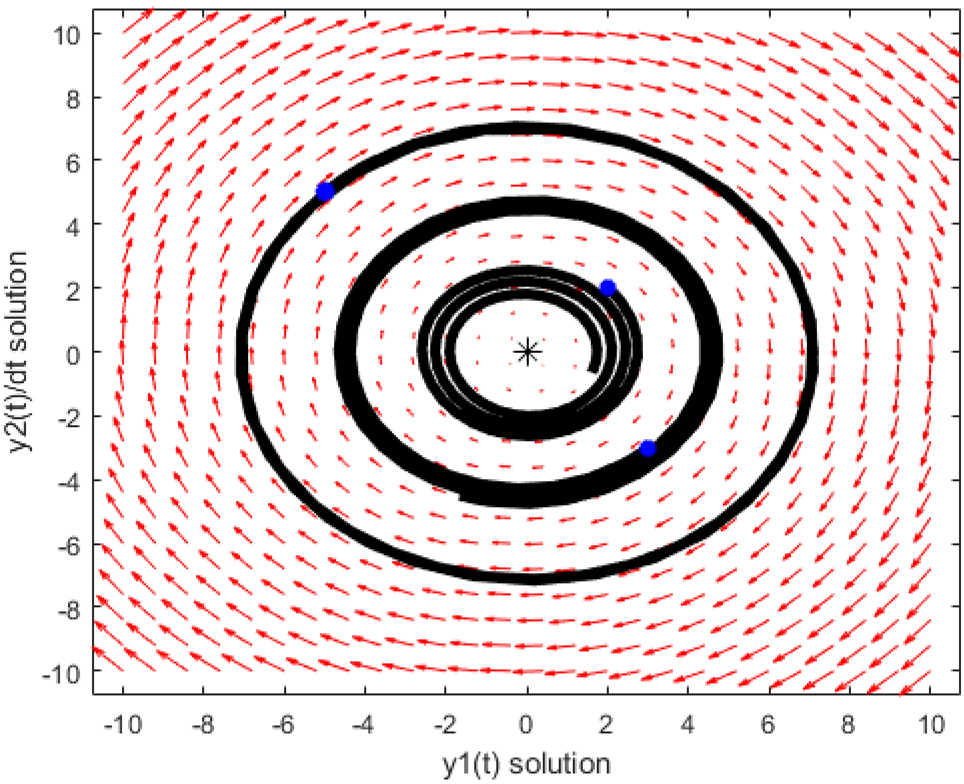


Figure 2.7: For small pertubation we get almost circular behaviour.

2.1 Applied Examples

2.1.1 Competing species

Suppose that in some closed environment there are two similar species competing for a limited food supply—for example, two species of fish in a pond that do not prey on each other but do compete for the available food. Let x and y be the populations of the two species at time t.

As discussed before, we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation

$$\frac{\mathrm{d}x}{\mathrm{dt}} = x(\varepsilon_1 - \sigma_1 x)$$
$$\frac{\mathrm{d}y}{\mathrm{dt}} = y(\varepsilon_2 - \sigma_2 y)$$

However, when both species are present, each will tend to diminish the available food supply for the other. In effect, they reduce each other's growth rates and saturation populations:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$\frac{\mathrm{d}y}{\mathrm{dt}} = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x)$$

The α_1 is a measure of the degree to which species y interferes with species x and similarly for α_2. The values of the positive constants $\varepsilon_i, \sigma_i, \alpha_i$ depend on the particular species under consideration and, in general, must be determined from observations.

1. First we find the critical points

$$\begin{cases} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) = 0 \\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) = 0 \end{cases} \Rightarrow (0,0), (\frac{\varepsilon_1}{\sigma_1}, 0), (0, \frac{\varepsilon_2}{\sigma_2}), \text{ and } (\frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}).$$

For the last critical point to be a realistic steady state we require that both components are positive:

Case I: Both $\varepsilon_1\sigma_2 > \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 > \varepsilon_1\alpha_2$
which also imply $\sigma_1\sigma_2 > \alpha_1\alpha_2$. This happens if $\varepsilon_1 = \varepsilon_2 = 1, \sigma_1 = \sigma_2 = 2, \alpha_1 = \alpha_2 = 1$.

Case II: Both $\varepsilon_1\sigma_2 < \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 < \varepsilon_1\alpha_2$
which also imply $\sigma_1\sigma_2 < \alpha_1\alpha_2$. This happens if $\varepsilon_1 = \varepsilon_2 = 1, \sigma_1 = \sigma_2 = 1, \alpha_1 = \alpha_2 = 2$.

The unrealistic cases are:

Case III: $\varepsilon_1\sigma_2 > \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 < \varepsilon_1\alpha_2$
which also imply $\sigma_1\sigma_2 < \alpha_1\alpha_2$.This happens if $\varepsilon_1 = \varepsilon_2 = 1, \alpha_1 = \sigma_1 = 1, \sigma_2 = \alpha_2 = 2$.

Case IV: $\varepsilon_1\sigma_2 < \varepsilon_2\alpha_1$ and $\varepsilon_2\sigma_1 > \varepsilon_1\alpha_2$
which also imply $\sigma_1\sigma_2 < \alpha_1\alpha_2$. This happens if $\varepsilon_1 = \varepsilon_2 = 1, \sigma_1 = \alpha_1 = 2, \alpha_2 = \sigma_2 = 1$.

