$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x \qquad F = \begin{pmatrix} y \\ -\sin x \end{pmatrix}$$

1) Find critical pts $\quad x' = 0,\ y' = 0 \Rightarrow y = 0$

$\sin x = 0$

$\Rightarrow (k\pi, 0)$

$k \in \mathbb{Z}$.

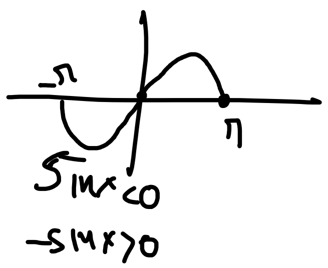
2) implicit sln

$$\frac{dy}{dx} = \frac{-\sin x}{y} \Rightarrow y^2 = \cos x + C.$$

3) nullcline analysis around the origin:

I) $\frac{dx}{dt} > 0, \frac{dy}{dt} > 0 \Rightarrow y > 0,\ -\sin x > 0$

$\Rightarrow y > 0,\ -\pi \le x < 0$



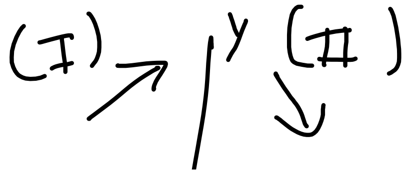
II) $x' > 0,\ y' < 0$

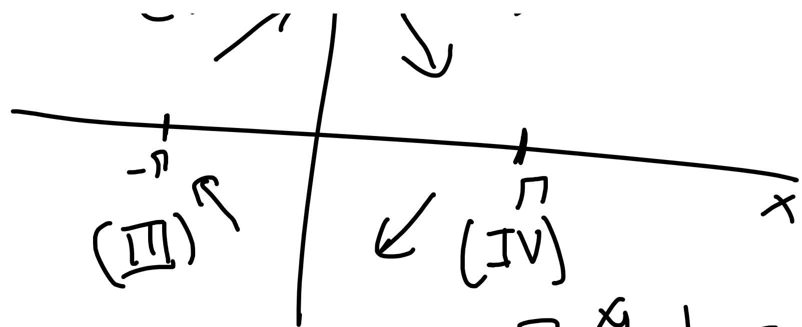
$\Rightarrow y > 0,\ \sin x > 0$

$\Rightarrow y > 0,\ 0 < x \le \pi$

III) $x' < 0,\ y' > 0 \Rightarrow y < 0,\ -\pi < x < 0$

IV) $x' < 0,\ y' < 0 \Rightarrow y < 0,\ 0 < x \le \pi$.





3) If we linearise we have $F = \begin{pmatrix} y \\ -\sin x \end{pmatrix}$ $J_F = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}\Bigg|_{(x_0, y_0)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

3.1) eigvl, eigvt $\quad \lambda = \frac{tr(A)}{2} \pm \frac{1}{2}\sqrt{tr(A)^2 - 4 \det A}$

$$= \pm i$$

$$v_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -i \\ -1 \end{pmatrix}$$

3.2) $x = c_1 e^{it} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} i \\ 1 \end{pmatrix}$

$x' = 2y, \quad y' = 8x$

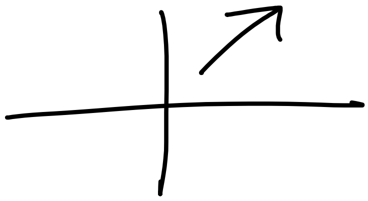
1) $\frac{y'}{x'} = \frac{8x}{2y} \Rightarrow y^2 = 4x^2 + C$

