Contents

1 Systems of ODEs

Consider system of equations:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \ldots + p_{1n}x_n + g_1(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \ldots + p_{nn}x_n + g_n(t) \end{aligned}$$

where $p_{ij}(t), g_i(t)$ are continuous functions. The continuity ensures that we have existence and uniqueness of solutions. Equivalently we can rewrite this system as

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t).$$

where $\mathbf{P}(t)$ denote the matrix where the entry in the ij^{th} position is $p_{ij}(t)$ and $\mathbf{g}(t)$ is the n-vector with entries $g_k(t)$ for $1 \le k \le n$. For the homogeneous problem (i.e. $\mathbf{g} \equiv 0$) we can see by linearity that if $\mathbf{x}_1, \mathbf{x}_2$ are both solutions to that system, then any linear combination $\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also a solution. In fact, as with second order ODEs, we will show that any solution to the system is of that form if $\{\mathbf{xn}_i\}$, for $1 \le i \le n$ are linearly independent solutions to the system (that is, any solution can be expressed as a linear combination of the solutions $\mathbf{x}_1, \ldots, \boldsymbol{x}_n$ when they are linearly independent). Analogously to second order, a collection of solutions $\mathbf{x}_1, ..., \mathbf{x}_n$ is called *linearly independent* if there exists t_* s.t.

$$\det[\mathbf{X}(t)] := \det \begin{bmatrix} x_{11}(t_*) & x_{12}(t_*) & \cdots & x_{1n}(t_*) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t_*) & x_{n2}(t_*) & \cdots & x_{nn}(t_*) \end{bmatrix} \neq 0,$$

where $\mathbf{x}_i = i^{\text{th}}\ row = (x_{1i}, ..., x_{ni})$.

General solution

Consider $\mathbf{x}_1, \ldots, \mathbf{x}_n$ linearly independent solutions of the system $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$ where $\mathbf{P}(t)$ is an $n \times n$ matrix and $\mathbf{v}(t)$ is any solution for the nonhomogeneous problem $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t)$. Then for any other solution $\mathbf{y}$ of the nonhomogeneous problem, there exist unique constants $\{c_i\}$ s.t.

$$\mathbf{y} = c_1\mathbf{x}_1 + ... + c_n\mathbf{x}_n + \mathbf{v}(t).$$

Proof. We will follow the same ideas as in the analogous second order result. We begin by proving an auxiliary result about the homogeneous equation. Let $\boldsymbol{\varphi}$ be a solution for the above homogeneous problem and $\mathbf{K} := \boldsymbol{\varphi}(t_*)$. Consider the system of equations given by

$$\mathbf{X}(t_*)\mathbf{c} := \begin{bmatrix} x_{1,1}(t_*) & x_{1,2}(t_*) & \cdots & x_{1,n}(t_*) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}(t_*) & x_{n,2}(t_*) & \cdots & x_{n,n}(t_*) \end{bmatrix} c = \mathbf{K}$$

for unknown vector $\mathbf{c}$. Then, by linear independence of the solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$, the matrix, $\mathbf{X}(t_*)$ is invertible and so we can solve for $\mathbf{c}$ by inverting $\mathbf{X}(t_*)$ and multiplying on the right by $\mathbf{K}$ (i.e $\mathbf{c} = \mathbf{X}^{-1}(t_*)\mathbf{c}$). Let $\boldsymbol{\zeta}(t) := \mathbf{X}(t)\mathbf{c}$ where $\mathbf{c}$ is the vector obtained from the above discussion. Then we have that $\boldsymbol{\zeta}(t_*) = \mathbf{K} = \boldsymbol{\varphi}(t_*)$. Therefore, by existence and uniqueness $\boldsymbol{\zeta}(t) = \boldsymbol{\varphi}(t)$ for all t.

Next let $\mathbf{y}$ be a solution for the nonhomogeneous problem. Then $\mathbf{y} - \mathbf{v}$ is a solution of the homogeneous problem and thus, by the above discussion, $\exists\, \mathbf{a} \in \mathbb{R}^n$ s.t.

$$\mathbf{y} = \mathbf{a} \cdot \mathbf{x} + \mathbf{v}.$$

$\square$

2 Homogeneous linear systems with constant coefficients

Consider the homogeneous system of n-ODEs

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

where $\mathbf{A}$ is $n \times n$ matrix with constant real entries. As with second order we make the ansatz $\boldsymbol{x}(t) = \boldsymbol{\xi} e^{\lambda t}$ where $\boldsymbol{\xi}$ is a fixed n-vector (to be chosen precisely later). Then, we observe that if $\boldsymbol{\xi}$ is chosen so that $\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$ (i.e $\boldsymbol{\xi}$ is an eigenvector of $\boldsymbol{A}$) we get

$$\mathbf{A}\mathbf{x}(t) = \mathbf{A}\boldsymbol{\xi}e^{\lambda t} = \lambda\boldsymbol{\xi}e^{\lambda t} = \boldsymbol{\xi}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\right) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t).$$

Such $\boldsymbol{\xi}, \lambda$ are called $\mathbf{A}'$s eigenvector and eigenvalue respectively (as noted earlier). We will now obtain the general solution. First we will assume that all eigenvalues $\{\lambda_i\}_{i=1}^n$ of $\mathbf{A}$ are real and distinct from each other; in the other sections we study the other cases. Let $\{\boldsymbol{\xi}_i\}_{i=1}^n$ be the corresponding eigenvectors. Then the solutions $\{\boldsymbol{\xi}_i e^{\lambda_i t}\}_{i=1}^n$ are linearly independent:

$$\det \begin{bmatrix} \xi_{11}e^{\lambda_1 t} & \cdots & \xi_{1n}e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ \xi_{n1}e^{\lambda_1 t} & \cdots & \xi_{nn}e^{\lambda_n t} \end{bmatrix} = e^{(\lambda_1 + \ldots + \lambda_n)t} \det \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & \ddots & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} \neq 0$$

where the last step follows because when all the eigenvalues of a matrix are distinct, then its eigenvectors will be linearly independent. Thus, from the above result we obtained the general solution.

Example —presenting the method

Consider two connected tanks A and B containing 1000L of well-mixed salt-water with $x(t), y(t)$ kilogram amounts of salt respectively. Let IP,OP denote the L/min-rate of salt-free water entering and exiting the two tanks and $P1, P2$ the L/min-rate of saltwater getting exchanged between the two tanks.

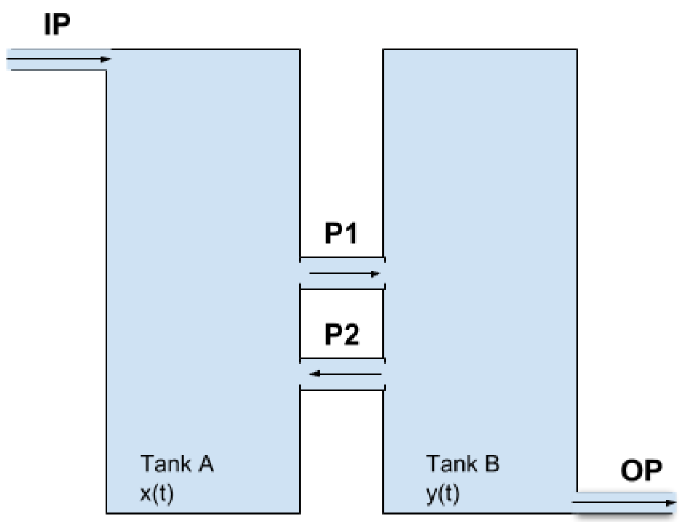


Figure 2.1: Tanks A and B containing salt

To keep the volume of water constant in the two tanks we set IP = OP = $1(L/\text{min})$. Let the rates $P1 = 1(L/\text{min})$ and $P2 = 2(L/\text{min})$ be constant in time. The concentration of salt in each tank is $\frac{x(t)}{1000}$kg/L, $\frac{y(t)}{1000}$kg/L respectively. Therefore, for tank A the rate of change of the amount of salt:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{Input rate} - \text{Output rate} = 2\cdot\frac{y(t)}{1000} - 1\cdot\frac{x(t)}{1000}$$

and for tank B we must also subtract the draining of salt from pipe OP

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{Input rate} - \text{Output rate} = 1\cdot\frac{x(t)}{1000} - 2\cdot\frac{y(t)}{1000} - 1\cdot\frac{y(t)}{1000}.$$

In matrix form our system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{1000}\begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

1. First we compute the eigenvalues

$$\det\begin{bmatrix} -1-\lambda & 2 \\ 1 & -3-\lambda \end{bmatrix} = 0 \;\Rightarrow (-1-\lambda)(-3-\lambda) - 2\cdot 1 = 0$$
$$\Rightarrow \lambda_1 = -2+\sqrt{3},\ \lambda_2 = -2-\sqrt{3}.$$

2. Second we find the corresponding eigenvectors. To find $\boldsymbol{\xi}_1 := \binom{\xi_{1,1}}{\xi_{2,1}}$ we solve the system (up to multiples):

$$\begin{bmatrix} -1-\lambda_1 & 2 \\ 1 & -3-\lambda_1 \end{bmatrix}\begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By solving the system directly we obtain the solution (up to multiples). For example, we rewrite the above to get:

$$\begin{cases} (1-\sqrt{3})\xi_{1,1} + 2\xi_{2,1} = 0 \\ \xi_{1,1} + (-1-\sqrt{3})\xi_{2,1} = 0 \end{cases} \implies \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}.$$

Similarly to obtain $\boldsymbol{\xi}_2$ we have to solve

$$\begin{bmatrix} -1-(-2-\sqrt{3}) & 2 \\ 1 & -3-(-2-\sqrt{3}) \end{bmatrix}\begin{pmatrix} \xi_{1,2} \\ \xi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we get

$$\boldsymbol{\xi}_2 = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}.$$

3. Therefore, by the discussion above the general solution will be

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{1000}c_1\cdot\boldsymbol{\xi}_1 e^{\lambda_1 t} + \frac{1}{1000}c_2\cdot\boldsymbol{\xi}_2 e^{\lambda_2 t} = c_1\cdot\begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}\frac{e^{(-2+\sqrt{3})t}}{1000} + c_2\cdot\begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}\frac{e^{(-2-\sqrt{3})t}}{1000}.$$

4. Since $2 > \sqrt{3}$, both the eigenvalues are negative and in turn the salt concentrations $x(t), y(t)$ will go to zero as $t \to +\infty$. This is reasonable because through pipe IP we are injecting salt-free water that over time transports the tanks' salt out through pipe OP.

5. Next we study the stability. Since $-2+\sqrt{3} > -2-\sqrt{3}$, we get $e^{(-2+\sqrt{3})t} > e^{(-2-\sqrt{3})t}$ and so as $t \to +\infty$ the first eigenvector $\binom{1+\sqrt{3}}{1}$ will dominate.

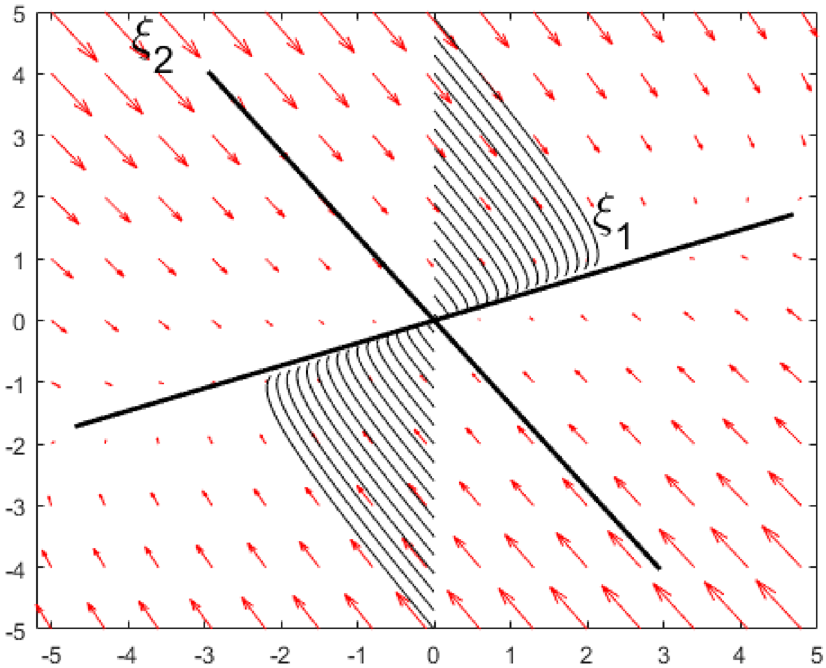


Figure 2.2: Solutions converge to line defined by vector $\xi_1 = \binom{1+\sqrt{3}}{1}$ and then to $(0, 0)$.

In other words, for large t we will have

$$x(t) \approx (1+\sqrt{3}) \cdot e^{(-2+\sqrt{3})t} > 1 \cdot e^{(-2+\sqrt{3})t} \approx y(t).$$

This is reasonable because $P2 > P1$ and so as $t \to +\infty$ the salt concentration x(t) in tank A will be greater than that of tank B.

Method formal steps

1. Starting with a matrix $\mathbf{A}$ we first compute its eigenvalues $\{\lambda_i\}$ eg. for matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have two eigenvalues:

$$det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\Rightarrow \lambda = \frac{(a+d)}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)} = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr(A)^2 - 4det(A)}.$$

2. Second for each eigenvalue, we find the corresponding eigenvector. Continuing the example from the previous step, we find the eigenvector $\boldsymbol{\xi}_1 := \binom{\xi_{1,1}}{\xi_{2,1}}$ by solving the system:

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here are the general formulas for eigenvectors for 2D systems:

- If $c \neq 0$ then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}.$$

- If $b \neq 0$ then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}.$$

- If $b = c = 0$ then

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3. Then the general solution will be of the form

$$x(t) = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + ... + c_n \boldsymbol{\xi}_n e^{\lambda_n t}.$$

4. Finally we study the stability for 2D systems:

   - If $\lambda_1 \neq \lambda_2$ and both positive then (0,0) will be a **nodal source** and solutions will be moving away from it (**unstable**).
   - If $\lambda_1 \neq \lambda_2$ and both negative then (0,0) will be a **nodal sink** and solutions will be moving towards it (**asymptotically stable**).
   - If $\lambda_1 \neq \lambda_2$ and with opposite signs then (0,0) will be a **saddle point** and solutions will be moving away from it along one eigenvector and towards it along the other eigenvector (**unstable**).
   - If one of the eigenvalues is zero eg. $\lambda_1 = 0$ and $\lambda_2 < 0$ then the line defined by ξ_1 will be a nodal source (**asymptotically stable**).
   - If one of the eigenvalues is zero eg. $\lambda_1 = 0$ and $\lambda_2 > 0$ then the line defined by ξ_1 will be a nodal sink (**asymptotically unstable**).

Examples

- We will exhibit each of the above stability cases by studying the IVP problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mathbf{x}$$

with $\mathbf{x}(0) = \binom{2}{2}$.

1. First we find the eigenvalues. For diagonal matrices this is immediate: $\lambda_1 = a, \lambda_2 = b$.
2. Next we find the corresponding eigenvectors:

$$\begin{bmatrix} a - \lambda_1 & 0 \\ 0 & b - \lambda_1 \end{bmatrix} \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \xi_{2,1} = 0$$

   therefore $\boldsymbol{\xi}_1 = \binom{1}{0}$. Similarly, we obtain $\boldsymbol{\xi}_2 = \binom{0}{1}$.
3. Therefore, the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{bt}.$$

4. Finally, using the initial condition we obtain

$$\mathbf{x}(t) = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{bt}.$$

5. Next we study the stability. The origin is a special point for dynamics because if $\mathbf{x}(t_*) = 0$ then $\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}(t_*) = A \cdot \binom{0}{0} = \binom{0}{0}$ and so it is a critical point.