2. We linearize the system by 2D-Taylor expanding

$$F(x,y) = \begin{pmatrix} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{pmatrix}$$

around critical point (x_0, y_0):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ y \end{pmatrix} &= F(x,y) = J_F(x_0,y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x-x_0, y-y_0)\|^2) \\ &= \begin{pmatrix} \varepsilon_1 - 2\sigma_1 x_0 - \alpha_1 y_0 & -\alpha_1 x_0 \\ -\alpha_2 y_0 & \varepsilon_2 - \alpha_2 x_0 - 2\sigma_2 y_0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x-x_0, y-y_0)\|^2). \end{aligned}$$

3. We determine the stability behaviour around each of the critical points.

   (a) At $(x_0, y_0) = (0,0)$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + O(\|(x,y)\|^2).$$

   Therefore, the eigenvalues are $\lambda_1 = \varepsilon_1 > 0, \lambda_2 = \varepsilon_2 > 0$ and so the origin (0,0) is an unstable source node.

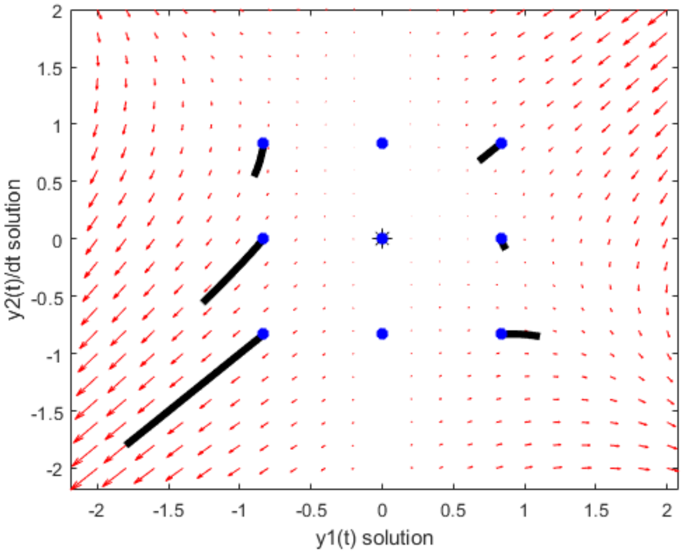


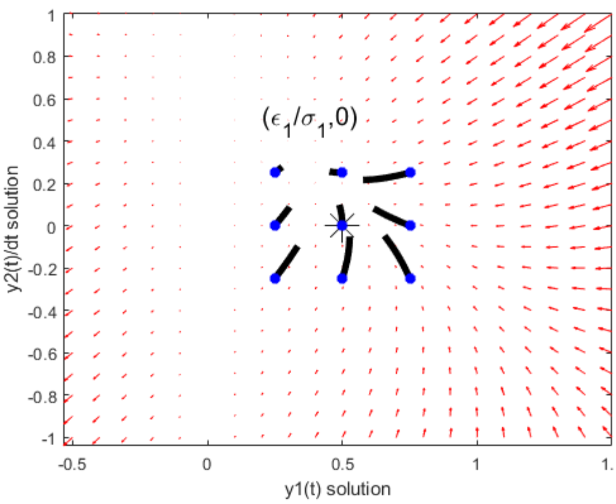
Figure 2.8: The origin is an unstable source node

(b) At $(x_0, y_0) = (\frac{\varepsilon_1}{\sigma_1}, 0)$ we have

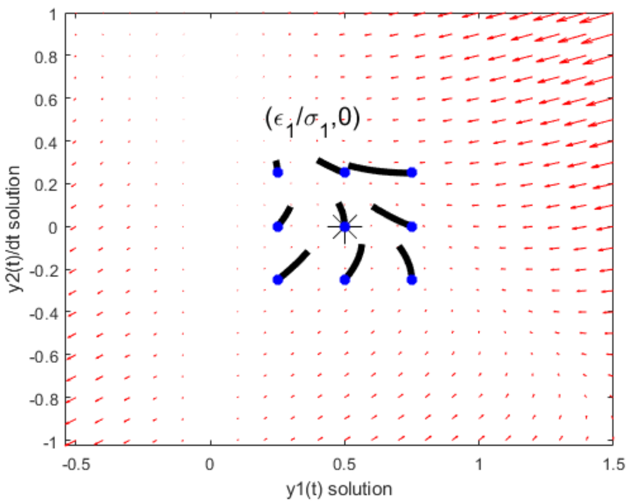
$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\varepsilon_1 & \frac{-\alpha_1\varepsilon_1}{\sigma_1} \\ 0 & \varepsilon_2 - \frac{\alpha_2\varepsilon_1}{\sigma_1} \end{pmatrix}\begin{pmatrix} x - x_0 \\ y \end{pmatrix} + O(\|(x - x_0, y)\|^2).$$

Therefore, the eigenvalues are $\lambda_1 = -\varepsilon_1 < 0, \lambda_2 = \frac{\sigma_1\varepsilon_2 - \alpha_2\varepsilon_1}{\sigma_1}$ and so

- in cases I,IV, the $(\frac{\varepsilon_1}{\sigma_1}, 0)$ is an unstable saddle node.