$= \frac{4x}{\cancel{3}y}$

2) $x' > 0, \; y' > 0 \Rightarrow 2y > 0$

$| \nearrow \quad 8x > 0$

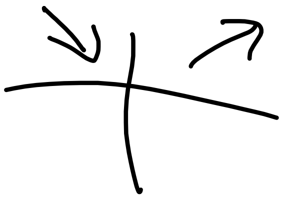
$8x<0$



$x'>0,\ y'<0$

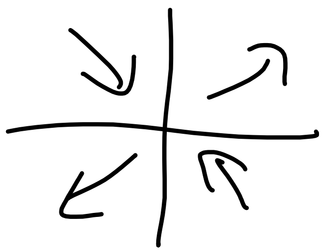
$\Downarrow$

$2y>0,\ 8x<0$



3) $x'<0,\ y'>0$

$\Rightarrow y<0,\ x>0$



4) $x'<0,\ y'<0$

$\Rightarrow y<0,\ x<0$

5) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

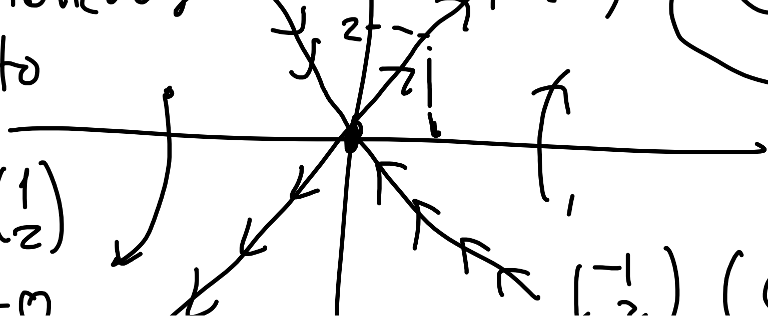
eigenpairs are $\left(4, \begin{pmatrix} 4 \\ 8 \end{pmatrix}\right), \left(-4, \begin{pmatrix} -4 \\ 8 \end{pmatrix}\right)$

$x = c_1 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

asymptotically stable to the seand ct $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

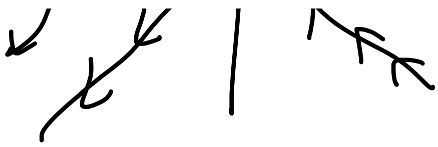
$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(0,0) is a saddle



$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $(c_1=0)$

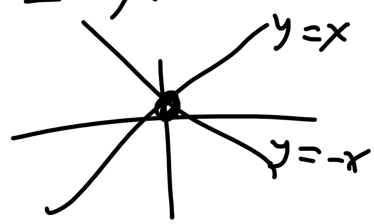
[illegible] (2)
for $c_1 \neq 0$



$\begin{pmatrix}-1\\2\end{pmatrix}$ $(c_1 = 0)$

$x' = -x + y$, $y' = -x - y$

1) $x' = 0$ and $y' = 0 \Rightarrow y = x$, $y = -x$
$\Rightarrow (0,0)$



2) paramedon sln

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{-x-y}{-x+y} \underset{\left(\cdot \frac{1}{x}\right)}{=} \frac{1 + y/x}{1 - y/x}$$

Let $v = y/x \Rightarrow \frac{dv}{dx} = \frac{dy}{dx} \cdot \frac{1}{x} + y\left(-\frac{1}{x^2}\right)$

$$\Rightarrow v' \cdot x + v = \frac{dy}{dx}$$

So $\frac{dv}{dx} \cdot x + v = \frac{dy}{dx} = \frac{1+v}{1-v} \Rightarrow$

$$x \cdot v' = \frac{1+v^2}{1-v} \Rightarrow \int \frac{1-v}{1+v^2}\, dv = \int \frac{1}{x}\, dx$$

$$\int \frac{1}{1+v^2}\, dv - \int \frac{v}{1+v^2}\, dv = \arctan(v) - \frac{1}{2}\ln(1+v^2)$$

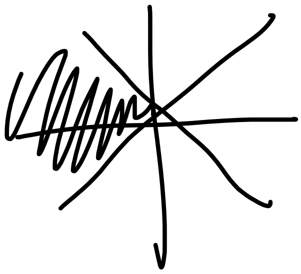
$\Rightarrow v = \frac{y}{x}$ the sln is

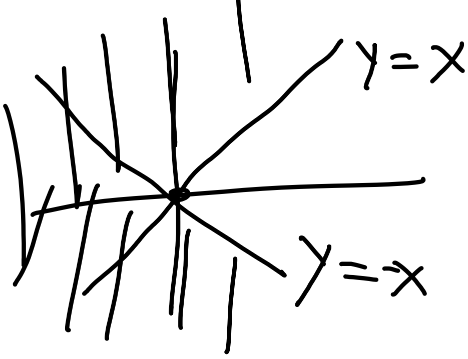
$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{x}\right)^2\right) = \ln(x) + C$$