(a) If $a \neq b$ and both positive we obtain that the solutions diverge to infinity

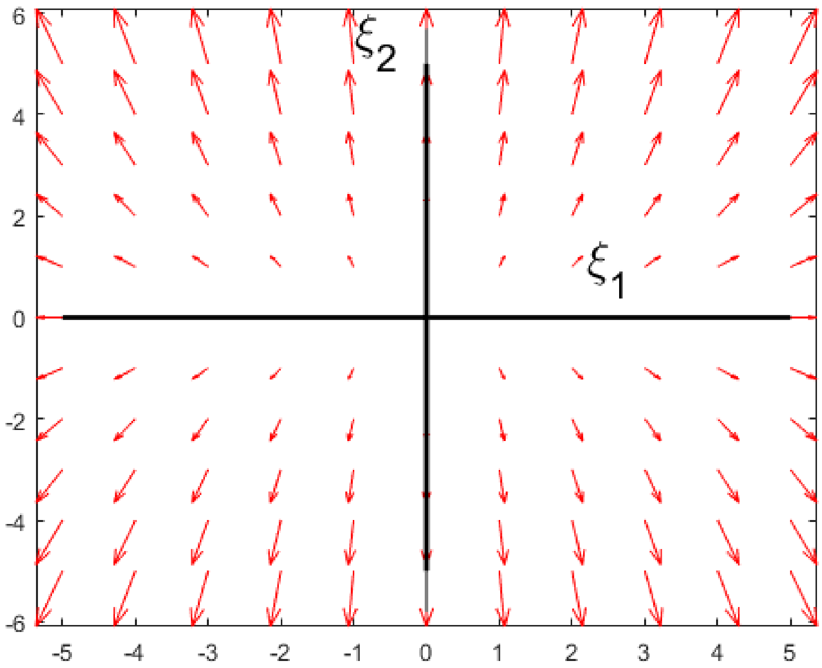


Figure 2.3: $a = 1$, $b = 3$

(b) If $a \neq b$ and both negative we obtain that they both converge to the source (0,0)

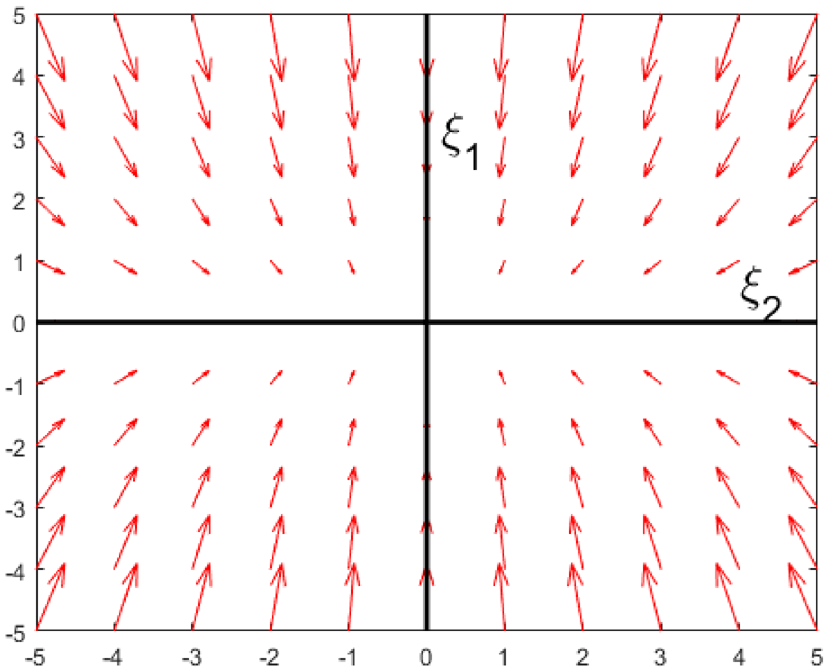
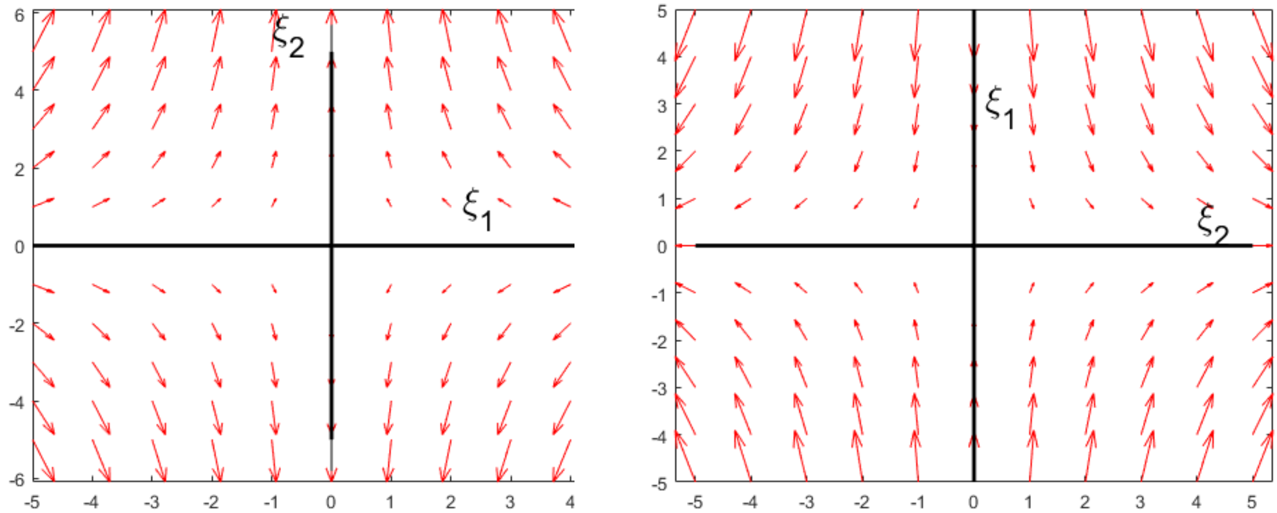


Figure 2.4: $a = -1$, $b = -3$

(c) If $a \neq b$ with different signs we obtain that (0,0) is a saddle point: if $a = -1, b = 3$ then the solutions converge to the origin if they start on the linear span of ξ_1 (x-axis) otherwise they diverge to infinity.



(a) $a = -1,\ b = 3$

(b) $a = 1,\ b = -3$

(d) If $a = 0$ and $b < 0$ then the linear span of ξ_1 (x-axis) will attract all the solutions:

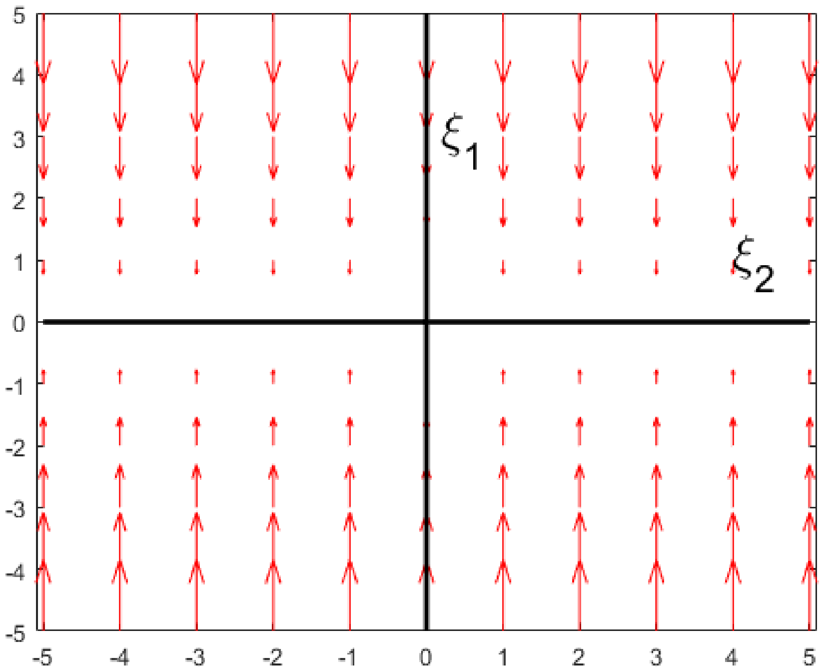


Figure 2.6: $a = 0,\ b = -3$

(e) If $a = 0$ and $b > 0$ then the linear span of ξ_1 (x-axis) will repel all the solutions:

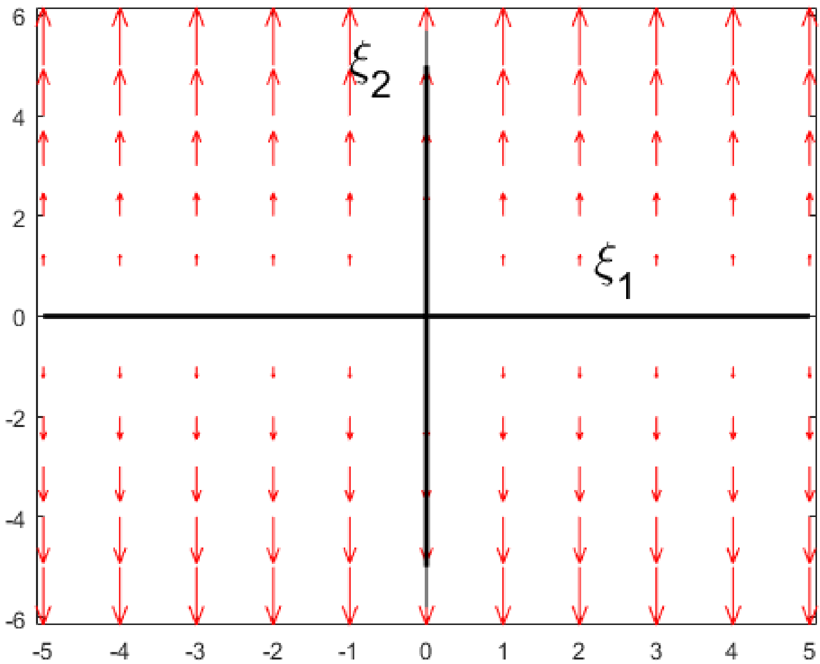


Figure 2.7: $a = 0$, $b = 3$

Applied examples

- **Richardson Arms race model**: Consider countries A, B with $x(t), y(t)$ amount of weaponry respectively. The model for the rate of change of weaponry is:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -a \cdot x + b \cdot y + e_1$$
$$\frac{\mathrm{d}y}{\mathrm{dt}} = c \cdot x - d \cdot y + e_2$$

The constants a, b, c, d are nonnegative. The constants b, c represent the fear magnitude eg. when $y(t)$ goes up then country A will increase its rate of weapon production by $b \cdot y(t)$. The constants a, d represent the fatigue factor because some countries decide on a lower rate of production given the amount of weapons they currently possess. For simplicity the constant e_1 represents the distrust country A has for country B and the reverse for e_2. But they can represent other factors not accounted for such as revenge, degradation of weapons, etc. So if we have no interaction i.e. $a = b = 0$ but positive amount of distrust $e_1 > 0$ then we still have a steady rate of weapon production $x'(t) = e_1 > 0$.
For simplicity $e_1 = -1, e_2 = -1$ consider the following matrix system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

1. A constant solution for this nonhomogeneous problem is $v(t) = \binom{1}{1}$, which we obtained by setting $\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \binom{0}{0}$ and solving for $\mathbf{x}$. Therefore, as explained above the general solution will be:

$$\mathbf{x} = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\xi}_2 e^{\lambda_2 t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

2. The eigenvalues for this matrix are the solutions to

$$0 = \lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 4\lambda - 5$$
$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -5.$$

3. The corresponding eigenvectors are $\boldsymbol{\xi}_1 = \binom{1}{1}$ and $\boldsymbol{\xi}_2 = \binom{-1}{2}$. So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-5t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4. Therefore, $\binom{1}{1}$ becomes a saddle point. That is, if a solution starts from $\binom{1}{1}$ in a direction parallel to $\binom{-1}{2}$ (i.e. choose $c_1 = 0$), then the solution will converge to the constant $\binom{1}{1}$ at an exponential rate (like e^{-5t}). For example, this happens if $\mathbf{x}(0) = \binom{\frac{1}{2}}{2}$. However for $c_1 \neq 0$ the solution will diverge to infinity like e^t in the direction $\binom{1}{1}$ away from the starting point $\binom{1}{1}$.

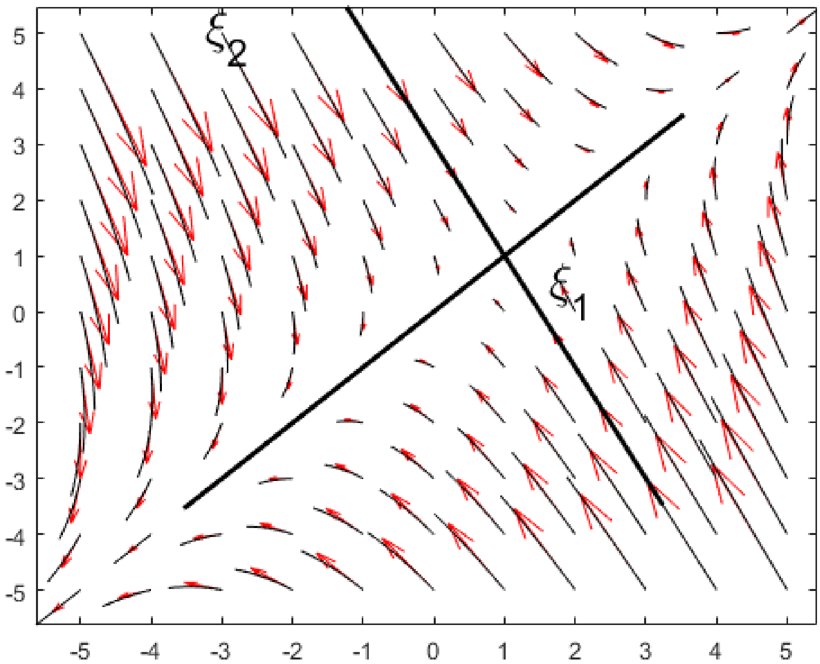


Figure 2.8: The $\binom{1}{1}$ is a saddle point

5. This is reasonable because if the amount of distrust is negative $e_1 = -1, e_2 = -1 < 0$ (i.e. positive trust), then for appropriate initial conditions the solutions will converge to peaceful coexistence $(x(t), y(t)) \to (1, 1)$.
6. To make sense of the special role $c_1 = 0$ plays we have to study the critical level sets. We have $x'(t) \geq 0$ and $y'(t) \geq 0$ when $-x + 2y - 1 \geq 0$ and $4x - 3y - 1 \geq 0$ respectively; call these lines as L_1, L_2. The first inequality happens when we are above L_1 and the second when we are below L_2. This is the region enclosed between the lines on the right. So in the direction of ξ_1 both countries are increasing their rate without stop . On the other hand, in the direction of ξ_2, which is above or below both lines, the rates will have opposite signs (In this case, one country is increasing their amount of weaponry while the other is decreasing it. The effect of one country decreasing their stock of weaponry interferes with the other countries desire to have more weapons. This is because the country that was originally increasing its stock of weapons will see the other country deplete its stock and so will have less incentive to create more.).

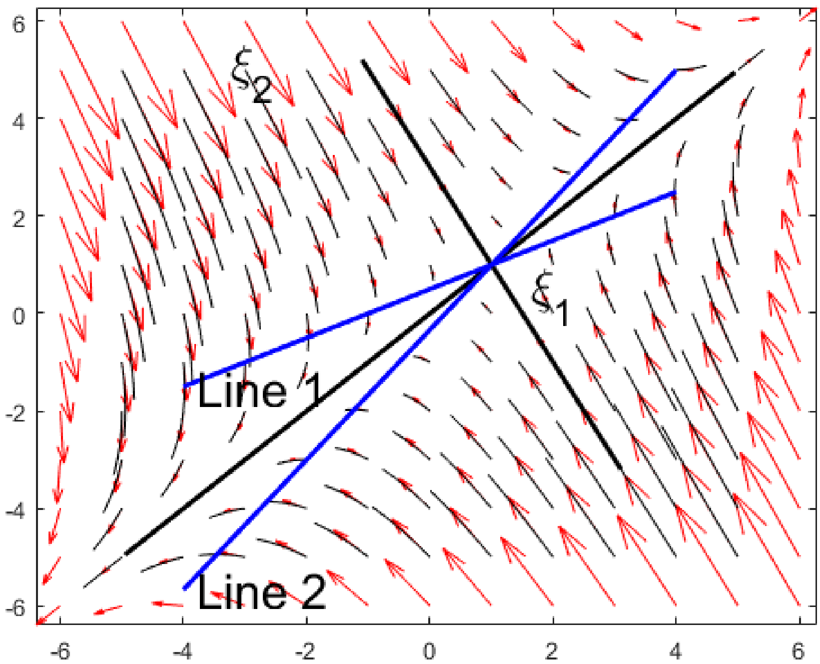


Figure 2.9: The lines L_1, L_2 separate into regions of stability and instability.

