Outline

1. Laplace transform for ODEs
2. Laplace transform for systems o
   - Method formal steps
3. Lyapunov method

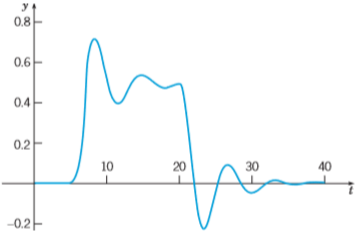

Figure: discontinuous forcing

Laplace transform for 1d ODEs

The Laplace transform of continuous functions f(t) with at most exponential growth, that is $|f(t)| \leq ce^{at}$ for $c, a > 0$, is defined as:

$$\mathcal{L}\Big\{f\Big\}(s) := \int_0^\infty e^{-st} f(t) dt,$$

where $s > a$.

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where $s > a$. By integration by parts we can easily check that we have:

$$\mathcal{L}\Big\{f'\Big\}(s) = s\mathcal{L}\Big\{f\Big\} - f(0)$$

so for the second derivative we have

$$\mathcal{L}\Big\{f''\Big\}(s) = s\mathcal{L}\Big\{f'\Big\} - f'(0) = s^2\mathcal{L}\Big\{f\Big\} - f'(0) - sf(0).$$

Laplace transform of vectors

Consider the Laplace transform of vectors $\mathcal{L}\Big\{\mathbf{x}\Big\}(s)$ defined componentwise

$$\mathcal{L}\Big\{\mathbf{x}\Big\}(s) := \begin{pmatrix} \mathcal{L}\Big\{x_1\Big\}(s) \\ \vdots \\ \mathcal{L}\Big\{x_n\Big\}(s) \end{pmatrix}.$$

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Therefore, as with the usual Laplace transform we obtain:

$$\mathcal{L}\left\{\mathbf{x}'\right\}(s) = s\mathcal{L}\left\{\mathbf{x}\right\}(s) - \mathbf{x}(0).$$

Consider the nonhomogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x} + \mathbf{g}(t).$$

1. Taking the Laplace transform of each term in the above equation we have:
$$s\mathcal{L}\Big\{\mathbf{x}\Big\}(s) - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\Big\{\mathbf{x}\Big\}(s) + \mathcal{L}\Big\{\mathbf{g}\Big\}(s).$$
2. For simplicity we assume that $\mathbf{x}(0) = 0$.
3. We then obtain the system:
$$(s\mathbf{I} - \mathbf{A})\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \mathcal{L}\Big\{\mathbf{g}\Big\}(s).$$
4. By inverting the matrix we obtain:
$$\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathcal{L}\Big\{\mathbf{g}\Big\}(s).$$
5. Then we do inverse Laplace transform of each component using known Laplace transform relations.

All the transforms we will need

$$\mathcal{L}\Big\{cos(at)\Big\}(s) = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\Big\{1\Big\}(s) = \frac{1}{s}$$

$$\mathcal{L}\Big\{e^{at}\Big\}(s) = \frac{1}{s-a}$$

$$\mathcal{L}\Big\{t^n e^{at}\Big\}(s) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\Big\{f_{step}(t,b)\Big\} = \frac{1-e^{-bs}}{s}$$

$$\mathcal{L}\Big\{e^{a(t-b)} f_{heavy}(t,b)\Big\} = \frac{e^{-bs}}{s-a}.$$

Presenting the method

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} f_{step}(t, 1) \\ t \end{pmatrix}.$$

Presenting the method

1. First we compute the Laplace transform of $\mathbf{g}$:

$$\mathcal{L}\left\{\begin{pmatrix} f_{step} \\ 3t \end{pmatrix}\right\}(s) = \begin{pmatrix} \dfrac{1-e^{-s}}{s} \\ \frac{1}{s^2} \end{pmatrix}$$

2. Then we compute the $(s\mathbf{I}-\mathbf{A})^{-1}$:

$$(s\mathbf{I}-\mathbf{A})^{-1}\mathcal{L}\left\{g\right\} = \begin{pmatrix} \frac{1-e^{-s}}{(s-2)s} - \frac{1}{s^2(s-2)(s-1)} \\ \frac{1}{s^2(s-1)} \end{pmatrix}.$$

Presenting the method

1. So by the cover-up method, we get:

$$\frac{1-e^{-s}}{(s-2)s} - \frac{1}{s^2(s-2)(s-1)}$$
$$= \frac{1-e^{-s}}{-2s} + \frac{1-e^{-s}}{2(s-2)} - \frac{1}{2s^2} + \frac{1}{(s-1)} - \frac{1}{4(s-2)} - \frac{3}{4s}.$$

2. Then we have

$$x_1(t) = -\frac{1}{2} f_{step}(t,1) + \frac{1}{2}(e^t - e^{2(t-1)} f_{heavy}(t,1))$$
$$- \frac{1}{2}t + e^t - \frac{1}{4}e^t - \frac{3}{4}$$

3. and

$$x_2(t) = -t^2 - 1 + e^t.$$

In class example

Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 \\ 0 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} f_{step}(t) \\ e^{-t} \end{pmatrix}.$$

- $$\mathcal{L}\Big\{1\Big\}(s) = \frac{1}{s}, \mathcal{L}\Big\{e^{at}\Big\}(s) = \frac{1}{s-a}, \mathcal{L}\Big\{t^n e^{at}\Big\}(s) = \frac{n!}{(s-a)^{n+1}}$$
- $$\mathcal{L}\Big\{f_{step}(t,b)\Big\} = \frac{1-e^{-bs}}{s}, \mathcal{L}\Big\{e^{a(t-b)} f_H(t,b)\Big\} = \frac{e^{-bs}}{s-a}.$$
- $$1 - f_H(t,b) = f_{step}(t,b) = \begin{cases} 1, & 0 \le t \le b \\ 0, & t > b \end{cases}$$

1. First we compute the Laplace transform of $\mathbf{g}$:

$$\mathcal{L}\left\{\begin{pmatrix} f_{step} \\ e^{-2t} \end{pmatrix}\right\}(s) = \begin{pmatrix} \dfrac{1-e^{-s}}{s} \\ \frac{1}{s+2} \end{pmatrix}$$

2. Then we compute the $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathcal{L}\left\{g\right\} = \begin{pmatrix} \frac{1-e^{-s}}{(s+1)s} + \frac{1}{(s+3)(s+1)(s+2)} \\ \frac{1}{(s+3)(s+2)} \end{pmatrix}.$$

1. For the first component we have:

$$\frac{1-e^{-s}}{(s+1)s}+\frac{1}{(s+3)(s+1)(s+2)}$$
$$=\frac{1-e^{-s}}{s+1}-\frac{1-e^{-s}}{s}+\frac{1}{(s+3)2}+\frac{-1}{(s+2)}+\frac{1}{(s+1)2}$$

for second

$$\frac{-1}{(s+3)}+\frac{1}{s+2}.$$

2. So the first component is

$$x_1=e^{-t}-e^{-(t-1)}f_H(t,1)-f_{step}(t,1)+\frac{1}{2}e^{-3t}-e^{-2t}+2e^{-t}.$$

3. the second component is

$$x_2(t)=-e^{-3t}+e^{-2t}$$

We return to the damping-free pendulum system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\frac{g}{L} sin(x).$$

Consider the total energy of the system:

$$\begin{aligned} E(x,y) &= \text{Potential } + \text{ Kinetic} \\ &= U(x,y) + K(x,y) \\ &:= mgL(1 - cos(x)) + \frac{1}{2} mL^2 y^2. \end{aligned}$$

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