---

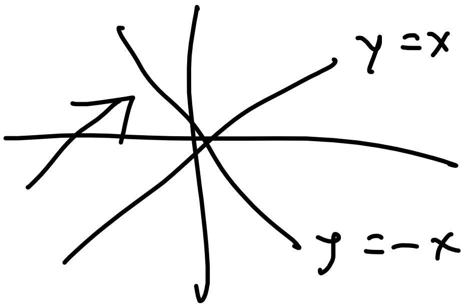
3) nullcline analysis

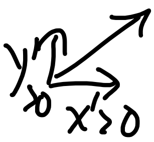
$$\begin{array}{ccc} & y>x & y<-x \\ & \Uparrow & \Uparrow \\ (I)\quad x'>0,\ y'>0 \Rightarrow & -x+y>0, & -x-y>0 \end{array}$$





$x'>0$
$y'>0$
$\Downarrow$

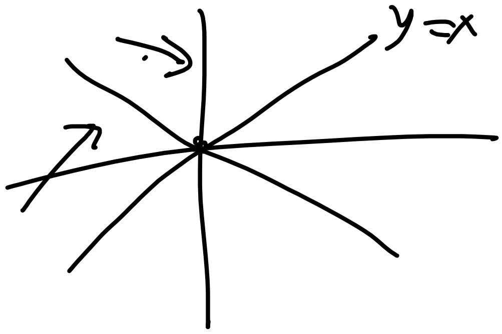


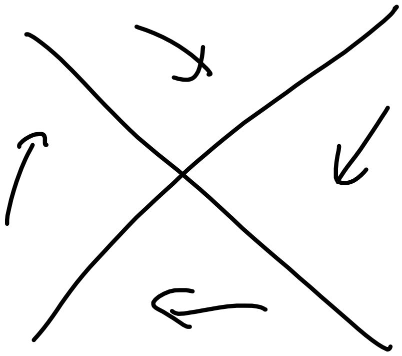


II) $x'>0,\ y'<0$

$\Rightarrow -x+y>0,\ -x-y<0$

$y>x\ ,\ -x<y$





4) $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

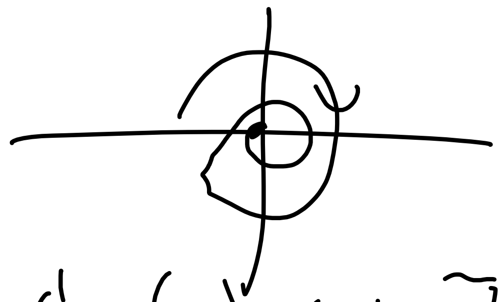
$$\lambda = \frac{\operatorname{Tr} A}{2} \pm \frac{1}{2}\sqrt{[\operatorname{Tr}(A)]^2 - 4\det(A)}$$

$$= -1 \pm \sqrt{1-(1+1)} \Rightarrow \lambda = -1 \pm i$$

$$v_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -i \\ -1 \end{pmatrix}$$

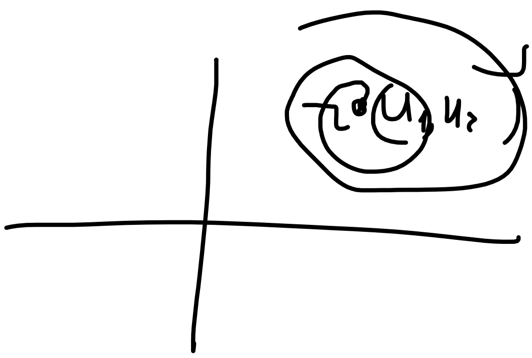
$$\Rightarrow x = c_1 e^{(-1+i)t}\begin{pmatrix} i \\ -1 \end{pmatrix} + c_2 e^{(-1-i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

so we get spiral sink.