As a reference the general solution for

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \begin{bmatrix} -a & b \\ c & -d \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.1}$$

is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \frac{\lambda_1 + a}{b} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ \frac{\lambda_2 + a}{b} \end{pmatrix} e^{\lambda_2 t} + \begin{pmatrix} \frac{e_1 d + e_2 b}{ad - bc} \\ \frac{e_1 a + e_2 c}{ad - bc} \end{pmatrix}. \tag{2.2}$$

- Consider the parallel circuit displayed in Figure 2.10 *capacitance* C (eg. battery), *resistance* R (eg. light bulb) and *inductance* L (eg. coil used for storing energy). Note that there is no voltage source.

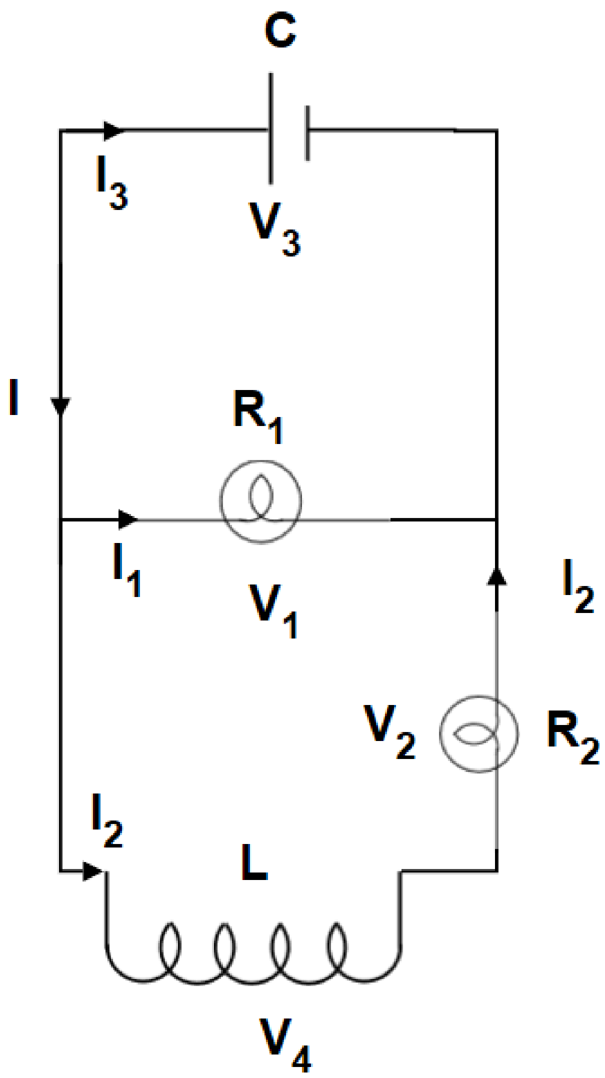


Figure 2.10: Parallel LRC circuit.

Let V be the *voltage drop* and I the *current* passing through the circuit. Here is a quick summary of the laws governing such systems:

- **Ohm's law**(OL): For the resistance we have $V = R \cdot I$.
- For the capacitance we have $I_3 = C \cdot \frac{\mathrm{d}V_3}{\mathrm{dt}}$.
- **Faraday's law** and **Lenz's law**(FLL)**:** For the inductance we have $V_4 = L \cdot \frac{\mathrm{d}I_2}{\mathrm{dt}}$.
- **Kirchhoff's current law**(KCL): $-I_3 = I = I_1 - I_2$; this is the conservation of energy law for circuits.
- **Kirchhoff's Voltage law**(KVL): The sum of voltages in a loop is zero. Thus, in the upper loop $V_3 = V_1$ and in the lower loop $V_2 + V_4 + V_1 = 0$.

We can express all these relations in a system of odes that will describe the above circuit system. We have

$$CV_1' \overset{(KVL)}{=} CV_3' = I_3 \overset{(KCL)}{=} -I \overset{(KCL)}{=} -I_1 + I_2 \overset{(OL)}{=} -\frac{V_1}{R_1} + I_2$$

where we first applied Kirchhoff's voltage law and then current law. We also have

$$LI_2' \overset{(FLL)}{=} V_4 \overset{(KVL)}{=} -V_1 - V_2 \overset{(OL)}{=} -V_1 - R_2 I_2.$$

Therefore, we have a system for I_2, V_1:

$$\begin{aligned} CV_1' &= I_2 - \frac{V_1}{R_1} \\ LI_2' &= -R_2 I_2 - V_1. \end{aligned}$$

We rewrite this in matrix form:

$$\begin{pmatrix} V_1' \\ I_2' \end{pmatrix} = \begin{bmatrix} -\frac{1}{CR_1} & \frac{1}{C} \\ \frac{-1}{L} & \frac{-R_2}{L} \end{bmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix}.$$

Suppose, for example, $R_1 = \frac{3}{5}, R_2 = 1, L = 1, C = \frac{1}{3}$ then

$$\begin{pmatrix} V_1' \\ I_2' \end{pmatrix} = \begin{bmatrix} -5 & 3 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix}.$$

1. First we find the eigenvalues. $\lambda_1 = -4, \lambda_2 = -2$.
2. Second we find the eigenvectors. By solving the system or using the formulas we obtain
$$\boldsymbol{\xi}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
3. Therefore, the general solution is
$$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}.$$
4. Therefore the current and voltage will go to zero as $t \to +\infty$. This is reasonable because there is no voltage source and so eventually electricity will dissipate by passing through the light bulbs.

2.1 Complex eigenvalues

Consider the system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

where now matrix $\mathbf{A}$ will have at least one pair of complex eigenvalues.

2.1.1 Example - Presenting the method

Consider the following parallel circuit with capacitance C (eg. battery), resistance R (eg. light bulb) and inductance L (eg. coil used for storing energy).

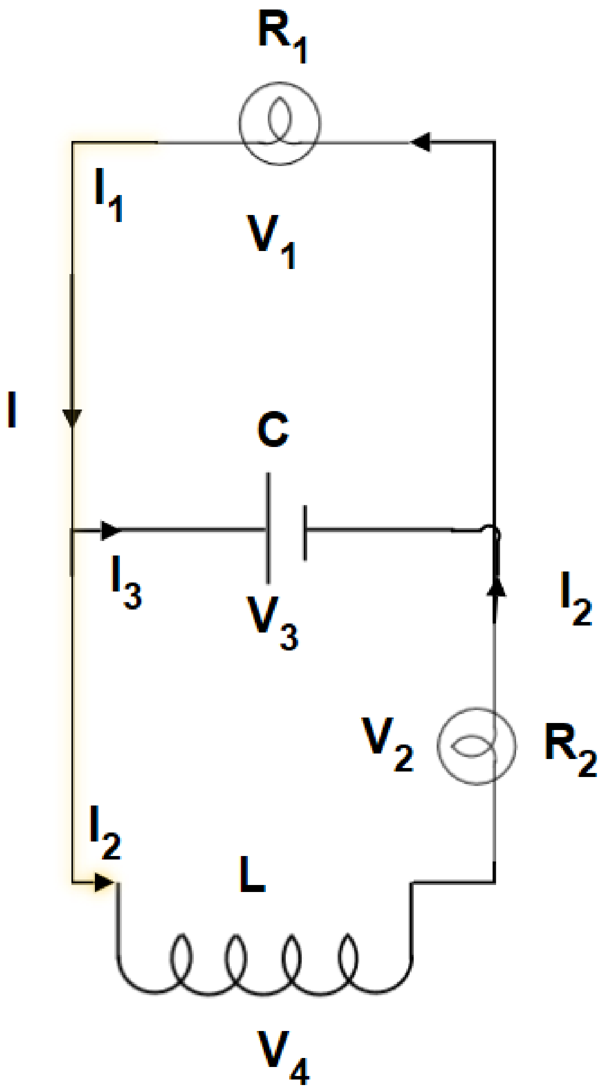


Figure 2.11: Parallel LRC circuit.

Let V be the voltage drop and I the current passing through the circuit. We also define the counterclockwise orientation as the positive one. Here is a quick summary of the laws governing such systems:

- **Ohm's law**(OL): $V = R \cdot I$.
- $C \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = I$.
- **Faraday's law** and **Lenz's law**(FLL): $L \cdot \frac{\mathrm{d}I}{\mathrm{d}t} = V$.
- **Kirchhoff's current law**(KCL): $I = I_2 + I_3$; this is the conservation of energy law for circuits.
- **Kirchhoff's Voltage law**(KVL): sum of voltages in a loop is zero. Thus, $V_1 = V_3$ and $V_3 + V_2 + V_4 = 0$.

We can express all these relations in a system of odes that will describe the circuit system depicted in Figure 2.11. We have

$$CV_1' \overset{(KVL)}{=} CV_3' = -I_3 = -I_1 + I_2 = -\frac{V_2}{R} + I_2 = -\frac{V_1}{R_1} + I_2$$

where we first applied Kirchhoff's voltage and current law and then Ohm's law. Note that we used $C(-V_3)' = I_3$ because V_3 is in the clockwise direction. We also have $LI_2' = V_4 = -V_2 - V_1 = -R_2 I_2 - V_1$. Therefore, we have a system for I_2, V_1:

$$CV' = I - \frac{V}{R_1}$$
$$LI' = -V + R_2 I.$$

We rewrite this in matrix form:

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -\frac{1}{CR_1} & -\frac{1}{C} \\ \frac{1}{L} & \frac{R_2}{L} \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

Let $R_1 = R_2 = 4, L = 8, C = \frac{1}{2}$ then

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -\frac{1}{2} & 2 \\ -1/8 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

1. First we find the eigenvalues. $\lambda_1 = \frac{-1+i}{2}, \lambda_2 = \frac{-1-i}{2}$.
2. Second we find the eigenvectors. By solving the system or using the formulas we obtain
$$\boldsymbol{\xi}_1 = \begin{pmatrix} -4i \\ 1 \end{pmatrix} \text{ and } \boldsymbol{\xi}_2 = \begin{pmatrix} 4i \\ 1 \end{pmatrix}.$$
3. Therefore, the general solution is
$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 \begin{pmatrix} -4i \\ 1 \end{pmatrix} e^{(\frac{-1+i}{2})t} + c_2 \begin{pmatrix} 4i \\ 1 \end{pmatrix} e^{(\frac{-1-i}{2})t}.$$
To obtain a real valued general solution it suffices to take real and imaginary parts of one of the basis elements:
$$\begin{pmatrix} \frac{-1+i}{2} \\ 1 \end{pmatrix} e^{(\frac{-1+i}{2})t} = e^{-t/2} \begin{pmatrix} -4ie^{it/2} \\ e^{it/2} \end{pmatrix})$$
$$= e^{-t/2} \begin{pmatrix} -4i(cos(t/2) + isin(t/2) \\ (cos(t/2) + isin(t/2) \end{pmatrix})$$
$$= e^{-t/2} (\begin{pmatrix} 4sin(t/2) \\ cos(t/2) \end{pmatrix} + i \begin{pmatrix} -4cos(t/2) \\ sin(t/2) \end{pmatrix}).$$
So we take
$$u(t) = e^{-t/2} (\begin{pmatrix} 4sin(t/2) \\ cos(t/2) \end{pmatrix} \text{ and } v(t) = e^{-t/2} \begin{pmatrix} -4cos(t/2) \\ sin(t/2) \end{pmatrix}.$$
Indeed by computing their wronskian we get
$$W(u, v) = \begin{bmatrix} e^{-t/2} 4sin(t/2) & -e^{-t/2} 4cos(t/2) \\ e^{-t/2} cos(t/2) & e^{-t/2} sin(t/2) \end{bmatrix} = 4e^{-t/2}(cos^2(t/2) + sin^2(t/2)) = 4e^{-t/2} \neq 0.$$
4. Since there is no voltage source, as expected the solutions will converge to the origin.

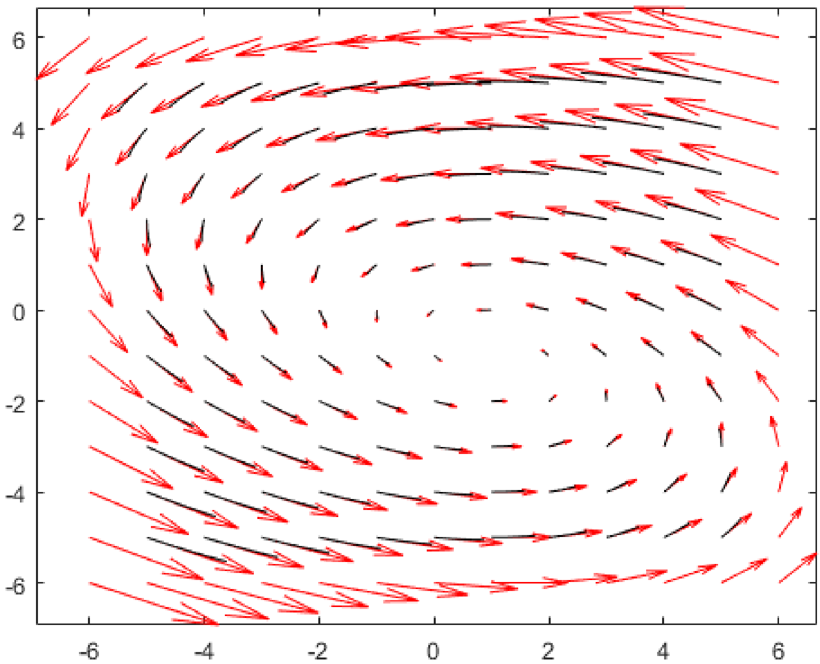


Figure 2.12: Phase portrait for (V, I).

5. To understand the periodicity we will describe the circuit's analogy with a mass spring.
   (a) When the spring is compressed we are storing energy in the form of atomic-bond energy or potential energy (i.e. the spring tries to regain its original position).
   (b) Then that energy is released into kinetic energy.
   (c) When the spring mass returns, it compress the springs and so the cycle begins again.

   In a circuit, a charged capacitor (battery) is analogous to a compressed spring and an inductor is analogous to the inertia mass.

   (a) The charged capacitor releases the electrical energy into the circuit which the inductor converts into magnetic field energy (analogous to kinetic energy).
   (b) When the capacitor is fully discharged, the magnetic field energy creates a counter current (by Faraday's law), which then charges the capacitor in the opposite direction.
   (c) The oppositely charged capacitor starts releasing a current in the opposite direction and so the cycle starts again.

