Outline

1. Variation of parameters

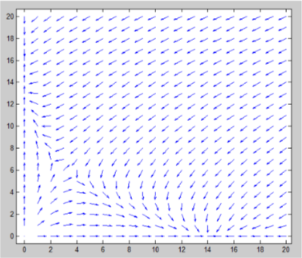

Figure: competing species separatrix

General nonhomogeneous systems

$$\mathbf{x}'(t) = \mathbf{P(t)}\mathbf{x}(t) + \mathbf{g}(t), \tag{1}$$

where $P(t)$ is a matrix that also depends on time.

General nonhomogeneous systems

$$\mathbf{x}'(t) = \mathbf{P(t)}\mathbf{x}(t) + \mathbf{g}(t), \tag{1}$$

where $P(t)$ is a matrix that also depends on time.We make the ansatz

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{y}(t),$$

where Ψ is a matrix whose columns are the x_1, x_2 solutions for the homogeneous problem.

Variation of parameters: system to solve

Plugging in the ansatz we obtain:

$$\mathbf{\Psi}(t)\mathbf{y}'(t) = \mathbf{g}(t).$$

Variation of parameters: system to solve

Plugging in the ansatz we obtain:

$$\mathbf{\Psi}(t)\mathbf{y}'(t) = \mathbf{g}(t).$$

or equivalently

$$\begin{cases} x_{1,1}(t)v_1' + x_{1,2}(t)v_2' = g_1(t) \\ x_{2,1}(t)v_1' + x_{2,2}(t)v_2' = g_2(t) \end{cases}$$

Example

Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

in class example

Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}.$$

1. First we find the fundamental matrix $\mathbf{\Psi}$: the eigenvalue is $(0, \binom{1}{2})$ and $\eta = k\binom{1}{2} - \binom{0}{1}$, therefore, the solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 (t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \Rightarrow$$
$$\mathbf{\Psi}(t) = \begin{bmatrix} 1 & t \\ 2 & 2t-1 \end{bmatrix}.$$

2. The system is

$$\begin{cases} v_1' + t v_2' = t^{-3} \\ 2v_1' + (2t-1) v_2' = -t^{-2} \end{cases}$$

and we obtain

$$\mathbf{v}'(t) = \begin{pmatrix} -(t^{-1} + 2t^{-2} - t^{-3}) \\ t^{-2} + 2t^{-3} \end{pmatrix} \Rightarrow$$
$$\mathbf{v}(t) = \begin{pmatrix} \frac{4t-1}{2t^2} - log(t) \\ -\frac{t+1}{t^3} \end{pmatrix}.$$

Therefore, the nonhomogeneous solution is

$$\begin{aligned} \mathbf{x}_{nh}(t) &= \mathbf{\Psi}(t)\mathbf{v}(t) \\ &= \begin{bmatrix} 1 & t \\ 2 & 2t-1 \end{bmatrix} \begin{pmatrix} \frac{4t-1}{2t^2} - log(t) \\ -\frac{t+1}{t^3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1+t}{t^2} + \frac{-1+4t}{2t^2} - log(t) \\ -\frac{((1+t)(-1+2t))}{t^3} + 2(\frac{(-1+4t)}{2t^2} - log(t)) \end{pmatrix} \\ &= -log(t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t^{-2}\begin{pmatrix} -3/2 \\ -2 \end{pmatrix} + t^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t^{-3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) + \mathbf{x}_{nh}(t).$$

In class example

Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

The End