But if $x' = (\ )x + \tilde{v}$

$$x = c_1 x_1 + c_2 x_2 + \underbrace{A^{-1}\tilde{v}}_{u}$$



$$x' = y, \quad y' = -x + x^3/6$$

1) $x'=0$ and $y'=0 \Rightarrow y=0$

$-x+x^3/6=0$

$\Rightarrow (0,0), (\pm\sqrt{6},0)$

2) p.sln

$$\frac{dy}{dx} = \frac{-x+x^3/6}{y} \Rightarrow y^2 = -\frac{x^2}{2}+\frac{x^4}{24}+C.$$

3) null. anl.

(I) $x'>0,\ y'>0 \Rightarrow y>0$

$-x+x^3/6>0$

$\Rightarrow x(x^2-6)>0$

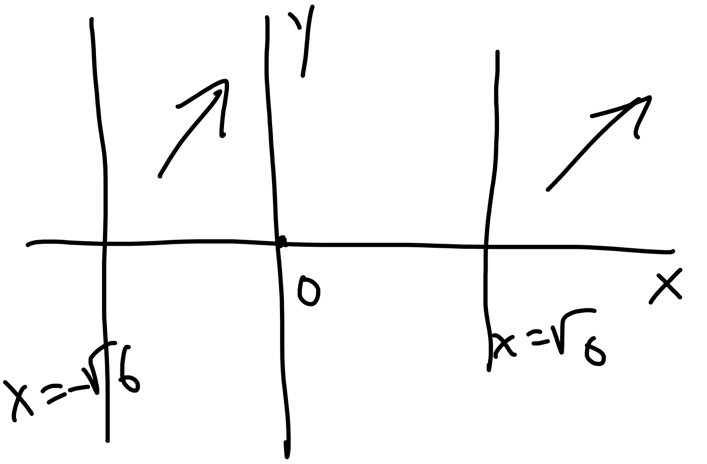
$\Rightarrow x>0$ and $x>\sqrt{6}$

or

($x<0$ and $x^2<6$)

$-\sqrt{6}<x<\sqrt{6}$

$x\in[-\sqrt{6},0]$

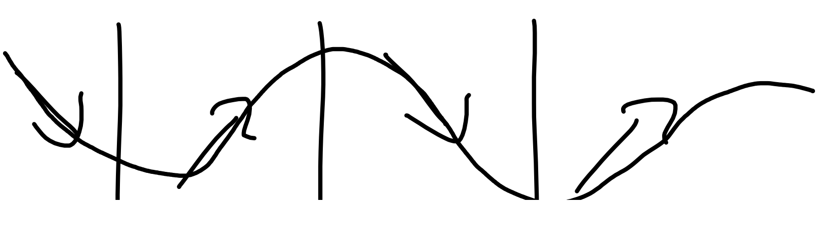


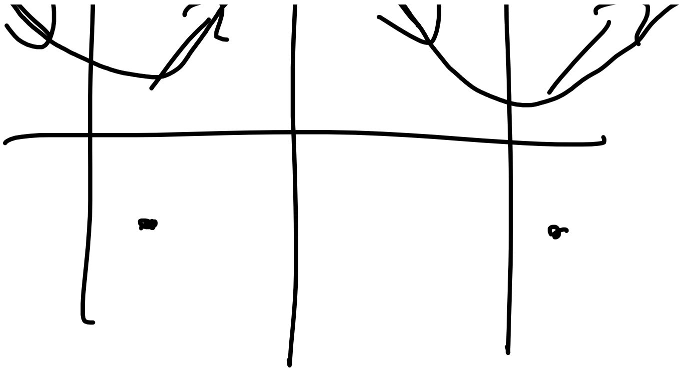
II) $x'>0,\ y'<0 \Rightarrow y>0$, $-x+\frac{x^3}{6}<0$