Method formal steps

1. Let $\lambda_1 = a + ib, \lambda_2 = a - ib$ be the complex eigenvalues and $\xi_1 = \binom{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}, \xi_2 = \binom{\varrho_1 e^{i\varphi_1}}{\varrho_2 e^{i\varphi_2}}$ the corresponding eigenvectors. Then the solution is

$$\mathbf{x} = e^{-at}(c_1 \boldsymbol{\xi}_1 e^{ibt} + c_2 \boldsymbol{\xi}_2 e^{-ibt}).$$

2. To obtain a real-valued solution (not all) it suffices to pick one of the terms above, say $e^{-at}\boldsymbol{\xi}_1 e^{ibt}$. Then its real and imaginary parts will also be solutions:

$$e^{-at}\boldsymbol{\xi}_1 e^{ibt} = e^{-at}\begin{pmatrix} r_1 e^{i(\theta_1+b)} \\ r_2 e^{i(\theta_2+b)} \end{pmatrix} = e^{-at}\begin{pmatrix} r_1 cos(\theta_1 + b) \\ r_2 cos(\theta_2 + b) \end{pmatrix} + ie^{-at}\begin{pmatrix} r_1 sin(\theta_1 + b) \\ r_2 sin(\theta_2 + b) \end{pmatrix} =: u + iv.$$

3. Sometimes we can even obtain a general solution. By computing the wronskian we obtain:

$$W[u,v] = e^{-at} r_1 r_2 (cos(\theta_1 + b) sin(\theta_2 + b) - cos(\theta_2 + b) sin(\theta_1 + b)).$$

So as we see depending on the choice of θ_1, θ_2, the solutions u,v might be linearly independent or dependent.

4. Stability results. We note that the crucial role of stability is played by the factor e^{-at}.

   - If $a < 0$, then the solutions will converge to the node sink $(0,0)$ and the spiral will be inward.
   - If $a > 0$, then the solutions will diverge away from the node source $(0,0)$ and the spiral will be outward.
   - If $a = 0$, then the solutions will be concentric circles centered at $(0,0)$.

2.2 Examples

- Consider the system

$$\mathbf{x}' = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

1. We first compute its eigenvalues: $\lambda^2 - a\lambda + 1 = 0 \Rightarrow \lambda = \frac{a \pm \sqrt{a^2-4}}{2}$. So to explore the complex case we assume that $a^2 < 4$.
2. Let ξ_1, ξ_2 be the corresponding eigenvectors: $\xi_1 = \binom{\lambda_1}{-1}, \xi_2 = \binom{\lambda_2}{-1}$

   – Assume $\alpha = -1 < 0$, then $\lambda = \frac{-2 \pm i\sqrt{3}}{2}$ and so the general solution will be

$$\mathbf{x} = e^{-t}(c_1 \boldsymbol{\xi}_1 e^{i\frac{\sqrt{3}}{2}t} + c_2 \boldsymbol{\xi}_2 e^{-i\frac{\sqrt{3}}{2}t}).$$

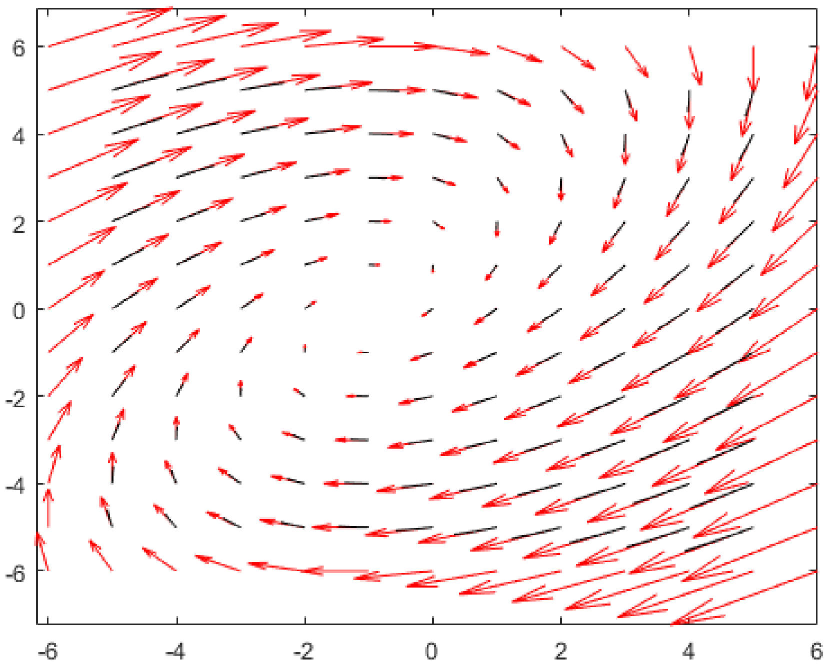


Figure 2.13: The solutions are converging towards the node sink (0,0).

– Assume $\alpha = 1 > 0$, then the general solution will be

$$\mathbf{x} = e^{t}(c_1 \boldsymbol{\xi}_1 e^{i\frac{\sqrt{3}}{2}t} + c_2 \boldsymbol{\xi}_2 e^{-i\frac{\sqrt{3}}{2}t}).$$

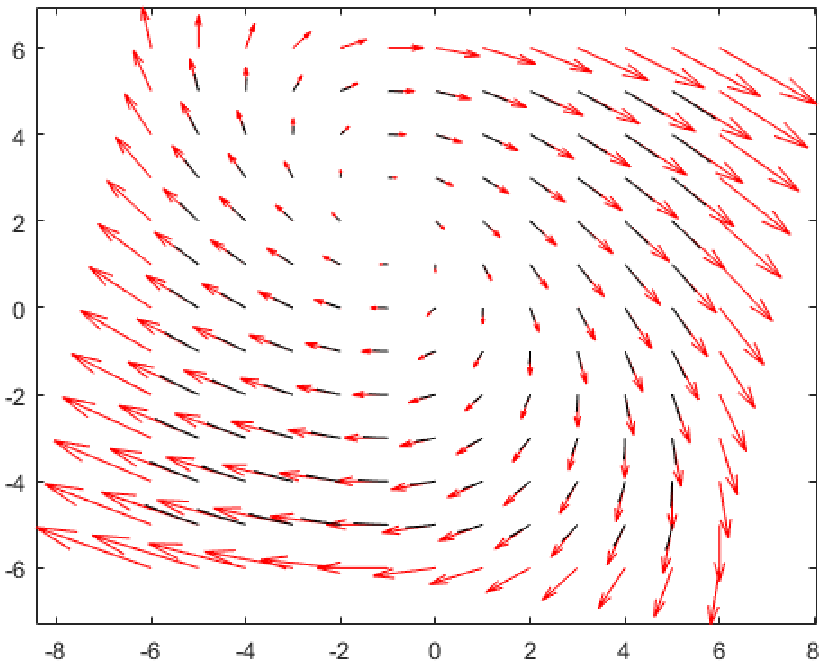


Figure 2.14: The solutions are spiraling outward from the node source $(0, 0)$.

– Assume $\alpha = 0$, then the general solution will be

$$\mathbf{x} = c_1 \boldsymbol{\xi}_1 e^{i\frac{\sqrt{3}}{2}t} + c_2 \boldsymbol{\xi}_2 e^{-i\frac{\sqrt{3}}{2}t}.$$

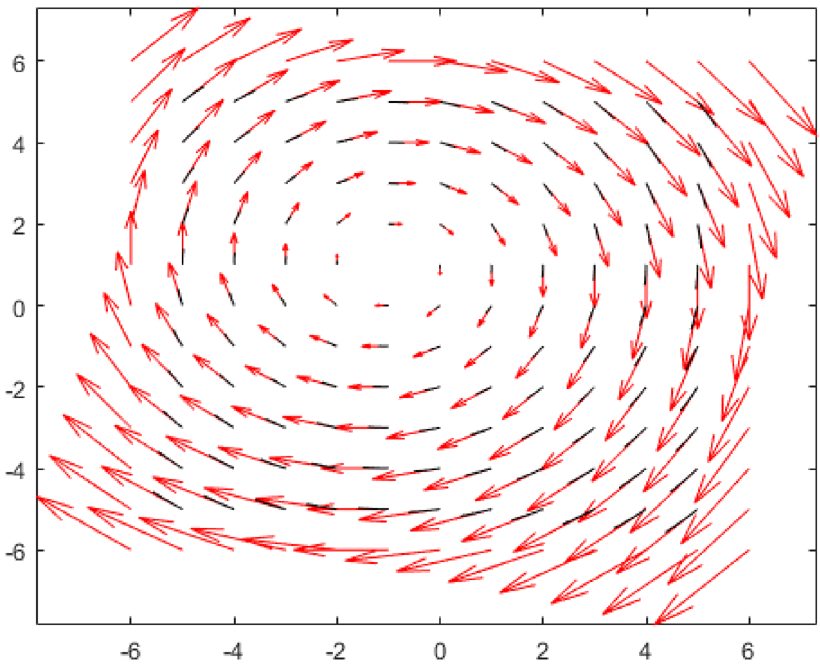


Figure 2.15: The solutions are concentric circles centered at $(0, 0)$.

2.3 Repeated eigenvalues

Consider the system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

where now matrix $\mathbf{A}$ will have at least two duplicated eigenvalues.

Example-presenting the method

Consider the following LRC circuit.

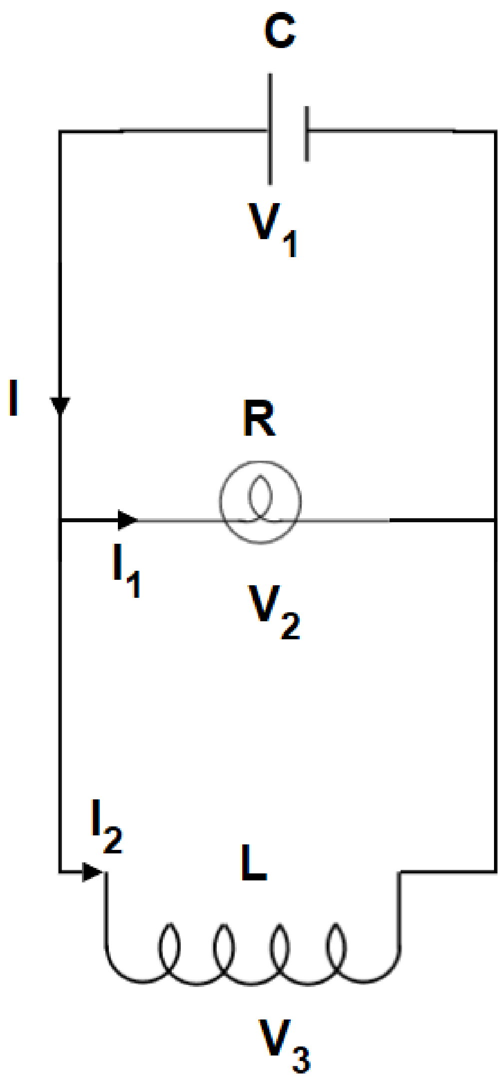


Figure 2.16: Parallel LRC circuit.

We have $CV_1' = -I = -I_1 - I_2 = -\frac{V_2}{R} - I_2$ and $LI_2' = V_3 = V_2 = V_1$. Therefore, the matrix system is:

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -\frac{1}{CR} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

1. The eigenvalues are $\lambda = -\frac{1}{2CR} \pm \frac{1}{2}\sqrt{(\frac{1}{CR})^2 - 4\frac{1}{CL}}$ and so if $(\frac{1}{CR})^2 - 4\frac{1}{CL} = 0 \Leftrightarrow L = 4R^2C$, we get a repeated eigenvalue $\lambda_1 = \lambda_2 = -\frac{1}{2CR}$. So assume $R = C = 1$ and $L = 4$, then the system is
$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{bmatrix} -1 & -1 \\ 1/4 & 0 \end{bmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

2. Then the eigenvalue is $\lambda = \frac{-1}{2}$ and the eigenvector is $\boldsymbol{\xi} = \binom{-2}{1}$. So we obtain the first term of the solution $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.

3. Similarly to second order odes with repeated roots, we make the ansatz
$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}),$$
where $\boldsymbol{\eta}$ is a yet undetermined vector.

4. Plugging in $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$ we obtain:
$$\lambda \boldsymbol{\xi} e^{\lambda t} t + (\boldsymbol{\xi} + \lambda \boldsymbol{\eta}) e^{\lambda t} = \mathbf{A}(\boldsymbol{\xi} e^{\lambda t} t + \boldsymbol{\eta} e^{\lambda t}).$$

5. Equating the coefficients of $e^{\lambda t} t$ and $e^{\lambda t}$ we get
$$\begin{aligned} \lambda \boldsymbol{\xi} &= \mathbf{A}\boldsymbol{\xi} \\ \boldsymbol{\xi} + \lambda \boldsymbol{\eta} &= \mathbf{A}\boldsymbol{\eta}. \end{aligned}$$

6. The first equation is always true by virtue of $\boldsymbol{\xi}$ being an eigenvector. We will use the second system to determine $\boldsymbol{\eta}$. In other words, we must solve the system
$$(\mathbf{A} - \lambda I)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

7. In our case we have
$$\begin{bmatrix} -1 & -1 \\ 1/4 & -0 \end{bmatrix} + \frac{1}{2} I) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{2} & -1 \\ 1/4 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

8. By solving the system we obtain
$$\eta_2 = 2 - \frac{\eta_1}{2}.$$
Therefore, for any $\eta_1 = k$ we obtain
$$\boldsymbol{\eta} = \begin{pmatrix} k \\ 2 - \frac{k}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + k \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}.$$

9. Returning above the ansatz solution will be:
$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} \cdot t + \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} + k \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \right\} e^{-t/2} \right] \\ &= a_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t/2} \cdot t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t/2} \right], \end{aligned}$$
where $a_1 := c_1 - k/2$.

10. Therefore, the (Voltage,Current) pair presents no periodicity and it simply goes to $(0, 0)$. However, for as $t \to +\infty$ the term $\binom{-2}{1} e^{-t/2} \cdot t$ will dominate and so the solutions will converge to the linear span of $\binom{-2}{1}$ and then along that line to the node sink $(0, 0)$.

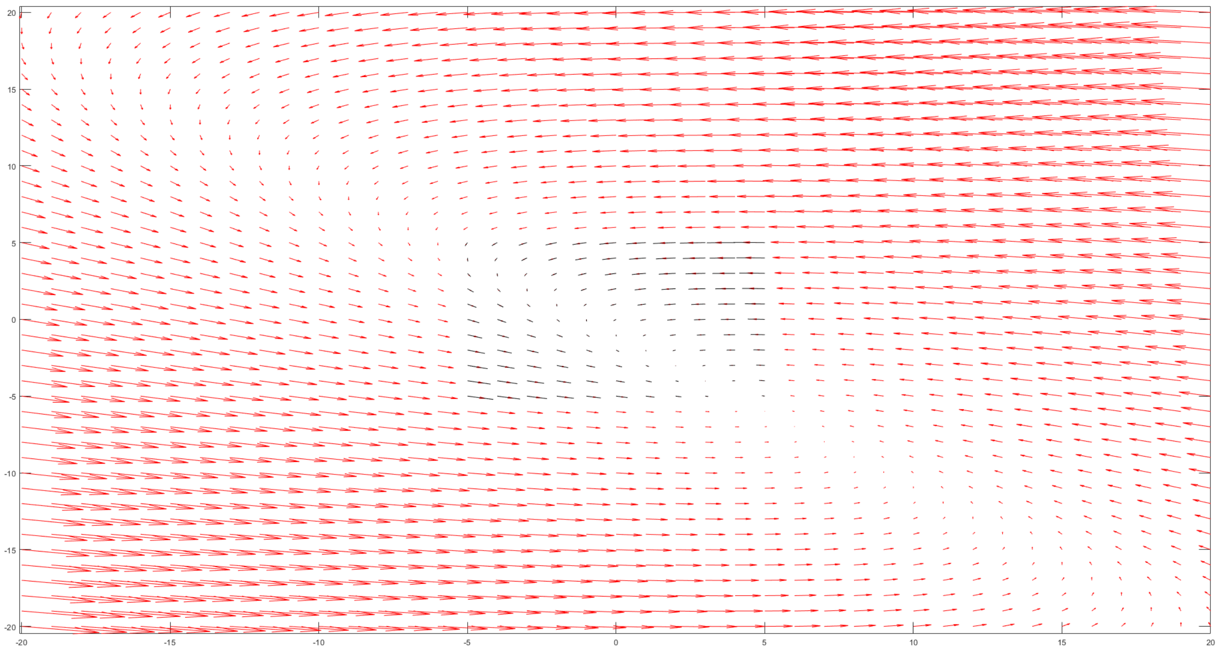


Figure 2.17: The linear span of $\binom{-2}{1}$ is a node and the origin will be the node sink.

