Outline

1. Autonomous systems and nullcli

2. Competing species

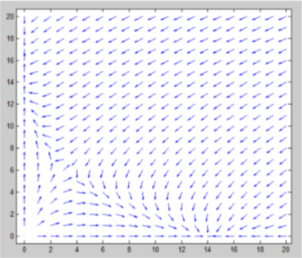

Figure: competing species separatrix

As in the 1D case we will study the following system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x, y), \frac{\mathrm{d}y}{\mathrm{dt}} = G(x, y),$$

where F, G are continuously differentiable functions. Here again we might not be able to obtain explicit solutions, but we can provide a qualitative analysis.

1. First, we find the critical points by setting

$$F(x, y) = 0 \text{ and } G(x, y) = 0.$$

2. Sometimes we can even solve such systems by taking their ratio:

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{\frac{\mathrm{d}y}{\mathrm{dt}}}{\frac{\mathrm{d}x}{\mathrm{dt}}} = \frac{G(x, y)}{F(x, y)}.$$

This ratio depends only on x and y (and not t), so methods from the first order section could be used.

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\omega^2 sin(x).$$

(1) First we find the critical points. We have $(k\pi, 0)$ for integer k.

(2) next we find the parametric solutions. The equation

$$\frac{dy}{dx} = \frac{-\omega^2 sin(x)}{y}$$

is separable and so we obtain:

$$y^2 = \omega^2 cos(x) + c.$$

(3) Next we do a detailed nullcline analysis (see handnotes) take $\omega^2 = 1$.

(4) The linearized system around the origin is

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \mathbf{x},$$

which indeed has concentric circles as its phase portrait.

In class example

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2y, \frac{\mathrm{d}y}{\mathrm{d}t} = 8x.$$

1. (5 minutes) Find parametric solution.
2. (10 minutes) Do nullcline analysis around the origin.
3. (10 minutes) For the linearized system

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix} \mathbf{x}$$

   find general solution and phase portrait.

In class example

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = -x + y, \frac{\mathrm{d}y}{\mathrm{dt}} = -x - y.$$

1. (5 minutes) Find parametric solution.
2. (10 minutes) Do nullcline analysis around the origin.
3. (10 minutes) For the linearized system

$$\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}$$

   find general solution and phase portrait.

In class example

Consider the system (Duffing's equation)

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -x + \frac{x^3}{6}.$$

1. (5 minutes) Find parametric solution.
2. (10 minutes) Do nullcline analysis around the origin.
3. (10 minutes) For the linearized system

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

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- As discussed before, we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation

$$\frac{\mathrm{d}x}{\mathrm{dt}} = x(\varepsilon_1 - \sigma_1 x)$$
$$\frac{\mathrm{d}y}{\mathrm{dt}} = y(\varepsilon_2 - \sigma_2 y)$$

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- The α_1 is a measure of the degree to which species y interferes with species x and similarly for α_2.

First we find the critical points

$$\begin{cases} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) = 0 \\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) = 0 \end{cases} \Rightarrow$$

$$(0,0), (\frac{\varepsilon_1}{\sigma_1}, 0), (0, \frac{\varepsilon_2}{\sigma_2}), \text{ and } (\frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}).$$

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For the last critical point to be a realistic steady state we require that both components are positive.

We linearize the system by 2D-Taylor expanding

$$F(x, y) = \begin{pmatrix} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{pmatrix}$$

around critical point (x_0, y_0):

Competing species: linearize

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around critical point (x_0, y_0):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ y \end{pmatrix} &= F(x, y) = J_F(x_0, y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2) \\ &= \begin{pmatrix} \varepsilon_1 - 2\sigma_1 x_0 - \alpha_1 y_0 & -\alpha_1 x_0 \\ -\alpha_2 y_0 & \varepsilon_2 - \alpha_2 x_0 - 2\sigma_2 y_0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x \end{aligned}$$

Competing species: around critical points

We determine the stability behaviour around each of the critical points.

(1) At $(x_0, y_0) = (0, 0)$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(\|(x, y)\|^2).$$

The End