$\Rightarrow x(x^2-6)<0$

$\Rightarrow x<0$ and $x^2>6$

or





or

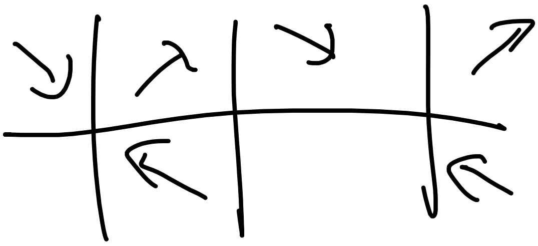
$x>0$ and $x^2<6$

$\Rightarrow$ $x<-\sqrt{6}$

or

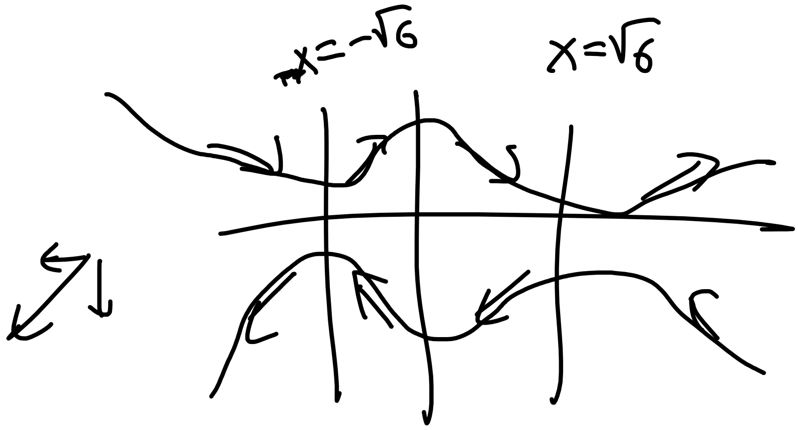
$0<x<\sqrt{6}$

III) $x'<0,\ y'>0$ $\Rightarrow$ $y<0$



$-x+\frac{x^3}{6}>0 \Rightarrow x$

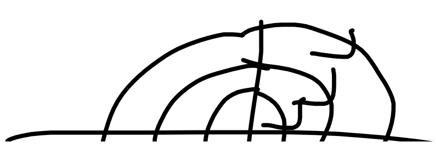
IV) $x'<0,\ y'<0$



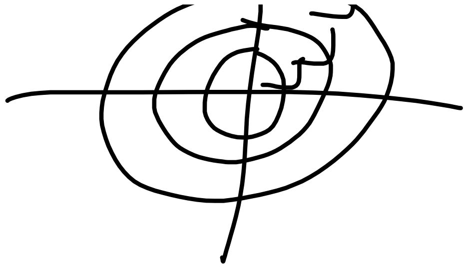
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenpairs $\left(i, \begin{pmatrix} i \\ -1 \end{pmatrix}\right), \left(-i, \begin{pmatrix} -i \\ -1 \end{pmatrix}\right)$

$$x = c_1 e^{it}\begin{pmatrix} i \\ -1 \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

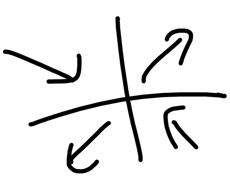


agrees with nullclines



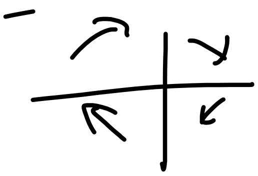
nullcln

(eg



)

saddle



$$X' = Ax + g \qquad P(t) = A$$

We guess that $x_{nh} = \Psi y(t) = [x_1 \;\; x_2] \cdot y(t)$

$$\Psi' y(t) + \Psi y' = A\Psi y + g \Rightarrow$$

$$\Psi' = [x_1', x_2']$$

$$\underbrace{\cancel{(\Psi' - A\Psi)}}_{0} y + \Psi y' = g \Rightarrow$$

$A\Psi = [Ax_1, Ax_2]$

as $Ax_i = x_i'$ $i = 1, 2$

$$\Psi y' = g \Rightarrow$$

Since $W(x_1, x_2) \neq 0 \Rightarrow \Psi^{-1}$ always exists

$$y' = \Psi^{-1} g(t)$$

$$x' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

1) Find eg fundamental matrix Ψ

$$\lambda = \frac{-\cancel{3}1}{2} \pm \frac{1}{2}\sqrt{\cancel{9}1 - 4(\cancel{4+2})} \qquad 4\,6 = 24$$

$$-2-4 \over -6$$

$$= \frac{-\cancel{3} \pm \sqrt{\cancel{33}25}}{2} = \begin{cases} \cancel{\ } \\ \cancel{\ } \end{cases} = \begin{cases} 2 \\ -3 \end{cases}$$

$$\frac{-\cancel{3}1 \pm 5}{2}$$

$$v_1 = \begin{pmatrix} 2+2 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3+2 \\ 4 \end{pmatrix}$$

$$X_{hom} = C_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\Psi = \begin{bmatrix} e^{2t} & -e^{-3t} \\ e^{2t} & 4e^{-3t} \end{bmatrix}$$