Method formal steps

1. We first find the repeated eigenvalue λ and its eigenvector $\boldsymbol{\xi}$. So the first term of the solution will be $\mathbf{x}_1 := \boldsymbol{\xi} e^{\lambda t}$.
2. For the second term we make the ansatz
$$\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}.$$
3. Plugging this into our system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ we obtain the stystem:
$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi}.$$
4. By determining $\boldsymbol{\eta}$ we obtain:
$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\boldsymbol{\xi} e^{\lambda t} + c_2(\boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}).$$

Examples

- Consider system
$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

    1. We first find the eigenvalues:
    $$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})} = 2$$
    and so we have a repeated eigenvalue.

2. Second, we find the corresponding eigenvector:

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

3. Assuming the solution is of the form $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ and plugging into our ODE we obtain:

$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi} \Longrightarrow \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   Solving this system gives us:

$$\eta_1 + \eta_2 = -1 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} k \\ -k-1 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

   where k is any real number. We can rewrite $\boldsymbol{\eta}$ as:

$$\boldsymbol{\eta} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

4. Therefore, the general solution is:

$$\begin{aligned} \mathbf{x} &= c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \\ &= c_1 e^{2t} \boldsymbol{\xi} + c_2 (\boldsymbol{\xi} e^{2t} \cdot t + \boldsymbol{\eta} e^{2t}) \\ &= c_1 e^{2t} \boldsymbol{\xi} + c_2 \left[\boldsymbol{\xi} e^{2t} \cdot t + \left\{ k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} e^{2t} \right] \\ &= e^{2t} \left[(c_1 + kc_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \right]. \end{aligned}$$

5. The vector $\boldsymbol{\xi}_1 = \binom{1}{-1}$ dominates the long term behaviour due to the extra term $\binom{1}{-1}t$ (provided we do not choose $c_2 = 0$). So we see that, essentially, all solutions are diverging away from the linear span of $\binom{1}{-1}$.

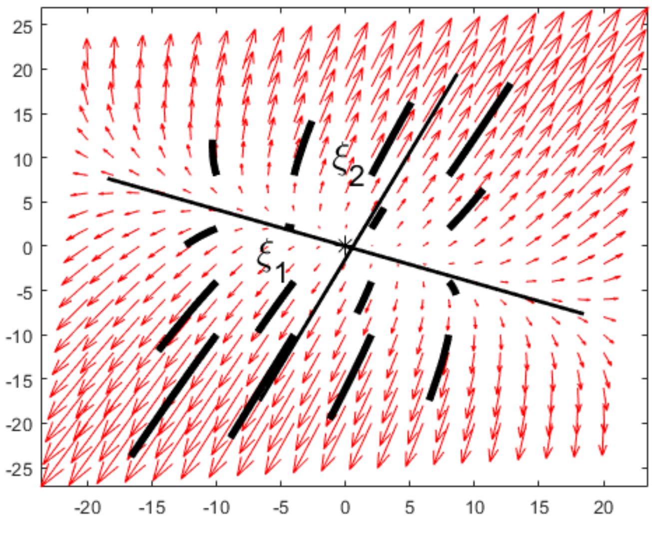


Figure 2.18: The phase portrait for $\mathbf{x} = (x_1, x_2)$ with $\boldsymbol{\xi}_1 = \binom{1}{-1}$, $\boldsymbol{\xi}_2 = \binom{0}{-1}$.

- Consider the system
$$\mathbf{x}' = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \mathbf{x}$$
with $\mathbf{x}(0) = \binom{1}{3}$.

1. First we find the eigenvalues:
$$\lambda = \frac{\mathrm{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})} = \frac{3}{2}$$
and so we have a repeated eigenvalue.
2. Second, we find a corresponding eigenvector:
$$\begin{bmatrix} 2-\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1-\frac{3}{2} \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
3. Assuming the solution is of the form $\mathbf{x}_2 := \boldsymbol{\xi} e^{\lambda t} \cdot t + \boldsymbol{\eta} e^{\lambda t}$ and plugging into our ODE we obtain:
$$(\mathbf{A} - \lambda \mathbf{I}_2)\boldsymbol{\eta} = \boldsymbol{\xi} \Longrightarrow \begin{bmatrix} 2-\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1-\frac{3}{2} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
Solving this system gives us:
$$\eta_1 + \eta_2 = 2 \Longrightarrow \boldsymbol{\eta} = \begin{pmatrix} k \\ -k+2 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$
where k is any real number. We can rewrite $\boldsymbol{\eta}$ as:
$$\boldsymbol{\eta} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$
Therefore, the general solution is:
$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \\ &= c_1 e^{\frac{3t}{2}}\boldsymbol{\xi} + c_2(\boldsymbol{\xi} e^{\frac{3t}{2}} \cdot t + \boldsymbol{\eta} e^{\frac{3t}{2}}) \\ &= c_1 e^{\frac{3t}{2}}\boldsymbol{\xi} + c_2\left[\boldsymbol{\xi} e^{\frac{3t}{2}} \cdot t + \left\{k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\} e^{\frac{3t}{2}}\right] \\ &= e^{\frac{3t}{2}}\left[(c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}\right]. \end{aligned}$$
If we now use the initial condition we obtain the system of equations
$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = (c_1 + kc_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
which can be solved to obtain that $c_1 = 1 - 2k$ and $c_2 = 2$.
4. The vector $\boldsymbol{\xi}_1 = \binom{1}{-1}$ dominates the long term behaviour due to the extra term $\binom{1}{-1}t$. So we see that all solutions to the IVP are diverging away from the linear span of $\binom{1}{-1}$.

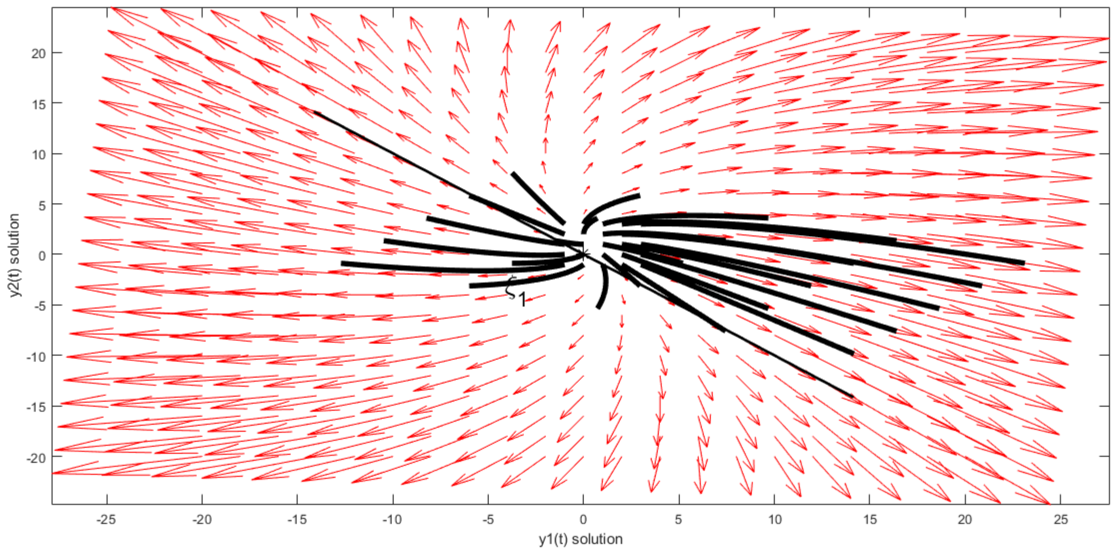
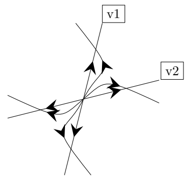
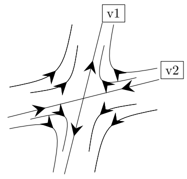
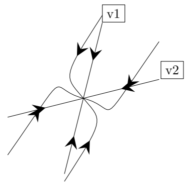
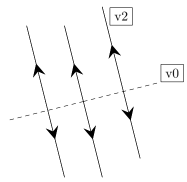
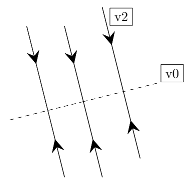
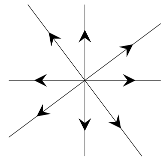
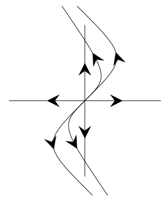
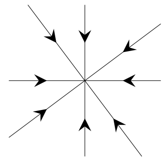
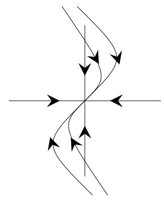
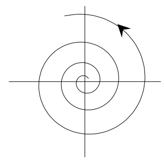
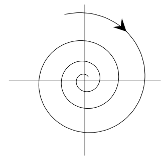
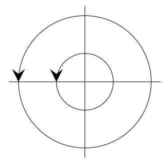


Figure 2.19: The phase portrait for $\mathbf{x} = (x_1, x_2)$ with $\boldsymbol{\xi}_1 = \binom{1}{-1}$, $\boldsymbol{\xi}_2 = \binom{0}{2}$.

2.4 Stability

Summary of the stability results.

Eigenvalues	**Type of critical point**	**Stability**	**Sample phase portrait**
$\lambda_1 > \lambda_2 > 0$	Nodal source	Unstable	
$\lambda_1 > 0 > \lambda_2$	Saddle point	Unstable	
$0 > \lambda_1 > \lambda_2$	Nodal sink	Asymptotically stable	
$\lambda_1 = 0, \lambda_2 > 0$	v0 line	Unstable and source	
$\lambda_1 = 0, \lambda_2 < 0$	v0 line	Unstable and sink	

$\lambda = \lambda_1 = \lambda_2 > 0$ and $A = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$	Proper node	Unstable	
$\lambda_1 = \lambda_2 > 0$ and A is not diagonal	Improper node	Asymptotically unstable	
$\lambda = \lambda_1 = \lambda_2 < 0$ and $A = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$	Proper node	Stable	
$\lambda_1 = \lambda_2 < 0$ and A is not diagonal	Improper node	Asymptotically stable	
$\lambda_1 = a + ib, \lambda_2 = a - ib$	Spiral		
a>0		Unstable	
0>a		Stable	
$\lambda_1 = ib, \lambda_2 = -ib$	Center	Unstable	

Classification of Phase Portraits in the $(\det A, \operatorname{Tr} A)$-plane

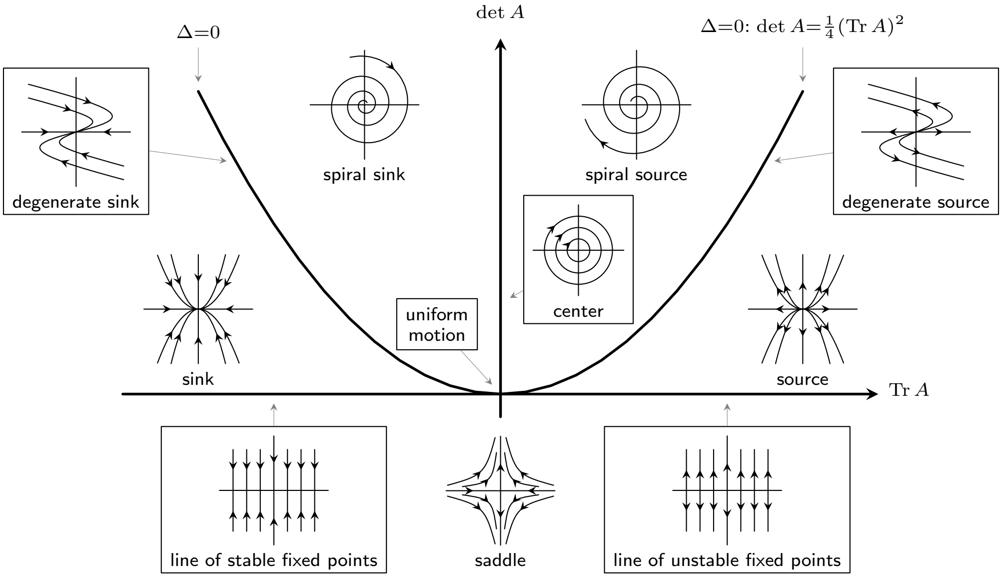


Figure 2.20: Classification of phase portraits.

2.4.1 Stability of Eigenvalue Dependence

In this section we study the limiting behaviour of solutions as distinct eigenvalues of $\mathbf{A}$ become repeated. Specifically, we demonstrate that the solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has distinct eigenvalues converges pointwise, after being suitably prepared, to the solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has repeated eigenvalues. We focus our analysis on the 2×2 case. Consider the problem

$$\mathbf{x}'(t) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mathbf{x}(t) \tag{2.3}$$

where $\lambda \in \mathbb{R}$ as well as the perturbed problem

$$\mathbf{x}'(t) = \begin{pmatrix} \lambda + \epsilon & 1 \\ 0 & \lambda - \epsilon \end{pmatrix} \mathbf{x}(t) \tag{2.4}$$

where $\epsilon > 0$. Notice that the perturbed problem has a matrix with distinct eigenvalues $\lambda + \epsilon$ and $\lambda - \epsilon$. One might hope that if we take a sequence of solutions to the perturbed problems as $\epsilon \to 0^+$ then, in the limit, we obtain a solution to the limiting problem (2.3). As we will see, this only works if we choose the sequence of solutions appropriately. To begin, we note that the solution $\mathbf{x}_\epsilon(t)$ to the perturbed problem (2.4) is

$$\mathbf{x}_\epsilon(t) = c_1(\epsilon) e^{(\lambda+\epsilon)t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(\epsilon) e^{(\lambda-\epsilon)t} \begin{pmatrix} 1 \\ -2\epsilon \end{pmatrix}$$

for each $\epsilon > 0$, where $c_1(\epsilon)$ and $c_2(\epsilon)$ are constants that may depend on ϵ (note that we are considering a sequence of solutions with no initial conditions and so we are free to choose these constants as we please). We now wish to show that as $\epsilon \to 0^+$ the above family of solutions tends to the solution of (2.3). Observe that we can write this solution as

$$\mathbf{x}_\epsilon(t) = \begin{pmatrix} c_1(\epsilon) e^{(\lambda+\epsilon)t} + c_2(\epsilon) e^{(\lambda-\epsilon)t} \\ -2\epsilon c_2(\epsilon) e^{(\lambda-\epsilon)t} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} c_1(\epsilon) e^{\epsilon t} + c_2(\epsilon) e^{-\epsilon t} \\ -2\epsilon c_2(\epsilon) e^{-\epsilon t} \end{pmatrix}.$$

To ensure that the second component converges to an interesting value, $k \in \mathbb{R}$, of our choosing we see that we must require that $c_2(\epsilon) = \frac{-k}{2\epsilon}$. Updating the family of solutions with this choice of $c_2(\epsilon)$ we obtain

$$\mathbf{x}_\epsilon(t) = e^{\lambda t}\begin{pmatrix} c_1(\epsilon)e^{\epsilon t} - \frac{k}{2\epsilon}e^{-\epsilon t} \\ ke^{-\epsilon t} \end{pmatrix}.$$

Observe that the first component now has a term that diverges as $\epsilon \to 0^+$. Thus, we must choose $c_1(\epsilon)$ in a way that combats this divergent term. In accordance with the above logic we choose $c_1(\epsilon) = \frac{k}{2\epsilon} + c_3(\epsilon)$ where we will decide later how to choose $c_3(\epsilon)$. Updating our family of solutions we obtain

$$e^{\lambda t}\begin{pmatrix} \left(\frac{k}{2\epsilon} + c_3(\epsilon)\right)e^{\epsilon t} - \frac{k}{2\epsilon}e^{-\epsilon t} \\ ke^{-\epsilon t} \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_3(\epsilon)e^{\epsilon t} + k\left(\frac{e^{\epsilon t} - e^{-\epsilon t}}{2\epsilon}\right) \\ ke^{-\epsilon t} \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_3(\epsilon)e^{\epsilon t} + kt\left(\frac{e^{\epsilon t} - e^{-\epsilon t}}{2\epsilon t}\right) \\ ke^{-\epsilon t} \end{pmatrix}.$$

We now choose $c_3(\epsilon) = a$ for some constant $a \in \mathbb{R}$ so that the first term in the first component converges. Observe also that, by L'Hôpital's rule we have

$$\lim_{\epsilon \to 0^+} \frac{e^{\epsilon t} - e^{-\epsilon t}}{2\epsilon t} = \lim_{\epsilon \to 0^+} \frac{2\epsilon}{2\epsilon} = 1.$$

Thus, we conclude that by letting ϵ tend to 0 from the right we obtain, for each $t \in \mathbb{R}$

$$\lim_{\epsilon \to 0^+} \mathbf{x}_\epsilon(t) = e^{\lambda t}\begin{pmatrix} a + kt \\ k \end{pmatrix} = ae^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ke^t\begin{pmatrix} t \\ 1 \end{pmatrix}$$

which is a solution to (2.3). Notice that a and k were arbitrary choices and so we can obtain any such solution. Notice, however, that we had to choose very specific behaviour for the coefficients to get this convergence.