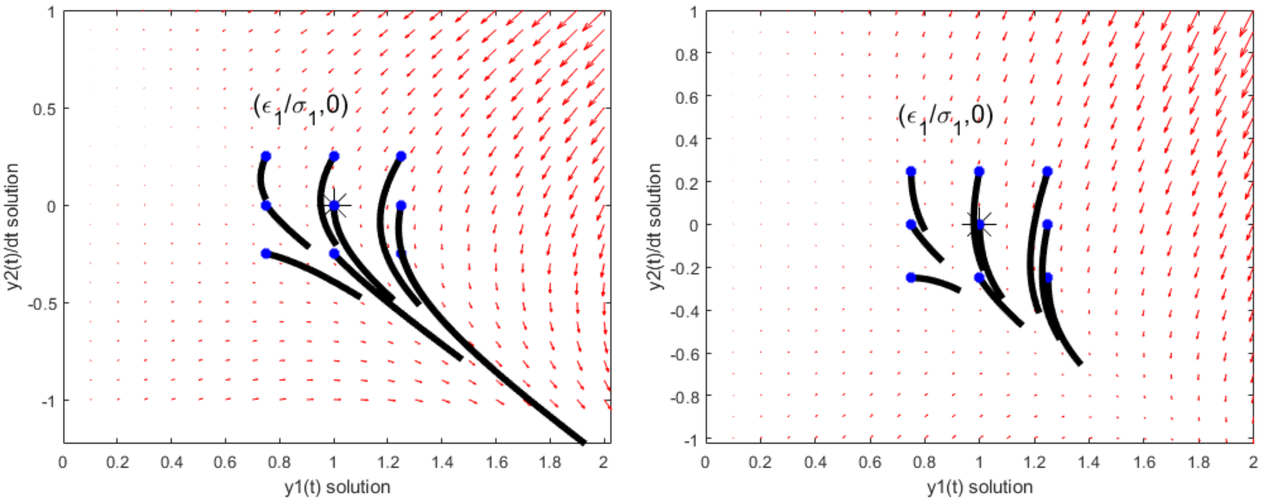


(a) Case I



(b) Case IV

- in cases II,III, the $(\frac{\varepsilon_1}{\sigma_1}, 0)$ is an stable sink node.



(a) Case II

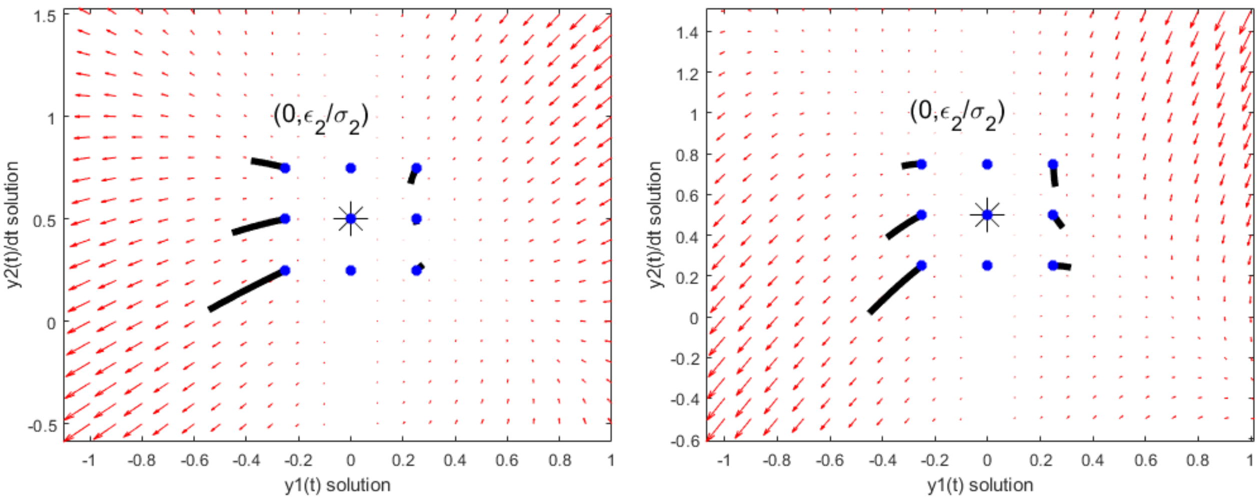
(b) Case III

(c) At $(x_0, y_0) = (0, \frac{\varepsilon_2}{\sigma_2})$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1 - \frac{\alpha_1 \varepsilon_2}{\sigma_2} & 0 \\ \frac{-\alpha_2 \varepsilon_2}{\sigma_2} & -\varepsilon_2 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y \end{pmatrix} + O(\|(x, y - y_0)\|^2).$$

Therefore, the eigenvalues are $\lambda_1 = \frac{\varepsilon_1 \sigma_2 - \alpha_1 \varepsilon_2}{\sigma_2}$, $\lambda_2 = -\varepsilon_2 < 0$ and so

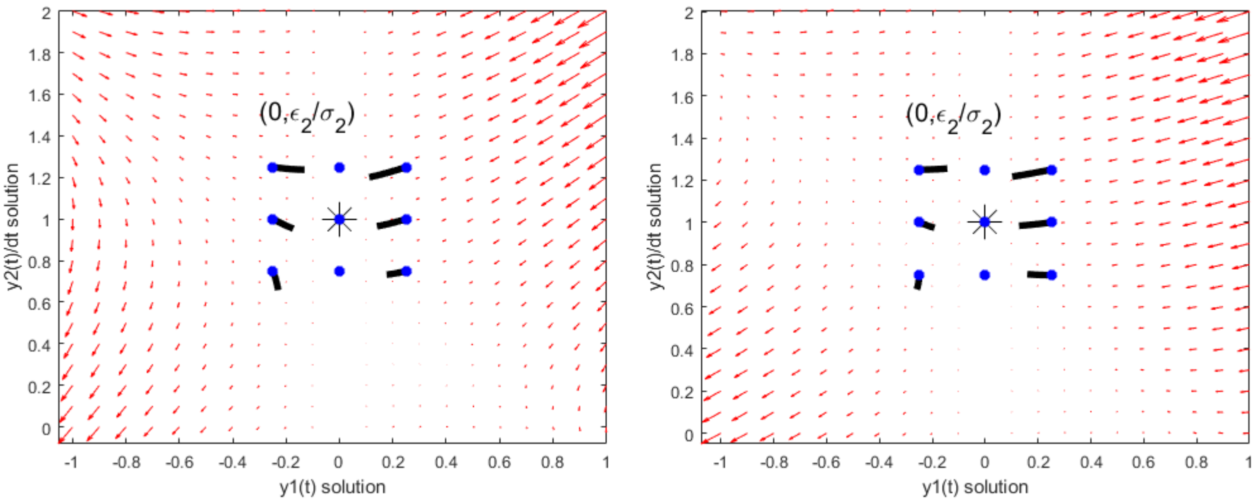
- in cases I,III, the $(0, \frac{\varepsilon_2}{\sigma_2})$ is an unstable saddle node.



