Weeks 11

Variations of these exercises will appear on the final. Most of the exercises are in the main textbook.

Locally linear

- Stability analysis
  1. Determine all critical points of the given system of equations.
  2. Find the corresponding linear system near each critical point.
  3. Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system
     - ☐ Consider the system
     $$\frac{\mathrm{d}x}{\mathrm{dt}} = 1 - xy, \frac{\mathrm{d}y}{\mathrm{dt}} = x - y^3.$$
     - ☐ Consider the system
     $$\frac{\mathrm{d}x}{\mathrm{dt}} = (1+x)sin(y), \frac{\mathrm{d}y}{\mathrm{dt}} = 1 - x - cos(y).$$
     - ☐ Consider the system
     $$\frac{\mathrm{d}x}{\mathrm{dt}} = y + x(1 - x^2 - y^2), \frac{\mathrm{d}y}{\mathrm{dt}} = -x + y(1 - x^2 - y^2).$$
     - ☐ Consider the system
     $$\frac{\mathrm{d}x}{\mathrm{dt}} = (1-y)(2x-y), \frac{\mathrm{d}y}{\mathrm{dt}} = (2+x)(x-2y).$$
- (*)Consider the autonomous system
  $$\frac{\mathrm{d}x}{\mathrm{dt}} = x, \frac{\mathrm{d}y}{\mathrm{dt}} = -2y + x^3.$$
  1. Show that the critical point (0, 0) is a saddle point.
  2. Sketch the trajectories for the corresponding linear system, and show that the trajectory for which $x \to 0, y \to 0$ as $t \to +\infty$ is given by x = 0.
  3. Determine the trajectories for the nonlinear system for $x \neq 0$ by integrating the equation for dy/dx. Show that the trajectory corresponding to $x = 0$ for the linear system is unaltered, but that the one corresponding to $y = 0$ is $y = x^3/5$. Sketch several of the trajectories for the nonlinear system.
- (*) In this problem we show how small changes in the coefficients of a system of linear equations can affect the nature of a critical point when the eigenvalues are equal.
  1. Consider the system
  $$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}.$$
  Show that the eigenvalues are $r_1 = -1, r_2 = -1$ so that the critical point (0, 0) is an asymptotically stable node. Now consider the system
  2. Now consider the system
  $$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -\varepsilon & -1 \end{pmatrix} \mathbf{x}.$$
  where $|\varepsilon|$ is arbitrarily small. Show that if $\varepsilon > 0$, then the eigenvalues are $-1 \pm i\sqrt{\varepsilon}$ so that the asymptotically stable node becomes an asymptotically stable spiral point. If $\varepsilon < 0$, then the roots are $-1 \pm \sqrt{|\varepsilon|}$, and the critical point remains an asymptotically stable node.

Competing species

- (*)For the following systems

  1. Find the critical points.
  2. For each critical point, find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.
  3. Sketch the trajectories in the neighborhood of each critical point.
  4. Determine the limiting behavior of x and y as $t \to +\infty$ by doing a nullcline analysis.
  5. (**)Interpret the results in terms of the populations of the two species.

  - [ ] $$\frac{\mathrm{d}x}{\mathrm{dt}} = x(1.5 - 0.5 * y), \frac{\mathrm{d}y}{\mathrm{dt}} = y(-0.5 + x).$$

  - [ ] $$\frac{\mathrm{d}x}{\mathrm{dt}} = x(1 - 0.5 * x - 0.5 * y), \frac{\mathrm{d}y}{\mathrm{dt}} = y(-0.25 + 0.5 * x).$$

- (*) Consider the system

  $$\frac{\mathrm{d}x}{\mathrm{dt}} = x(1 - \sigma x - 0.5 * y), \frac{\mathrm{d}y}{\mathrm{dt}} = y(-0.75 + 0.25 * x),$$

  where $\sigma > 0$.

  1. Find all of the critical points. How does their location change as σ increases from zero? Observe that there is a critical point in the interior of the first quadrant only if $\sigma < 1/3$.
  2. Determine the type and stability property of each critical point. Find the value $\sigma_1 < 1/3$ where the nature of the critical point in the interior of the first quadrant changes. Describe the change that takes place in this critical point as σ passes through σ_1.
  3. (**) Describe the effect on the two populations as σ increases from zero to 1/3.