3 Nonhomogeneous linear systems

In this section we study nonhomogeneous first order systems of equations building off of the previous work on homogeneous first order systems.

3.1 Diagonalization Method

Consider nonhomogeneous linear first order systems:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a vector of continuous functions and $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with eigenvalues $\{\lambda_i\}_{i=1,\ldots,n}$. The latter assumption means that if $\mathbf{T}$ has the eigenvectors of $\mathbf{A}$ as columns, then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$ is a diagonal matrix.

Using diagonalization Plugging in $\mathbf{x} = \mathbf{T}\mathbf{y}$ for some yet unknown $\mathbf{y}$ we obtain

$$\begin{aligned} \mathbf{T}\mathbf{y}' &= \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t) \\ \implies \mathbf{y}' &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t). \end{aligned}$$

As a result, we decoupled the system. From this decoupled system we obtain the first order equations:

$$y_i' = \lambda_i y_i(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_i \quad \text{for } i = 1, ..., n.$$

For $h_i(t) := (\mathbf{T}^{-1}\mathbf{g}(t))_i$ we have (by the method of integrating factors)

$$y_i(t) = e^{\lambda_i t}\left[\int_0^t e^{-\lambda_i s} h_i(s) ds + c_i\right].$$

Therefore, we found the solution $\mathbf{x} = \mathbf{T}\mathbf{y}$.

3.1.1 Method formal steps

1. As usual we first find the eigenvalues λ_1, λ_2 and corresponding eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

2. Form the change of basis matrix $\mathbf{T} := [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$ and find the solution to the following two first order odes

$$\begin{aligned} y_1' &= \lambda_1 y_1(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_1 \\ y_2' &= \lambda_2 y_2(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_2. \end{aligned}$$

3. By integrating factor the solutions are

$$\begin{aligned} y_1(t) &= e^{\lambda_1 t}\left[\int_0^t e^{-\lambda_1 s}(\mathbf{T}^{-1}\mathbf{g}(s))_1 ds + c_1\right] \\ y_2(t) &= e^{\lambda_1 t}\left[\int_0^t e^{-\lambda_2 s}(\mathbf{T}^{-1}\mathbf{g}(s))_2 ds + c_2\right]. \end{aligned}$$

4. We obtain the original solution by undoing the change of basis:

$$\mathbf{x} = \mathbf{T}\mathbf{y}.$$

Example-Presenting the method Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

1. First we find the eigenvalues

$$\lambda = \frac{\operatorname{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\big(\operatorname{Tr}(\mathbf{A})\big)^2 - 4\det(\mathbf{A})} \implies \lambda_1 = -1,\ \lambda_2 = 1.$$

2. The corresponding eigenvectors are, respectively, $v_1 = (1,3)^{\mathrm{T}}, v_2 = (1,1)^{\mathrm{T}}$ and so the change of basis matrix $\mathbf{T}$ that diagonalizes our matrix is:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}.$$

3. Therefore, as argued above, the solution will be

$$\mathbf{x} = \mathbf{T}\mathbf{y},$$

where

$$y_1(t) = e^{-t}\int_0^t e^s h_1(s)\mathrm{d}s + c_1e^{-t} \text{ and } y_2(t) = e^t\int_0^t e^{-s}h_2(s)\mathrm{d}s + c_2e^t$$

with

$$h_1(t) := \frac{1}{2}(t - e^t) \text{ and } h_2(t) := \frac{1}{2}(3e^t - t).$$

First we find the y_i (note that $\sinh(t)$ is $\frac{e^t - e^{-t}}{2}$):

$$y_1 = c_1e^{-t} + \frac{1}{2}e^{-t}\big[e^t\{t - \sinh(t) - 1\} + 1\big] \text{ and } y_2 = c_2e^t + \frac{1}{2}e^t\big[3t + e^{-t}(t+1) - 1\big].$$

4. Therefore, we obtain, since $\mathbf{x} = \mathbf{T}\mathbf{y}$

$$\begin{cases} x_1 = c_1e^{-t} + c_2e^t + \frac{1}{2}e^{-t}[1 + e^t\{-1 + t - \sinh(t)\}] + \frac{1}{2}e^t[-1 + 3t + e^{-t}(1+t)] \\ x_2 = 3c_1e^{-t} + c_2e^t + \frac{3}{2}e^{-t}[1 + e^t\{-1 + t - \sinh(t)\}] + \frac{1}{2}e^t[-1 + 3t + e^{-t}(1+t)] \end{cases}$$

$$\implies \mathbf{x} = \left[c_1e^{-t} + \frac{1}{2}e^{-t}\{1 + e^t(-1 + t - \sinh(t))\}\right]\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \left[c_2e^t + \frac{1}{2}e^t\{-1 + 3t + e^{-t}(1+t)\}\right]\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Examples

- Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ cos(t) \end{pmatrix}.$$

  1. First we find the eigenvalues

$$\lambda = \frac{\operatorname{Tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\big(\operatorname{Tr}(\mathbf{A})\big)^2 - 4\det(\mathbf{A})} \implies \lambda_1 = -2 + i\sqrt{5},\ \lambda_2 = -2 - i\sqrt{5}.$$

2. The corresponding eigenvectors are, respectively, $v_1 = (i\sqrt{5}, 1)^{\mathrm{T}}, v_2 = (-i\sqrt{5}, 1)^{\mathrm{T}}$ and so the change of basis matrix $\mathbf{T}$ that diagonalizes our matrix is:

$$\mathbf{T} = \begin{pmatrix} i\sqrt{5} & -i\sqrt{5} \\ 1 & 1 \end{pmatrix}.$$

3. Therefore, as argued above, the solution will be

$$\mathbf{x} = \mathbf{T}\mathbf{y},$$

where

$$y_1(t) = e^{(-2+i\sqrt{5})t} \int_0^t e^{-(-2+i\sqrt{5})s} h_1(s)\mathrm{d}s + c_1 e^{(-2+i\sqrt{5})t} \text{ and } y_2(t) = e^{(-2-i\sqrt{5})t} \int_0^t e^{-(-2-i\sqrt{5})s} h_2(s)\mathrm{d}s$$

with
$$h_1(t) := -2icos(t) \text{ and } h_2(t) := cos(t).$$

4. First we find the y_i. We will do y_1 and y_2 is similar.

$$\begin{aligned}
&\int_0^t e^{-(-2+i\sqrt{5})s} h_1(s)\mathrm{d}s + c_1 = -2i \int_0^t e^{-(-2+i\sqrt{5})s} cos(s)\mathrm{d}s + c_1 \\
&= \frac{-2i}{1+(2-i\sqrt{5})^2} \Big\{ (e^{-t(-2+i\sqrt{5})}(sin(t) + (2-i\sqrt{5})cos(t)) - (2-i\sqrt{5}) \Big\} + c_1,
\end{aligned}$$

where we used the formula

$$\int e^{(a+ib)t} cos(s)ds = \frac{1}{1+(a+ib)^2}(e^{t(a+ib)}(sin(t) + (a+ib)cos(t)) - (a+ib)).$$

we simplify by setting $c_1 := \frac{-2i(2-i\sqrt{5})}{1+(2-i\sqrt{5})^2}$ to get

$$\begin{aligned}
y_1 &= e^{t(-2+i\sqrt{5})} \frac{-2i}{1+(2-i\sqrt{5})^2} \Big\{ e^{-t(-2+i\sqrt{5})}(sin(t) + (2-i\sqrt{5})cos(t)) \Big\} \\
&= \frac{-2i}{1+(2-i\sqrt{5})^2} \Big\{ sin(t) + (2-i\sqrt{5})cos(t) \Big\} \\
&= \frac{-2i}{-4i\sqrt{5}} \Big\{ 2cos(t) + sin(t) - i\sqrt{5}cos(t) \Big\} \\
&= \frac{1}{2\sqrt{5}} \Big\{ 2cos(t) + sin(t) - i\sqrt{5}cos(t) \Big\} \\
&=: u(t) + iv(t)
\end{aligned}$$

For y_2 we have

$$\begin{aligned}
y_2(t) &= \frac{1}{1+(2+i\sqrt{5})^2} \Big\{ sin(t) + (2+i\sqrt{5})cos(t) \Big\} \\
&= \frac{1}{4i\sqrt{5}} \Big\{ 2cos(t) + sin(t) + i\sqrt{5}cos(t) \Big\}
\end{aligned}$$

using that $i^{-1} = -i$ we obtain

$$\begin{aligned}
&= \frac{1}{4\sqrt{5}}\sqrt{5}cos(t) - \frac{i}{4\sqrt{5}}(2cos(t) + sin(t)) \\
&= \frac{1}{4}cos(t) - \frac{i}{4\sqrt{5}}(2cos(t) + sin(t)) \\
&=: \widetilde{u}(t) + i\widetilde{v}(t).
\end{aligned}$$

5. Undoing the change of basis we obtain:

$$\begin{aligned}
\mathbf{x}_{nh} &= \mathbf{Ty} \\
&= \begin{pmatrix} i\sqrt{5} & -i\sqrt{5} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u+iv \\ \widetilde{u}+i\widetilde{v} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{5}(\widetilde{v}-v+i(u-\widetilde{u})) \\ \widetilde{v}-v+i(u-\widetilde{u}) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{5}(\widetilde{v}-v) \\ \widetilde{v}-v \end{pmatrix} + i \begin{pmatrix} \sqrt{5}(u-\widetilde{u}) \\ u-\widetilde{u} \end{pmatrix}
\end{aligned}$$

we have that

$$\begin{aligned}
\widetilde{v}-v &= \frac{-1}{4\sqrt{5}}(2cos(t)+sin(t)) + cos(t)\frac{1}{2} \\
&= cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}} \\
&\text{and} \\
\widetilde{u}-u &= \frac{1}{4}cos(t) - \frac{1}{2\sqrt{5}}(2cos(t)+sin(t)) \\
&= cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{2\sqrt{5}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&= \begin{pmatrix} \sqrt{5}(cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}}) \\ cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}} \end{pmatrix} \\
&\qquad + i\begin{pmatrix} \sqrt{5}(cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{2\sqrt{5}}) \\ cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{2\sqrt{5}} \end{pmatrix}
\end{aligned}$$

6. Since we are only looking for a particular solution we only take the real part:

$$\mathbf{x}_{nh} = \begin{pmatrix} \sqrt{5}(cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}}) \\ cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}} \end{pmatrix}$$

7. Therefore, the general solution is:

$$\begin{aligned}
\mathbf{x} = &\left[c_1 e^{(-2+i\sqrt{5})t} \begin{pmatrix} i\sqrt{5} \\ 1 \end{pmatrix} + c_2 e^{(-2-i\sqrt{5})t} \begin{pmatrix} -i\sqrt{5} \\ 1 \end{pmatrix} \right] \\
&\qquad + \begin{pmatrix} \sqrt{5}(cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}}) \\ cos(t)\frac{1}{2}(1-\frac{1}{\sqrt{5}}) + sin(t)\frac{-1}{4\sqrt{5}} \end{pmatrix}
\end{aligned}$$

3.2 Using method of undetermined coefficients

If the components of $\mathbf{g}(t)$ are linear combinations of polynomial, exponential, or sinusoidal functions, then as before we assume that the solution $\mathbf{x}$ is a linear combination of the same type of functions.

3.2.1 Example-Presenting the method

Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1. Given $\mathbf{g}(t) = \binom{e^t}{t}$ we assume that the solution is of the form:

   $$\mathbf{x}(t) = \mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}.$$

   for some vectors **a, b, c,** and **d** to be found.

2. Plugging this into our system we obtain

   $$\mathbf{a}(te^t + e^t) + \mathbf{b}e^t + \mathbf{c} = \mathbf{x}' = \mathbf{A}\big(\mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}\big) + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   Therefore, we obtain algebraic equations for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$:

   $$\begin{aligned} \mathbf{a} &= \mathbf{A}\mathbf{a}, \\ \mathbf{a} + \mathbf{b} &= \mathbf{A}\mathbf{b} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ 0 &= \mathbf{A}\mathbf{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \mathbf{c} &= \mathbf{A}\mathbf{d}. \end{aligned}$$

   The first equation implies that $\mathbf{a}$ is an eigenvector for $\mathbf{A}$ with associated double eigenvalue $\lambda_1 = 1$ and so we can assume that

   $$\mathrm{a} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   because it will solve the second equation:

   $$\mathrm{b} = \begin{pmatrix} k \\ k - \frac{1}{2} \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   By solving the remaining systems we obtain

   $$\mathrm{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

   Therefore, the solution is

   $$\mathrm{x} = c_1e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^t + \left\{c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3.3 Integrating Factor

We now present an alternative method for solving nonhomogeneous first order systems. Specifically, we wish to solve the first order system described by

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)$$

where $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ is a column vector, $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ is continuous.