(a) Case I

(b) Case III

- in cases II,IV, the $(0, \frac{\varepsilon_2}{\sigma_2})$ is an stable sink node.



(a) Case II (b) Case IV

(d) At $(x_0, y_0) = (\frac{\varepsilon_1\sigma_2-\varepsilon_2\alpha_1}{\sigma_1\sigma_2-\alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1-\varepsilon_1\alpha_2}{\sigma_1\sigma_2-\alpha_1\alpha_2})$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_1 x_0 & -\alpha_1 x_0 \\ -\alpha_2 y_0 & \sigma_2 y_0 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y \end{pmatrix} + O(\|(x-x_0, y-y_0)\|^2).$$

Therefore, the eigenvalues are

$$\lambda_1 = \frac{-(\sigma_1 x_0 + \sigma_2 y_0) - \sqrt{(\sigma_1 x_0 + \sigma_2 y_0)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)x_0 y_0}}{2}$$

by rearranging

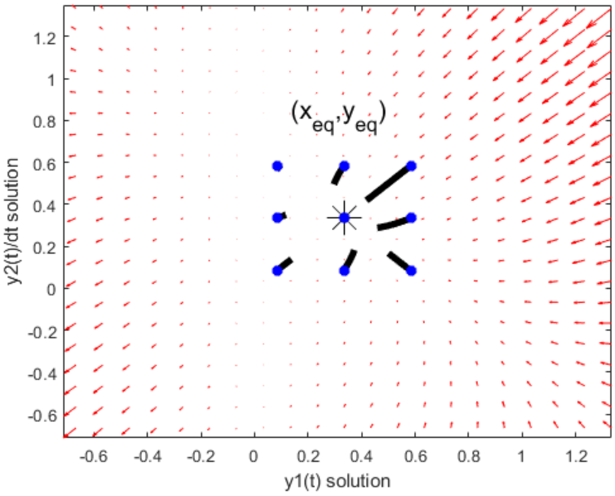
$$= \frac{-(\sigma_1 x_0 + \sigma_2 y_0) - \sqrt{(\sigma_1 x_0 - \sigma_2 y_0)^2 + 4\alpha_1\alpha_2 x_0 y_0}}{2}$$

$$\lambda_2 = \frac{-(\sigma_1 x_0 + \sigma_2 y_0) + \sqrt{(\sigma_1 x_0 + \sigma_2 y_0)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)x_0 y_0}}{2}$$

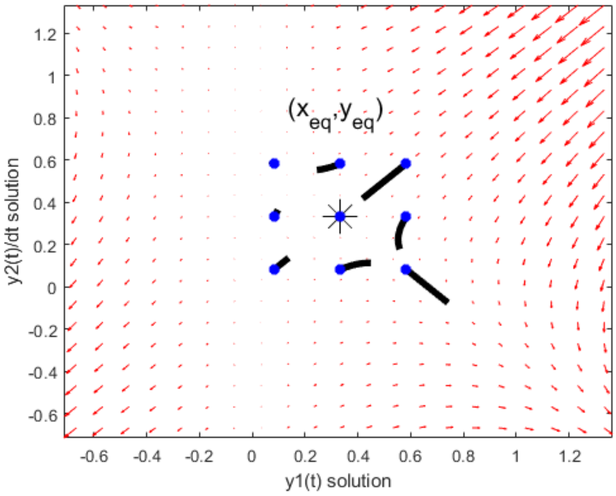
by rearranging

$$= \frac{-(\sigma_1 x_0 + \sigma_2 y_0) + \sqrt{(\sigma_1 x_0 - \sigma_2 y_0)^2 + 4\alpha_1\alpha_2 x_0 y_0}}{2}.$$

- In cases I,II we have $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$. Thus, weobserve that the radicand is positive and so the eigenvalues will always be real. Therefore, we get $\lambda_1 < 0$ and $\lambda_2 < 0$ and in turn (x_0, y_0) is stable sink node.

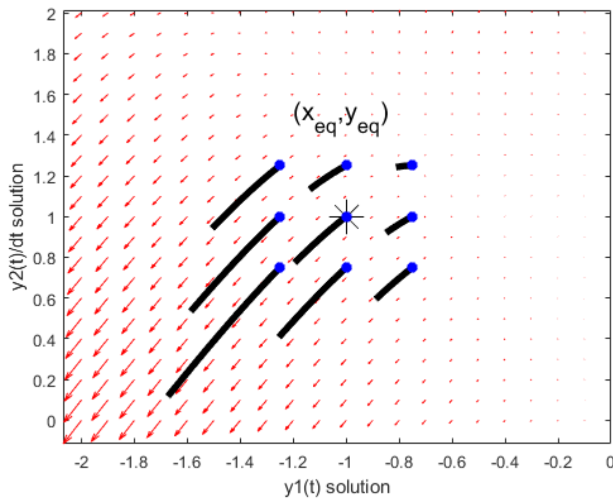


(a) Case I

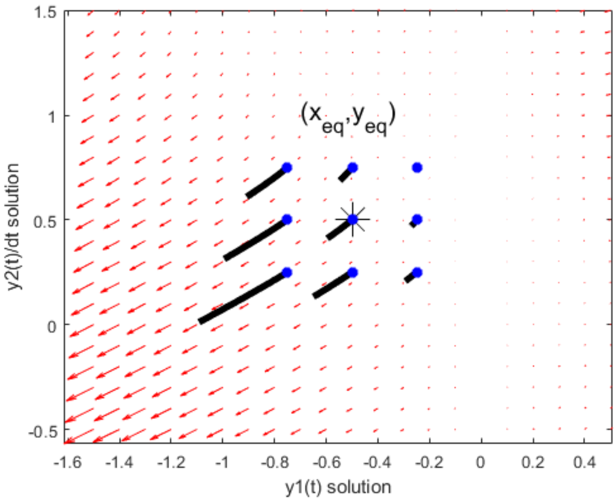


(b) Case II

- In cases III,IV the $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$ and so $\lambda_1 < 0, \lambda_2 > 0$. Thus, (x_0, y_0) is an unstable saddle node.