2) $X_{nh} = \Psi \cdot y(t) \Rightarrow \Psi y' = g(t)$

$$e^{2t} y_1' - e^{-3t} y_2' = g_1 = e^{-2t}$$
$$e^{2t} y_1' + 4e^{-3t} y_2' = g_2 = -2e^{t}$$

$R_2 - R_1$
$$\Rightarrow y_2' \, 5 e^{-3t} = -2e^{t} - e^{-2t}$$
$$\Rightarrow y_2' = -\frac{2}{5} e^{4t} - \frac{1}{5} e^{t}$$

$$\Rightarrow y_2 = -\frac{2}{20} e^{4t} - \frac{1}{5} e^{t}$$

$$\cancel{\Rightarrow e^{2t} y_1' + 4 e^{-3}}$$

$$\xrightarrow{R_1 + \frac{1}{4}R_2} e^{2t} y_1' \left(1 + \frac{1}{4}\right) = e^{-2t} - \frac{1}{2} e^{t}$$

$$\Rightarrow y = \quad ' = \frac{4}{5}\left(1 - \frac{1}{2} e^{-t}\right)$$

$$\Rightarrow y_1 = \frac{4}{5}\left(t + \frac{1}{2} e^{-t}\right)$$

$$y_2 = -\frac{1}{10} e^{4t} - \frac{1}{5} e^{t}$$

$$\Rightarrow y = \begin{pmatrix} \frac{4}{5} t + \frac{4}{10} e^{-t} \\ -\frac{1}{10} e^{4t} - \frac{1}{5} e^{t} \end{pmatrix}$$

3) $X_{nh} = \Psi \cdot y$

$$= \begin{bmatrix} e^{2t} & -e^{-3t} \\ e^{2t} & 4e^{-3t} \end{bmatrix} \begin{pmatrix} \frac{4}{5} t + \frac{1}{5} e^{-t} \\ -\frac{1}{10} e^{4t} - \frac{1}{5} e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} \cdot \frac{4}{5} t + \frac{1}{5} e^{t} + e^{t} \frac{1}{10} + \frac{1}{5} e^{-2t} \\ e^{2t} t \frac{4}{5} + \frac{1}{5} e^{t} + \left(-\frac{4}{10} e^{t} - \frac{4}{5} e^{-2t}\right) \end{pmatrix}$$

$$\Big(e^{2t} t \frac{4}{5} + \frac{1}{5} e^{t} + \big(\frac{4}{10} e^{t} - \frac{4}{5} e^{-2t}\big) \Big)$$

$$x = c_1 e^{-3t} \begin{pmatrix} -1 \\ 4 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + X_{nh}.$$

$$\lambda = -2, \quad \xi = \begin{pmatrix} -2-(-3) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda I)\eta = \xi \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \eta_1 - \eta_2 = 1 \Rightarrow \eta = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

set $\eta_1 = k$ $\quad \eta = \begin{pmatrix} k \\ k-1 \end{pmatrix}$

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \Big(e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \Big)$$

$$= (c_1 + k)\, \underline{e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} + c_2 \Big(\underline{e^{-2t}\, t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}}$$

$$\begin{pmatrix} te^{-2t} - 0 \\ te^{-2t} - e^{-2t} \end{pmatrix}$$

$$\Psi = \begin{bmatrix} e^{-2t} & t e^{-2t} \\ e^{-2t} & (t-1) e^{-2t} \end{bmatrix}$$

Solve $\begin{cases} e^{-2t} v_1' + t e^{-2t} v_2' = e^{-2t} \\ e^{-2t} v_1' + v_2'(t-1) e^{-2t} = -2e^{t} \end{cases}$