Observe that, by moving the term $\mathbf{Ax}(t)$ to the other side we obtain

$$\mathbf{x}'(t) - \mathbf{Ax}(t) = \mathbf{g}(t). \tag{3.1}$$

One might be reminded of the one-dimensional nonhomogeneous differential equation given by

$$x'(t) - ax(t) = g(t).$$

You may recall that to solve such an ODE we multiplied through by an integrating factor, $\mu(t)$, chosen carefully so that we may view the left hand side as the derivative of a product of functions. Specifically, we choose μ to satisfy $\mu'(t) = -a\mu(t)$. Multiplying through by μ and applying the above strategy leads to the solution, as one can check,

$$x(t) = e^{at}\left\{C + \int_0^t e^{-as} g(s)\mathrm{d}s\right\}.$$

Motivated by the philosophy that ODEs that look similar are probably solved by similar techniques we attempt to use a similar strategy for the first order system. Note that since we are working with matrices we have to be careful with what we multiply through since not all matrices can be multiplied together (recall that matrix multiplication only makes sense if the row and column sizes are appropriate). After some consideration we may anticipate that the function we desire is a function of the form $\boldsymbol{\mu} : \mathbb{R} \to M_{n\times n}(\mathbb{R})$. That is, a function such at each time t we obtain an $n \times n$ matrix $\boldsymbol{\mu}(t)$. Multiplying our equation on the left by $\boldsymbol{\mu}(t)$ gives

$$\boldsymbol{\mu}(t)\mathbf{x}'(t) - \boldsymbol{\mu}(t)\mathbf{Ax}(t) = \boldsymbol{\mu}(t)\mathbf{g}(t).$$

If we now choose $\boldsymbol{\mu}$ such that $\boldsymbol{\mu}'(t) = -\boldsymbol{\mu}(t)\mathbf{A}$, where this derivative is understood componentwise as in the case of $\mathbf{x}'$, then we could rewrite the equation as

$$\big(\boldsymbol{\mu x}\big)'(t) = \boldsymbol{\mu}(t)\mathbf{g}(t)$$

and then integrating[1] we obtain

$$\boldsymbol{\mu}(t)\mathbf{x}(t) = \mathbf{C} + \int_0^t \boldsymbol{\mu}(s)\mathbf{g}(s)\mathrm{d}s$$

[1] It is worth pondering what integration would mean here since the product of $\boldsymbol{\mu}$ and $\mathbf{g}$ is a column vector and not a scalar. One is usually taught that integrating a continuous scalar function f is the result of taking a limit of Riemann sums. That is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x.$$

In view of this, we notice that if $\mathbf{f} : [a, b] \to \mathbb{R}^m$ is now a vector-valued function then $\mathbf{f}$ is simply a column of scalar-valued functions $\big(f_1, \ldots, f_m\big)^T$. Observe that in this case it makes sense to write

$$\sum_{i=1}^n \mathbf{f}(x_i)\Delta x = \sum_{i=1}^n \begin{pmatrix} f_1(x_i) \\ \vdots \\ f_m(x_i) \end{pmatrix} \Delta x = \begin{pmatrix} \sum_{i=1}^n f_1(x_i)\Delta x \\ \vdots \\ \sum_{i=1}^n f_m(x_i)\Delta x. \end{pmatrix}$$

where $\mathbf{C}$ is a vector of constants of integration. Finally, if we are lucky enough that $\boldsymbol{\mu}(t)$ is invertible for all t then we can solve for $\mathbf{x}$ to obtain

$$\mathbf{x}(t) = \big(\boldsymbol{\mu}(\boldsymbol{t})\big)^{-1}\left(\mathbf{C} + \int_0^t \boldsymbol{\mu}(s)\mathbf{g}(s)\mathrm{d}s\right). \tag{3.2}$$

We now try to find our candidate $\boldsymbol{\mu}$. Recall that we needed to solve the equation

$$\boldsymbol{\mu}'(t) = -\boldsymbol{\mu}(t)\mathbf{A}. \tag{3.3}$$

Given that the one-dimensional case had $\mathbf{A}$ on the left hand side of the candidate function one might find it odd to have the matrix $\mathbf{A}$ on the right for this computation. To fix this, we define $\mathbf{y}(t) = \big(\boldsymbol{\mu}(\boldsymbol{t})\big)^T$. Transposing equation (3.3) we obtain

$$\mathbf{y}'(t) = \big(\boldsymbol{\mu}'(t)\big)^T = \big(-\boldsymbol{\mu}(t)\mathbf{A}\big)^T = -\mathbf{A}^T\big(\boldsymbol{\mu}(\boldsymbol{t})\big)^T = -\mathbf{A}^T\mathbf{y}(t).$$

Observe that if we write $\mathbf{y}$ is columns then the above can be understood as

$$\begin{bmatrix}\mathbf{y}_1'(t) & \cdots & \mathbf{y}_n'(t)\end{bmatrix} = \mathbf{A}^T\begin{bmatrix}\mathbf{y}_1(t) & \cdots & \mathbf{y}_n(t)\end{bmatrix} = \begin{bmatrix}-\mathbf{A}^T\mathbf{y}_1(t) & \cdots & -\mathbf{A}^T\mathbf{y}_n(t)\end{bmatrix}$$

which means that each column $\mathbf{y}_i$ satisfies the first order homogeneous system

$$\mathbf{y}_i'(t) = -\mathbf{A}^T\mathbf{y}_i(t).$$

Suppose now we choose the columns of $\mathbf{y}$ to be n-linearly independent solutions (i.e fundamental solutions) of this first order homogeneous system. Then we will have found $\mathbf{y}$ which means we have found $\boldsymbol{\mu}$ by transposing. Specifically, choose $\mathbf{y}_1 \ldots, \mathbf{y_n}$ to be n-linearly independent solutions. Now we let $\mathbf{y}(t) = \begin{bmatrix}\mathbf{y}_1(t) & \cdots & \mathbf{y}_n(t)\end{bmatrix}$. This tells us that

$$\boldsymbol{\mu}(t) = \big(\mathbf{y}(t)\big)^T = \begin{bmatrix}\big(\mathbf{y}_1(t)\big)^T \\ \vdots \\ \big(\mathbf{y}_n(t)\big)^T\end{bmatrix}.$$

One can check that equation $\big(3.3\big)$ is satisfied. Since we chose that the solutions $\mathbf{y}_i$ are all linearly independent then $\boldsymbol{\mu}(t)$ is invertible for all t. In particular the formula given in equation $\big(3.2\big)$ is valid.[2] Note that the above technique results in a more general answer than the technique given by using diagonalization since we did not assume anything about the matrix $\mathbf{A}$. However, we can see that the cost of generality is that obtaining the solution is more challenging.

To make the above construction more notationally clear we use the concept of a

Now taking limits suggests the definition

$$\int_a^b \mathbf{f}(x)\Delta x := \begin{pmatrix} \int_a^b f_1(x)\mathrm{d}x \\ \vdots \\ \int_a^b f_m(x)\mathrm{d}x. \end{pmatrix}$$

[2]One might try to check that the formula for the solution given in equation $\big(3.2\big)$ is in fact correct. Note, however, that this computation is actually somewhat sophisticated since you have to differentiate the function $\big(\boldsymbol{\mu}(t)\big)^{-1}$ which requires computing the derivative of the function that assigns the inverse of a matrix.

matrix exponential in the following section outlining the steps to implementing the above construction. The matrix exponential, $e^{t\mathbf{A}}$, (whose formula for a 2×2 matrix depends on whether its eigenvalues are complex, real and repeated, or real and distinct) is the matrix whose columns are the fundamental solutions to the problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and whose value at $t = 0$ is the identity. Notice that the notation was deliberately chosen to remind you of the scalar ODE $x' = ax$ whose solution (up to a constant) is e^{ta}.

3.3.1 Method formal steps

1. As usual we first find the eigenvalues λ_1, λ_2 of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
2. Identifying the exponential of $\mathbf{A}$:
   - If the eigenvalues are distinct then
   $$e^{t\mathbf{A}} := e^{\lambda_1 t}\frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}_2) - e^{\lambda_2 t}\frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_1\mathbf{I}_2).$$
   - If $\lambda = \lambda_1 = \lambda_2$ then
   $$e^{t\mathbf{A}} := e^{\lambda t}\mathbf{I}_2 + e^{\lambda t}t(\mathbf{A} - \lambda\mathbf{I}_2).$$
   - If $\lambda_1 = a + ib, \lambda_2 = a - ib$ then
   $$e^{t\mathbf{A}} := \frac{e^{at}}{b}\{b\cos(bt)\mathbf{I}_2 + \sin(bt)(\mathbf{A} - a\mathbf{I}_2)\}.$$
3. We compute $e^{-t\mathbf{A}}$ by inverting the matrix $e^{t\mathbf{A}}$ (see the linear algebra appendix)
4. Finally we obtain the general solution for our system (using identities proven in the linear algebra appendix):
$$\mathbf{x}(t) = \big(\boldsymbol{\mu}(\boldsymbol{t})\big)^{-1}\Big(\mathbf{C} + \int_0^t \boldsymbol{\mu}(s)\mathbf{g}(s)\mathrm{d}s\Big) = exp\big\{t\mathbf{A}\big\}\Big(\mathbf{C} + \int_0^t exp\big\{-s\mathbf{A}\big\}\mathbf{g}(s)\mathrm{d}s\Big) \tag{3.4}$$

3.3.2 Example-Presenting the method

Consider the system
$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}. \tag{3.5}$$
Notice that the matrix $\mathbf{A}$ in equation (3.5) has only 1 as an eigenvalue but the only solution to
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{v} = \mathbf{v}$$
is
$$\mathbf{v} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
for $a \in \mathbb{R}$. So this matrix is not diagonalizable and hence the diagonalization method does not apply. However, the integrating factor technique will still work. Following step 1 we notice that we have a repeated eigenvalue $\lambda = 1$. By step 2 we obtain that
$$e^{t\mathbf{A}} = e^t\mathbf{I}_2 + e^t t(\mathbf{A} - \mathbf{I}_2) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

By step 3 we learn that

$$(e^{t\mathbf{A}})^{-1} = e^{-t\mathbf{A}} = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

Finally, by equation $\left(3.4\right)$ we learn that the general solution is:

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left(\mathbf{C} + \int_0^t \begin{bmatrix} e^{-s} & -se^{-s} \\ 0 & e^{-s} \end{bmatrix} \begin{bmatrix} e^s \\ 1 \end{bmatrix} \mathrm{d}s \right) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left(\mathbf{C} + \int_0^t \begin{bmatrix} 1 - se^{-s} \\ e^{-s} \end{bmatrix} \mathrm{d}s \right) \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left(\mathbf{C} + \begin{bmatrix} t - 1 + e^{-t} + te^{-t} \\ 1 - e^{-t} \end{bmatrix} \right) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \mathbf{C} + \begin{bmatrix} 2te^t - e^t + 1 \\ e^t - 1 \end{bmatrix} \end{aligned}$$

One can check that this solves equation $\left(3.5\right)$ since

$$\mathbf{x}'(t) = \begin{bmatrix} e^t & te^t + e^t \\ 0 & e^t \end{bmatrix} \mathbf{C} + \begin{bmatrix} 2te^t + e^t \\ e^t \end{bmatrix}$$

while

$$\mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) = \begin{bmatrix} e^t & te^t + e^t \\ 0 & e^t \end{bmatrix} \mathbf{C} + \begin{bmatrix} 2te^t \\ e^t - 1 \end{bmatrix} + \begin{bmatrix} e^t \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t + e^t \\ 0 & e^t \end{bmatrix} \mathbf{C} + \begin{bmatrix} 2te^t + e^t \\ e^t \end{bmatrix}.$$

3.4 Variation of Parameters

Given the complexity of the process to solve the inhomogeneous first order system found in section $\left(3.3\right)$ one might wish to find a simpler way of obtaining the solution. This is possible if one is more clever about how they proceed. Recall that for the scalar inhomogeneous equation $x'(t) = ax(t) + g(t)$ the general solution is

$$x(t) = e^{at} \left(C + \int_0^t e^{-as} g(s) \mathrm{d}s \right)$$

where C is a constant. Observe that the term Ce^{at} actually solves the homogeneous equation $x'(t) = ax(t)$. Thus, the part of this solution that is needed to solve the inhomogeneous equation is

$$e^{at} \int_0^t e^{-as} g(s) \mathrm{d}s.$$

Observe that this looks like the solution to the homogeneous equation multiplied by a new function. Inspired by this we might try to solve the system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t). \tag{3.6}$$

by using the ansatz

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{y}(t)$$

where $\mathbf{X}(t)$ is the $n \times n$ matrix consisting of n linearly independent solutions to the homogeneous equation and $\mathbf{y}$ is to be determined. Note that $\mathbf{X}$ plays the role of e^{at} from the scalar case. Thus, we desire that

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) = \mathbf{A}\mathbf{X}(t)\mathbf{y}(t) + \mathbf{g}(t)$$

but we have

$$\mathbf{x}'(t) = \mathbf{X}'(t)\mathbf{y}(t) + \mathbf{X}(t)\mathbf{y}'(t) = \mathbf{A}\mathbf{X}(t)\mathbf{y}(t) + \mathbf{X}(t)\mathbf{y}'(t).$$

Comparing these two equations we see that we must demand

$$\mathbf{X}(t)\mathbf{y}'(t) = \mathbf{g}(t)$$

which becomes after solving for $\mathbf{y}'(t)$, since $\mathbf{X}(t)$ is invertible,

$$\mathbf{y}'(t) = \big(\mathbf{X}(t)\big)^{-1}\mathbf{g}(t).$$

Integrating then gives

$$\mathbf{y}(t) = \mathbf{C} + \int_0^t \big(\mathbf{X}(s)\big)^{-1}\mathbf{g}(s)\mathrm{d}s.$$

Thus, we have found the solution

$$\mathbf{x}(t) = \mathbf{X}(t)\bigg(\mathbf{C} + \int_0^t \big(\mathbf{X}(s)\big)^{-1}\mathbf{g}(s)\mathrm{d}s\bigg).$$

There are a few of advantages to this solution over the one found in section $\big(3.3\big)$. First, notice that it is not too hard to verify that this does in fact solve $\big(3.6\big)$. Second, unlike the solution found in section $\big(3.3\big)$ this formula makes reference directly to the matrix of fundamental solutions to the homogeneous system. As a result of this, less computations are needed. In particualr, exponential matrix identities are not needed to make this expression simpler.

Method formal steps

1. Solve the homogeneous systems to find two linearly independent solutions $\mathbf{x}_1(t) = \binom{x_{1,1}(t)}{x_{1,2}(t)}$ and $\mathbf{x}_2(t) = \binom{x_{2,1}(t)}{x_{2,2}(t)}$ to form the fundamental matrix:

$$\mathbf{\Psi}(t) := \begin{bmatrix} x_{1,1}(t) & x_{2,1}(t) \\ x_{1,2}(t) & x_{2,2}(t) \end{bmatrix},$$

   which satisfies $\mathbf{\Psi}' = \mathbf{A}\mathbf{\Psi}$

2. We make the ansatz we have $\mathbf{x}_{nh}(t) = \mathbf{\Psi}(t)\cdot\mathbf{v}(t) = \mathbf{\Psi}(t)\cdot\binom{v_1(t)}{v_2(t)}$.

3. Plugging this guess to the equation we obtain the system:

$$\mathbf{\Psi}(t)\cdot\mathbf{v}'(t) = \mathbf{g}(t) \Rightarrow$$

$$\begin{bmatrix} x_{1,1}(t) & x_{2,1}(t) \\ x_{1,2}(t) & x_{2,2}(t) \end{bmatrix} \cdot \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \Rightarrow$$

   we obtain the system

$$\begin{cases} x_{1,1}(t)v_1' + x_{1,2}(t)v_2' = g_1(t) \\ x_{2,1}(t)v_1' + x_{2,2}(t)v_2' = g_2(t) \end{cases}$$

4. Solving this system for v_1', v_2' we then integrate to obtain v_1, v_2 and finally obtain the $\mathbf{x}_{nh}(t) = \mathbf{\Psi}(t)\cdot\mathbf{v}(t)$.

Examples

- Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

1. First we find the fundamental matrix $\mathbf{\Psi}$: the eigenpairs are $(-3, \binom{-1}{4}), (2, \binom{1}{1})$ and so the fundamental matrix is:

$$\mathbf{\Psi} = \begin{bmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{bmatrix}.$$

2. The system is

$$\begin{cases} -e^{-3t}v_1' + e^{2t}v_2' = e^{-2t} \\ 4e^{-3t}v_1' + e^{2t}v_2' = -2e^{t} \end{cases}$$

and we obtain

$$\mathbf{v}'(t) = \begin{bmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{bmatrix}^{-1} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \frac{1}{-5} \begin{pmatrix} e^{t} + 2e^{4t} \\ -4e^{-4t} + 2e^{-t} \end{pmatrix} \Rightarrow$$

$$\mathbf{v}(t) = \frac{1}{-5} \begin{pmatrix} e^{t} + e^{2t} \\ e^{-4t} - 2e^{-t} \end{pmatrix}.$$

Therefore, the nonhomogeneous solution is

$$\mathbf{x}_{nh}(t) = \frac{1}{-5} \begin{pmatrix} -2e^{t} - e^{-t} \\ -2e^{t} + 6e^{-2t} + 4e^{-t} \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 4 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{-5} \begin{pmatrix} -2e^{t} - e^{-t} \\ -2e^{t} + 6e^{-2t} + 4e^{-t} \end{pmatrix}.$$

3. From the above solution we note that the dominating term is $e^{2t}\binom{1}{1}$ and even the second dominating term $e^{t}\binom{1}{1}$ is along the same span. So since the rest of the terms go to zer, we expect the solution to converge to the linear span of $\binom{1}{1}$.