(a) Case III



(b) Case IV

4. We also do a nullcline analysis to predict how solutions will behave based on their initial data.

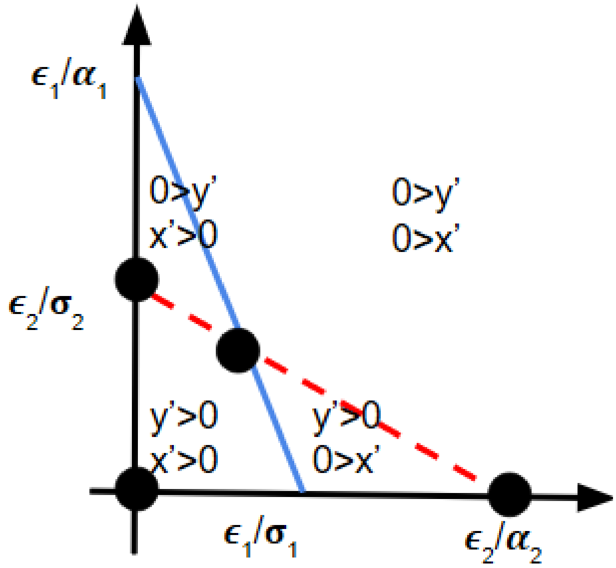


Figure 2.15: Case I

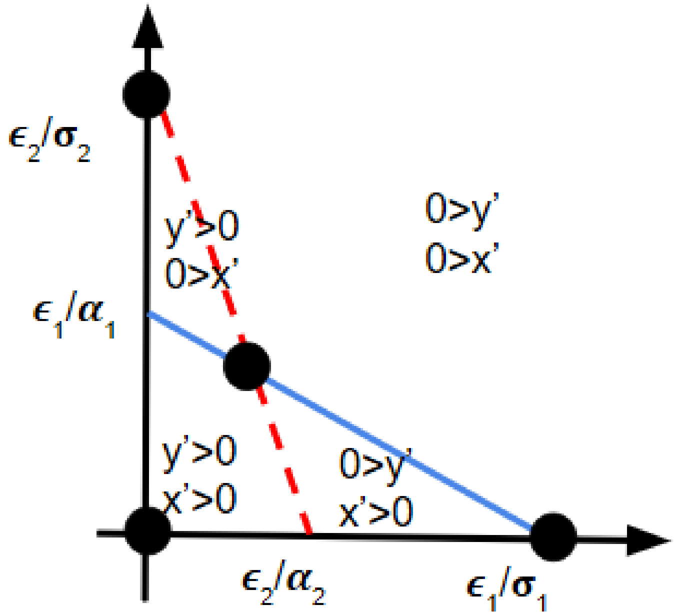


Figure 2.16: Case II

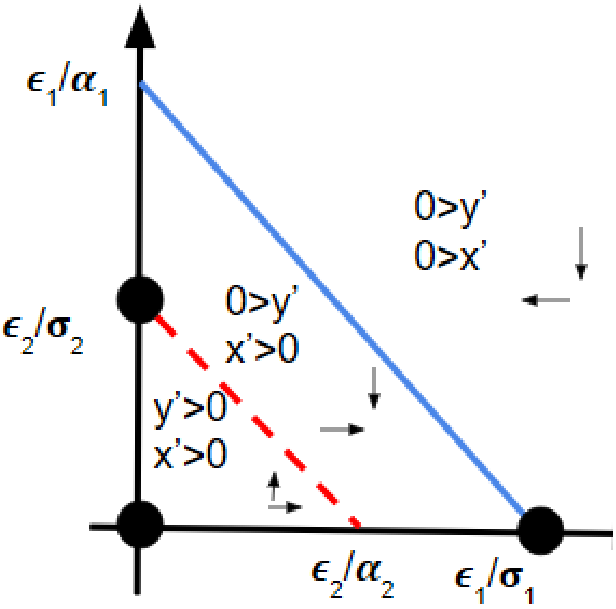


Figure 2.17: Case II

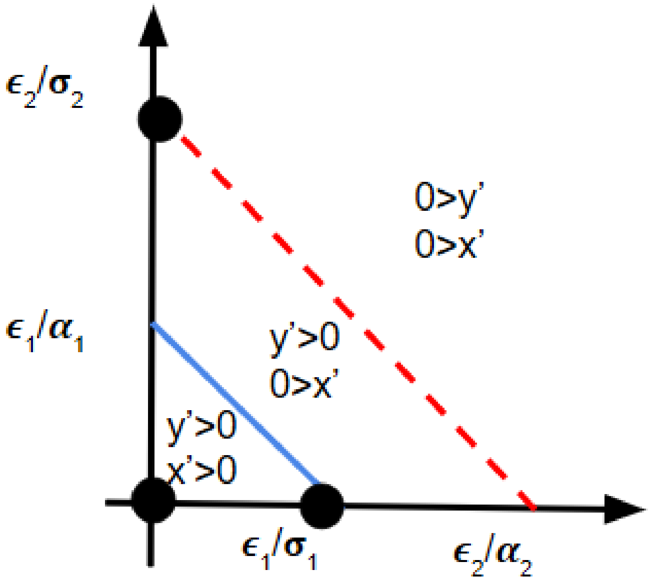


Figure 2.18: Case IV

(a) For simplicity lets start from case III and IV where the lines are well separated. First in case III we will show that all solutions tend to $(\frac{\varepsilon_1}{\sigma_1}, 0)$(i.e. second species dies out).

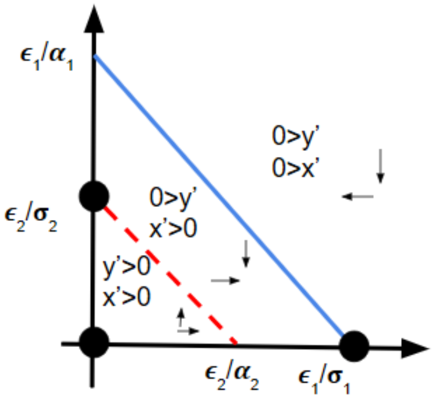


Figure 2.19: Case III

In the region where $0 > y', 0 > x'$ we will have the solutions flowing towards the left (left pointing arrow) and downwards (down pointing arrow). We similarly obtain the other arrows as shown in the figure. Therefore, solutions will escape the region where $y' > 0, x' > 0$, then once in the region $0 > y'x' > 0$ they will move south and rightwards till they hit the critical point $(\frac{\varepsilon_1}{\sigma_1}, 0)$.
The case IV similarly gives that $(0, \frac{\varepsilon_2}{\sigma_2})$ is the equilibrium point (i.e. first species dies out).

(b) Next we study case I. We will show that all solutions tend to the equilibrium point (both species coexist)

$$(x_0, y_0) = (\frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}).$$

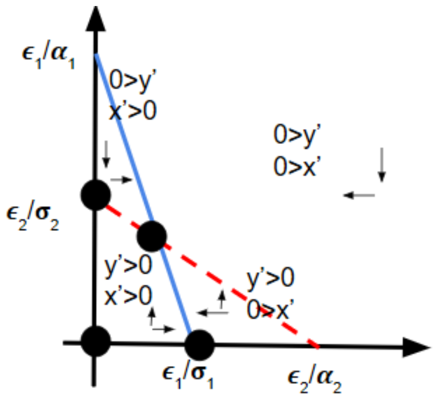


Figure 2.20: Case I

In the region where $0 > y', 0 > x'$ we will have the solutions flowing towards the left (left pointing arrow) and downwards (down pointing arrow). Solutions will escape the region where $y' > 0, x' > 0$, into either a) the region $0 > y'x' > 0$ in which they will move south and rightwards till they hit the equilibrium point or into b)the region $y' > 0, 0 > x'$ in which they will move north and leftwards till they hit the equilibrium point.
The case II similarly gives the same equilibrim point as the stable solution.

Price adjustment mechanism [SHS08, p. 6.8]

Consider the following system of differential equations:

$$p' = H_1(D_1(p,q) - S_1(p,q)), q' = H_2(D_2(p,q) - S_22(p,q)).$$

where p, q denote the prices of two different commodities with corresponding demand and supply D_i, S_i for $i = 1, 2$ and H_i are functions of one variable. Assume that $H_1(0) = H_2(0) = 0$ and that $H_1' > 0, H_2' > 0$.

Walras's law and the *ttonnement* mechanism

Here, we consider the question of stability of a pure exchange, competitive equilibrium with an adjustment mechanism known as t^atonnement and directly inspired by the work of $LonWalras$ (1874), one of the founding fathers of mathematical economics.
The basic idea behind the t^atonnement mechanism is the same assumed in the rudimentary price adjustment mechanism models, namely that prices of commodities rise and fall in response to discrepancies between demand and supply (the so-called 'law of demand and supply').
In the present case, demand is determined by individual economic agents maximising a utility function subject to a budget constraint, given a certain initial distribution of stocks of commodities. The model can be described schematically as follows.

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{dt}} = \mathbf{f}(\mathbf{p}) = \begin{pmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \end{pmatrix},$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions with all their derivatives continuous as well.

1. A price point $\mathbf{p}_0$ is called an equilibrium if

$$\begin{aligned} &f_i(\mathbf{p}_0) \leq 0, p_i \geq 0, \text{and } p_j > 0 \text{for some j} \\ &\text{or } f_i(\mathbf{p}_0) < 0, \mathbf{p}_0 = \mathbf{0}. \end{aligned}$$

   The first case makes economic sense (i.e. at least one price is nonzero) and so by *equilibrium point* we will mean the first case.

2. (Hypothesis **H**)The hypothesis that agents maximise utility is that the functions $f_i(\mathbf{p})$ are **homogeneous of degree zero**, namely $f_i(\lambda\mathbf{p}) = \lambda^0 f_i(\mathbf{p}) = f_i(\mathbf{p})$ for any $\lambda > 0$.

3. (**Walras's law**)Consider that the budget constraint for each individual k takes the form

$$\sum_{i=1}^{2} p_i f_i^k(\mathbf{p}) = p_1 f_1^k(\mathbf{p}) + p_2 f_2^k(\mathbf{p}) = 0,$$

   where f_i^k denotes the excess demand by the kth economic agent for the ith commodity, i.e., the difference between the agent's demand for, and the agent's initial endowment of, that commodity. In general for m commodities by summing over all N economic agents we have:

$$\sum_{k=1}^{N} \sum_{i=1}^{m} p_i f_i^k(\mathbf{p}) = \sum_i p_{i=1}^m f_i(\mathbf{p}) = 0.$$

   This law states that, in view of the budget constraints, for any set of semipositive prices p (not necessarily equilibrium prices), the value of aggregate excess demand, evaluated at those prices, must be zero.