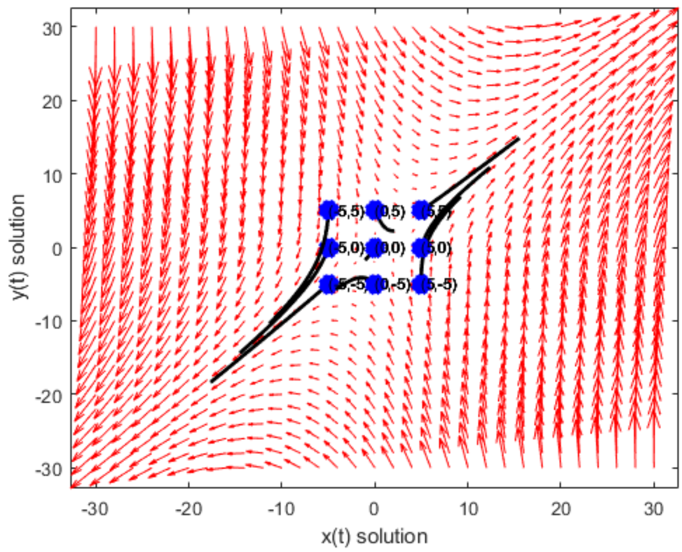


Figure 3.1: Phase portrait

- Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

1. First we find the fundamental matrix $\mathbf{\Psi}$: the eigenvalue is $(-2, \binom{1}{1})$ and $\eta = k\binom{1}{1} - \binom{0}{1}$, therefore, the solution is

$$\begin{aligned} \mathbf{x}(t) =& c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}) \Rightarrow \\ \mathbf{\Psi}(t) =& \begin{bmatrix} e^{-2t} & te^{-2t} \\ e^{-2t} & e^{-2t}(t-1) \end{bmatrix}. \end{aligned}$$

2. The system is

$$\begin{cases} e^{-2t}v_1' + te^{-2t}v_2' = e^{-2t} \\ e^{-2t}v_1' + e^{-2t}(t-1)v_2' = -2e^{t} \end{cases}$$

and we obtain

$$\mathbf{v}'(t) = \begin{pmatrix} 1 - t(1+2e^{3t}) \\ 1 + 2e^{3t} \end{pmatrix} \Rightarrow$$

$$\mathbf{v}(t) = \begin{pmatrix} t - \frac{1}{2}t^2 - \frac{2}{3}te^{3t} + \frac{1}{9}e^{3t} \\ t + \frac{2}{3}e^{3t} \end{pmatrix}.$$

Therefore, the nonhomogeneous solution is

$$\mathbf{x}_{nh}(t) = \begin{pmatrix} e^{-2t}(-t^2/2 - 2/3e^{3t}t + t + e^{3t}/9) + e^{-2t}t(t + (2e^{3t})/3) \\ e^{-2t}(-t^2/2 - 2/3e^{3t}t + t + e^{3t}/9) + e^{-2t}(t-1)(t + (2e^{3t})/3) \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}) + \mathbf{x}_{nh}(t).$$

- Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}.$$

1. First we find the fundamental matrix $\mathbf{\Psi}$: the eigenvalue is $(0, \binom{1}{2})$ and $\eta = k\binom{1}{4} - \binom{0}{1}$, therefore, the solution is

$$\begin{aligned} \mathbf{x}(t) =& c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2(t \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \Rightarrow \\ \mathbf{\Psi}(t) =& \begin{bmatrix} 1 & t \\ 4 & 4t-1 \end{bmatrix}. \end{aligned}$$

2. The system is

$$\begin{cases} v_1' + tv_2' = t^{-3} \\ 4v_1' + (4t-1)v_2' = -t^{-2} \end{cases}$$

and we obtain

$$a = -(t^2 + 4t - 1)/t^3 and b = (t + 4)/t^3 and t != 0 \mathbf{v}'(t) = \begin{pmatrix} -(t^{-1} + 4t^{-2} - t^{-3}) \\ t^{-2} + 4t^{-3} \end{pmatrix} \Rightarrow$$

$$\mathbf{v}(t) = \begin{pmatrix} \frac{8t-1}{2t^2} - log(t) \\ -\frac{t+2}{t^3} \end{pmatrix}.$$

Therefore, the nonhomogeneous solution is

$$\begin{aligned} \mathbf{x}_{nh}(t) &= \mathbf{\Psi}(t)\mathbf{v}(t) \\ &= \begin{bmatrix} 1 & t \\ 4 & 4t - 1 \end{bmatrix} \begin{pmatrix} \frac{8t-1}{2t^2} - log(t) \\ -\frac{t+2}{t^3} \end{pmatrix} \\ &= \begin{pmatrix} (-\frac{2+t}{t^2} + \frac{-1+8t}{2t^2} - log(t) \\ -\frac{((2+t)(-1+4t))}{t^3} + \frac{4((-1+8t)}{2t^2} - log(t))) \end{pmatrix} \end{aligned}$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}) +$$

4 Problems

4.1 Real eigenvalues

- Find the general solution of the system. Describe the asymptotic behaviour (what is the dominating term and the limit). Draw the two eigenvector's spans and draw arrows towards the dominating term. Is it a saddle or a sink to the origin?

  1. $$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}.$$
  2. $$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}.$$

- Find the particular solution of the system. Describe the asymptotic behaviour (what is the dominating term and the limit). Draw the two eigenvector's spans and draw arrows towards the dominating term.

  1. $$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$
  2. $$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- For some $a \in [\frac{1}{2}, 2]$ consider the system
  $$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -a & -1 \end{pmatrix}\mathbf{x}.$$
  Find the general solution in terms of a. Determine the asymptotic behaviour for $a = \frac{1}{2}$ and for 2, and find the $a_* \in [\frac{1}{2}, 2]$,called the bifurcation value, where the asymptotic behaviour changes.

- (*)The amounts of salt $x_1(t), x_2(t)$ in the two tanks satisfy the equations
  $$\frac{\mathrm{d}x_1}{\mathrm{dt}} = -k_1x_1,\ \frac{\mathrm{d}x_2}{\mathrm{dt}} = k_1x_1 - k_2x_2 \text{ with } x_1(0) = 15,\ x_2(0) = 0,$$
  where $k_1 = \frac{r}{V_1} = \frac{1}{5}$, $k_2 = \frac{r}{V_2} = \frac{2}{5}$. Find the particular solution and determine the asymptotic behaviour. What does it tell you about the tank's salt concentration?

Figures/systems_tanks_real.PNG

Figure 4.1: The two brine tanks.

4.2 Complex eigenvalues

- Find the general solution of the system. Describe the asymptotic behaviour. Are the trajectories forming a spiral source, a spiral sink or concentric circles?

  1. $$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\mathbf{x}.$$

2.

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

3.

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\mathbf{x}.$$

- (*) Find the general solution of the system. Find the bifurcation value or values of α where the qualitative nature of the phase portrait for the system changes. Draw a phase portrait for a value of α slightly below, and for another value slightly above, each bifurcation value.

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}\mathbf{x}.$$

- (*)Consider the circuit

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & \frac{-1}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

Solve and determine long term behaviour. Is it asymptotically stable?

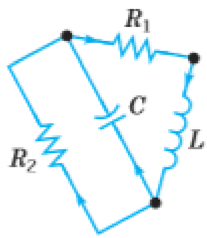


Figure 4.2: The circuit with complex eigenvalues.

4.3 Repeated eigenvalues

- Find the general solution of the system. Describe the asymptotic behaviour. Are the trajectories forming a source or sink behaviour wrt the origin?

1.

$$\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x}.$$

2.

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}.$$

3. Find the particular solution and determine the asymptotic behaviour as above:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

4.4 Differential inequalities

1. In this question we will show the following result: Suppose $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a function that is continuous on $[a, b]$, differentiable on (a, b), and satisfies, for $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ a diagonal matrix,

$$\mathbf{x}'(t) \leq \mathbf{A}\mathbf{x}(t)$$

for all $t \in (a, b)$, where the inequality means that each component of the left hand side is smaller than the corresponding component on the right hand side. Then

$$\mathbf{x}(t) \leq e^{(t-a)\mathbf{A}}\mathbf{x}(a).$$ [3]

[3] This is known as Grönwall's inequality. The principle involved here is that if a function grows no quicker than $\mathbf{A}\mathbf{x}(t)$ then the value of the function should not exceed the solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ which maximizes its growth.

(a) First show that if $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ then $\mathbf{Ax} \geq \mathbf{0}_{n\times 1}$ whenever $\mathbf{x} \geq \mathbf{0}_{n\times 1}$ if and only if $\mathbf{A}$ is a non-negative matrix (all entries in the matrix are non-negative).

(b) Next show that if $t \geq 0$ then $e^{t\mathbf{A}}$ is a non-negative matrix if and only if $\mathbf{A}$ has non-negative off diagonal entries.[4]

(c) Consider the function $\mathbf{w} : [a, b] \to \mathbb{R}$ defined by[5]

$$\mathbf{w}(t) = e^{(a-t)\mathbf{A}}\mathbf{x}(t).$$

Show, using (1b), that $\mathbf{w}'(t) \leq \mathbf{0}_{n\times 1}$ for $t \in (a, b)$.

(d) Conclude that each component of $\mathbf{w}$ is decreasing on $[a, b]$.

(e) Finally, conclude that $e^{(a-t)\mathbf{A}}\mathbf{x}(t) = \mathbf{w}(t) \leq \mathbf{w}(a) = \mathbf{x}(a)$ which can be rewritten as

$$\mathbf{x}(a) - e^{(a-t)\mathbf{A}}\mathbf{x}(t) \geq \mathbf{0}_{n\times 1}.$$

Use (1b) to conclude the desired inequality.

2. In this question we show that solutions to the initial value problem $\mathbf{x}'(t) = \mathbf{Ax}(t)$ for $t \in (a, b)$ with $\mathbf{x}(a) = \mathbf{x}_0$ are unique using Grönwall type inequalities. Suppose $\mathbf{x}, \mathbf{y} : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that they satisfy $\mathbf{x}'(t) = \mathbf{Ax}(t)$ on (a, b) as well as $\mathbf{y}'(t) = \mathbf{Ay}(t)$ on (a, b).

   (a) First show that, for $t \in (a, b)$

   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{x}(t) - \mathbf{y}(t)\|^2\big) = \big(\mathbf{x}(t) - \mathbf{y}(t)\big)^T \mathbf{A}\big(\mathbf{x}(t) - \mathbf{y}(t)\big).$$

   (b) Next use equation (4.1) and the previous step to conclude that

   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{x}(t) - \mathbf{y}(t)\|^2\big) \leq \lambda_n(\mathbf{A})\|\mathbf{x}(t) - \mathbf{y}(t)\|^2.$$

   (c) Argue that we obtain, for $t \in [a, b]$,

   $$\|\mathbf{x}(t) - \mathbf{y}(t)\|^2 \leq e^{2t\lambda_n(\mathbf{A})}\|\mathbf{x}(a) - \mathbf{y}(a)\|^2.$$

   (d) Deduce that if $\mathbf{x}(a) = \mathbf{y}(a)$ then $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [a, b]$.

4.5 Systems of ODEs and Quadratic forms

1. (a) In this problem we show that if $\mathbf{A}$ has only positive eigenvalues, $\mathbf{x}(0) \neq \mathbf{0}_{n\times 1}$, and if $\mathbf{x}'(t) = \mathbf{Ax}(t)$ then $\|\mathbf{x}(t)\| \to +\infty$ as $t \to +\infty$.

   i. First, show that

   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{x}(t)\|^2\big) = \big(\mathbf{x}(t)\big)^T \mathbf{Ax}(t).$$

   ii. Next, observe that

   $$\min_{\|\mathbf{x}\|=1} \big\{\mathbf{x}^T\mathbf{Ax}\big\} = \lambda_1(\mathbf{A})$$

   where $\lambda_1(\mathbf{A})$ denotes the smallest eigenvalue of $\mathbf{A}$. To see this, note that $\mathbf{A}$ is diagonalizable and so we can represent $\mathbf{x}$ as a linear combination of orthonormal

[4] Such matrices are called Metzler matrices.

[5] Some properties of the matrix exponential will be used here. Please refer to the linear algebra appendix for proofs of these properties.

eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. So we have

$$\mathbf{A}\mathbf{x} = \mathbf{A}\left(\sum_{i=1}^{n} c_1 \mathbf{u}_i\right) = \sum_{i=1}^{n} \lambda_i(\mathbf{A}) c_i \mathbf{u}_i$$

which means

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} \lambda_i c_i^2 \geq \lambda_1(\mathbf{A}) \sum_{i=1}^{n} c_i^2 = \lambda_1(\mathbf{A})\|\mathbf{x}\|^2 = \lambda_1(\mathbf{A})$$

and observe that $\mathbf{x}$ was an arbitrary unit vector. Note that equality can be obtained.

iii. Using the previous two questions observe that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{x}(t)\|^2\right) \geq \lambda_1(\mathbf{A})\|\mathbf{x}(t)\|^2.$$

Using the integrating factor $e^{-2t\lambda_1(\mathbf{A})}$ conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2t\lambda_1(\mathbf{A})}\|\mathbf{x}(t)\|^2\right) \geq 0.$$

iv. Conclude that

$$\|\mathbf{x}(t)\|^2 \geq \|\mathbf{x}(0)\|^2 e^{2t\lambda_1(\mathbf{A})}.$$

(b) In this problem we show that if $\mathbf{A}$ has all negative eigenvalues then $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$ if $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

i. Show that, as in the previous question,

$$\max_{\|\mathbf{x}\|=1} \left\{\mathbf{x}^T\mathbf{A}\mathbf{x}\right\} = \lambda_n(\mathbf{A}) \tag{4.1}$$

and conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{x}(t)\|^2\right) \leq \lambda_n(\mathbf{A})\|\mathbf{x}(t)\|^2.$$

ii. Conclude that

$$\|\mathbf{x}(t)\|^2 \leq \|\mathbf{x}(0)\|^2 e^{2t\lambda_n(\mathbf{A})}.$$

2. In this problem we will demonstrate how to find a solution with perpendicular trajectories in $\mathbb{R}^2$. Suppose $\mathbf{x}, \mathbf{y} : \mathbb{R} \to \mathbb{R}^2$ solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ and $\mathbf{y}'(t) = \mathbf{B}\mathbf{y}(t)$ respectively, where $\mathbf{A}, \mathbf{B} \in M_{2\times 2}(\mathbb{R})$. Assume also that, for all $\mathbf{a} \in \mathbb{R}^2$, that if $\mathbf{x}(0) = \mathbf{a} = \mathbf{y}(0)$ then $\mathbf{x}'(0) \perp \mathbf{y}'(0)$.

(a) Use the conditions given in the problem description to conclude that

$$\mathbf{a} \cdot \left(\mathbf{A}^T\mathbf{B}\mathbf{a}\right) = 0$$

for all $a \in \mathbb{R}^2$.

(b) Conclude that there is a constant $c \in \mathbb{R}$ such that

$$\mathbf{A}^T\mathbf{B} = c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) If $c = 0$ conclude that either $\mathbf{A} = \mathbf{0}_{2\times 2}$, $\mathbf{B} = \mathbf{0}_{2\times 2}$, or $\mathbf{A}^T\mathbf{B} = \mathbf{0}_{2\times 2}$ while $\mathbf{A}$ and $\mathbf{B}$ are both not the zero matrix. In the event that $\mathbf{A}^T\mathbf{B} = \mathbf{0}_{2\times 2}$ even though both $\mathbf{A}$ and $\mathbf{cB}$ are both not the zero matrix show that $\mathrm{im}(\mathbf{A}) \perp \mathrm{im}(\mathbf{B})$.

(d) Now we may assume that $c \neq 0$. By taking determinants show that both $\mathbf{A}$ and $\mathbf{B}$ are invertible. Finally, conclude that either

$$\mathbf{A}^T\mathbf{B} = c\mathbf{R}_{\frac{\pi}{2}}$$

or

$$\mathbf{A}^T\mathbf{B} = c\mathbf{R}_{\frac{-\pi}{2}}$$

for some $c > 0$ where $\mathbf{R}_{\pm\frac{\pi}{2}}$ denote rotation matrices at angles $\pm\frac{\pi}{2}$ respectively. Use this to conclude that

$$\mathbf{B} = c\big(\mathbf{A^{-1}}\big)^T\mathbf{R}_{\pm\frac{\pi}{2}}.$$