4. The Jacobian matrix for $\mathbf{f}$ is

$$D\mathbf{f}(\mathbf{p}_0) = \begin{pmatrix} \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_1(\mathbf{p}_0)}{\mathrm{dp}_2} \\ \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_1} & \frac{\mathrm{d}f_2(\mathbf{p}_0)}{\mathrm{dp}_2} \end{pmatrix}.$$

   We will need this matrix to be **Metzler**:

(a) Suppose that if the price of the ith commodity increases, while all the other prices remain constant, the excess demand for the ith commodity decreases (and vice versa). Suppose also that the effect of changes in the price of the ith commodity on its own excess demand is stronger than the combined effect of changes in the other prices (where the latter can be positive or negative). This can be formalised by assuming that

$$a_{ii} := \frac{\mathrm{d}f_i(\mathbf{p}_0)}{\mathrm{dp_i}} < 0$$

and the "strict diagonal dominance" (SDD) assumption that there exists a positive vector $d \in \mathbb{R}^m$ (in our case $m = 2$) s.t.

$$|a_{ii}|d_i > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}|d_j.$$

(b) Moreover, we have the "gross substitutability" (GS) assumption that if we start from equilibrium and the price of a commodity increases (decreases) while the prices of all other commodities remain constant, then the excess demand of all of the other commodities increases (decreases):

$$a_{ij} := \frac{\mathrm{d}f_i(\mathbf{p}_0)}{\mathrm{dp_j}} > 0, i \neq j.$$

5. The eigenvalues for this system are:

$$\begin{aligned}\lambda &= \frac{a_{11} + a_{22}}{2} \pm \frac{1}{2}\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \\ &= \frac{-|a_{11} + a_{22}|}{2} \pm \frac{1}{2}\sqrt{|(a_{11} - a_{22})^2 + 4a_{12}a_{21}|}\end{aligned}$$

3 Simulation code

Encoding the system If we start with the 2D system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = f_1(x, y, t) \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = f_2(x, y, t),$$

we first encode it as a function as follows:

```
f = @(t,y) [f_{1}(y(1),y(2),t);f_{2}(y(1),y(2),t)]; }
```

where $y(1) = x$ and $y(2) = y$. For example, for

$$\frac{\mathrm{d}x}{\mathrm{dt}} = x^2 + y \text{ and } \frac{\mathrm{d}y}{\mathrm{dt}} = sin(y)$$

we write

```
f = @(t,y) [y(1)^2+y(2);sin(y(2))]; }
```

Direction field The following matlab-function will generate the direction field for the above function f:

```
function vectfield(func,y1val,y2val,t)
if nargin==3
  t=0;
end
```

```
n1=length(y1val);
n2=length(y2val);
yp1=zeros(n2,n1);
yp2=zeros(n2,n1);
for i=1:n1
   for j=1:n2
      ypv = feval(func,t,[y1val(i);y2val(j)]);
      yp1(j,i) = ypv(1);
      yp2(j,i) = ypv(2);
   end
end
quiver(y1val,y2val,yp1,yp2,'r','Autoscalefactor', 3)%'MaxHeadSize');
axis tight;
```

This is an example of calling it for the above example $F = (x^2 + y, sin(y))$:

```
f = @(t,y) [y(1)^2+y(2);y(2)];
 vectfield(f,-1:0.1:1,-1:0.1:1);

```

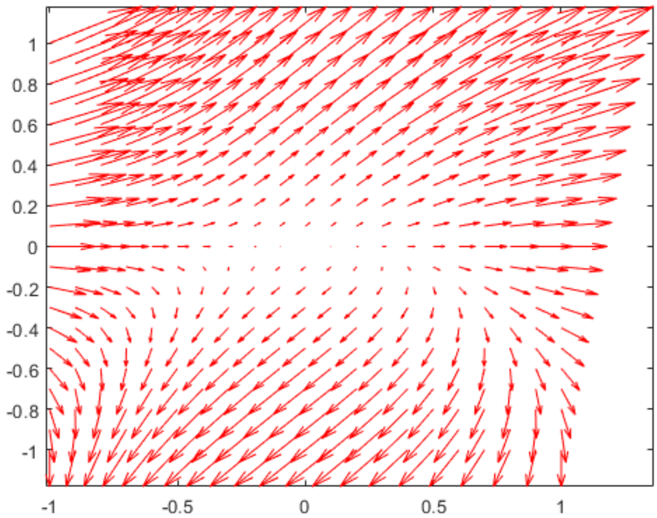


Figure 3.1: Direction field for system $\frac{dx}{dt} = x^2 + y$ and $\frac{dy}{dt} = sin(y)$.

Solving the ode system Given function f and initial data x_0, y_0, the following ode solver outputs a two dimensional solution run up to time T:

```
[ts,ys] = ode45(f,[0,T],[x0;y0]);
```

This is an example for the above function

```
x0=0.5;
y0=0.5;
T=4
f = @(t,y) [y(1)^2+y(2);y(2)];
vectfield(f,-1:0.1:1,-1:0.1:1);
hold on

plot(x0,y0,'o','MarkerFaceColor','b','MarkerSize',20)

hold on;
[ts,ys] = ode45(f,[0,T/10],[x0;y0]);
 plot(ys(:,1),ys(:,2),'k','Linewidth', 4)
xlabel('y1(t) solution')
     ylabel('y2(t)/dt solution')
```

that outputs:

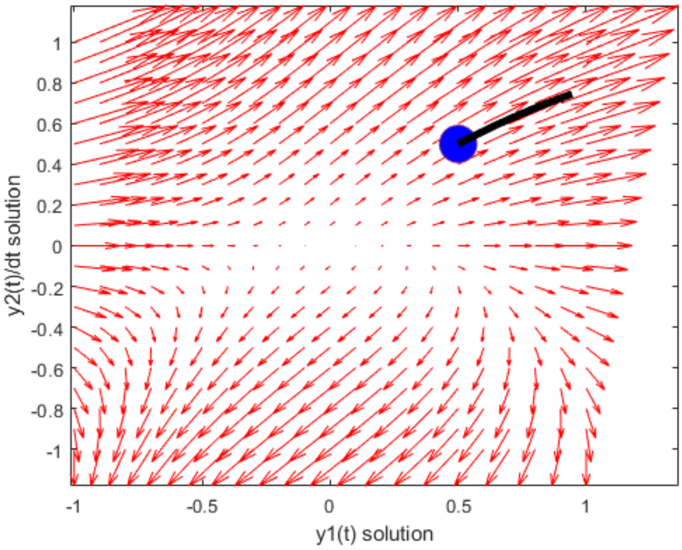


Figure 3.2: Solution curve for system $\frac{dx}{dt} = x^2 + y$ and $\frac{dy}{dt} = sin(y)$.

Plotting implicit solutions If we obtain the implicit solutions $f(x, y) = constant$ then we can plot them with

```
fimplicit(@(x,y) f(x,y)=k,'LineWidth',1,'Color','k')
```

For example, for $x^2 + y^2 - sin(y) = constant$ the following program

```
for k=1:10
f=@(x,y)x^2+y^2-sin(y)-k;
fimplicit(f,'LineWidth',1,'Color','k')
hold on;
end
```

outputs

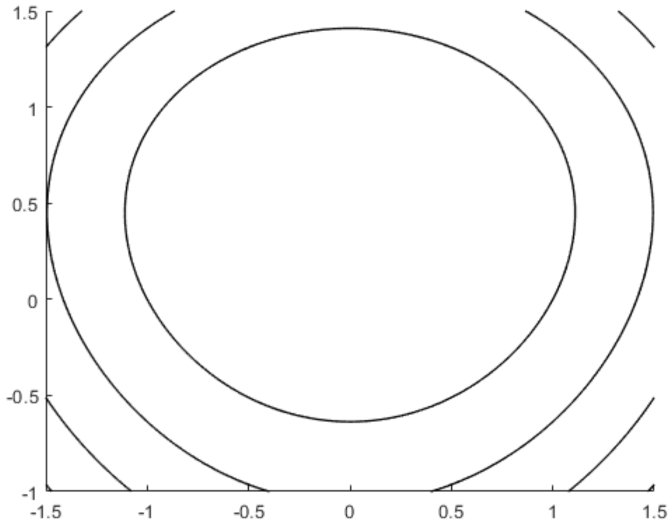


Figure 3.3: trajectories of implicit solutions.

Plotting eigenvectors for linear system Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

First we find the eigenvalues

```
[V,D]=eig(A);
```

The following code will generate linear spans for the eigenvectors and plot them:

```

A=[1 1; 0 2]
[V,D]=eig(A);

m1=1; %%% This two parameters scale the length of the line
m2=1;

xi11 = [-m1* V(1,1), m1*V(1,1)];
xi21 = [-m1*V(2,1), m1*V(2,1)];
pl1 = line(xi11,xi21,'Color','k','LineWidth',2);

x1 = V(1,1)/3;
y1 =V(2,1);
txt1 = ' \xi_1 ';
text(x1,y1,txt1,'FontSize',20)
hold on

xi12 = [-m2* V(1,2), m2*V(1,2)];
xi22 = [-m2*V(2,2), m2*V(2,2)];

pl2 = line(xi12,xi22,'Color','k','LineWidth',2);

x2 =  V(1,2)/3;
y2 =V(2,2)/3;
txt1 = '\xi_2 ';
text(x2,y2,txt1,'FontSize',20)
```

This is the full example with both the direction field and the eigenvector spans:

```
x0=0.5;
y0=0.5;
T=4

f = @(t,y) [y(1);2*y(2)];
vectfield(f,-1:0.1:1,-1:0.1:1);
hold on

plot(x0,y0,'o','MarkerFaceColor','b','MarkerSize',20)

hold on;
[ts,ys] = ode45(f,[0,T/10],[x0;y0]);
 plot(ys(:,1),ys(:,2),'k','Linewidth', 4)
hold on;

A=[1 1; 0 2]
[V,D]=eig(A);



m1=1;
m2=1;


xi11 = [-m1* V(1,1), m1*V(1,1)];
xi21 = [-m1*V(2,1), m1*V(2,1)];
pl1 = line(xi11,xi21,'Color','k','LineWidth',2);

x1 = V(1,1)/3;
y1 =V(2,1);
txt1 = ' \xi_1 ';
text(x1,y1,txt1,'FontSize',20)


hold on

xi12 = [-m2* V(1,2), m2*V(1,2)];
xi22 = [-m2*V(2,2), m2*V(2,2)];

pl2 = line(xi12,xi22,'Color','k','LineWidth',2);

x2 =  V(1,2)/3;
y2 =V(2,2)/3;
```

```
txt1 = '\xi_2 ';
text(x2,y2,txt1,'FontSize',20)
```

It outputs the following figure:

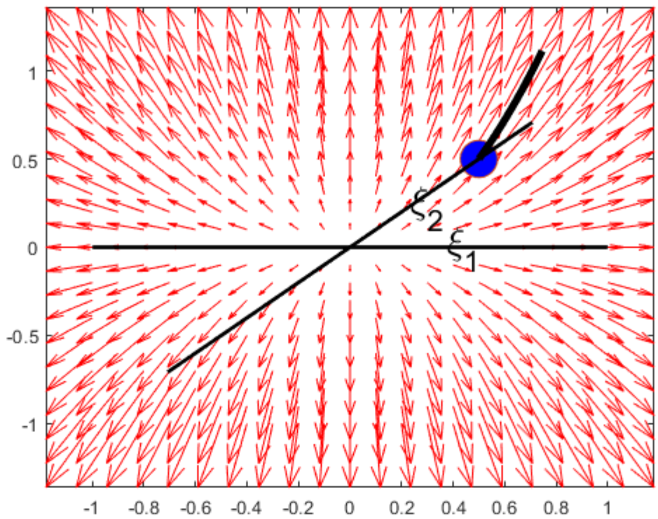


Figure 3.4: Eigenvectors spans and Direction field