Contents

1 Laplace transform for 1D ODEs

The *Laplace transform* of continuous functions $f(t)$ with at most exponential growth, that is $|f(t)| \leq ce^{at}$ for $a \geq 0$ and $c \geq 0$, is defined as:

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t,$$

where $s > a$. In future, we denote continuous functions such that $|f(t)| \leq ce^{at}$ for $a \geq 0$, $c \geq 0$, and for all $t \geq 0$ as $C([0,\infty), e^{at})$, where if $a = 0$ we understand this as the space of bounded continuous functions on $[0,\infty)$. Note that we have dropped the constant c from the definition of $C([0,\infty), e^{at})$ since only a affects the region of definition of the Laplace transform. By integrating by parts we can easily check that we have:

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\} - f(0)$$

so for the second derivative we have, by iterating the previous observation,

$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\} - f'(0) = s^2\mathcal{L}\{f\} - f'(0) - sf(0).$$

Continuing with the above observation we see that

$$\mathcal{L}\{f^{(m)}\}(s) = s^m\mathcal{L}\{f\} - \sum_{j=0}^{m-1} s^j f^{m-j-1}(0)$$

where $f^{(m)}$ denotes the m^{th} derivative of f. Another useful property is that the Laplace transform is linear on continuous function of exponential growth. First observe that if $f, g \in C([0,\infty), e^{at})$ then linear combination of f, g also belong to $C([0,\infty), e^{at})$. Thus, the Laplace transform is defined on $f + g$, for $f, g \in C([0,\infty), e^{at})$, and satisfies:

$$\mathcal{L}\{f + g\}(s) = \mathcal{L}\{f\}(s) + \mathcal{L}\{g\}(s)$$

for $s > a$.

Method formal steps

1. Starting from the equation $ay''(t) + by'(t) + cy(t) = g(t)$ we compute the laplace transform of both sides, assuming an exponential growth condition on y and g, to obtain:

$$a\mathcal{L}\{y''\}(s) + b\mathcal{L}\{y'\}(s) + c\mathcal{L}\{y\}(s) = \mathcal{L}\{g\}(s)$$

we obtain from above:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as + b)}{as^2 + bs + c}.$$

2. So by inverting the laplace transform (using linearity and known inversions) we can obtain solution $y(t)$ back. Note that inverting the laplce transform is permitted by Lerch's theorem (3) which says that if two functions, f_1 and f_2, have the same laplace transform then they are "essentially" equal.

3. The main computational aspect of this is splitting partial fractions to get the known relations. But Heaviside motivated by the same problem when computing the Laplace transform, came up with the cover-up method. In computing the coefficients below

$$\frac{p(s)}{(s - a_1) \cdots (s - a_n)} = \frac{A_1}{s - a_1} + ... + \frac{A_n}{s - a_n},$$

for polynomial $p(s)$, we see by rearranging that:

$$\frac{p(s)}{(s - a_1) \cdots (s - a_{i-1})(s - a_{i+1}) \cdots (s - a_n)} = \frac{A_1(s - a_i)}{s - a_1} + ... + A_i + ... + \frac{A_n(s - a_i)}{s - a_n},$$

and by setting $s = a_i$ we obtain the ith coefficient A_i:

$$A_i = \frac{p(a_i)}{(a_i - a_1) \cdots (a_i - a_{i-1})(a_i - a_{i+1}) \cdots (a_i - a_n)}.$$

Here is a table of known Laplace transforms (see section 3 for proofs):

function $f(t) = \mathcal{L}^{-1}\{g\}(t)$	Laplace transform $g(s) = \mathcal{L}\{f\}(s)$	Region of definition
constant a	$\frac{a}{s}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(bt)e^{at}$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$\cos(bt)e^{at}$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$f_{\text{step}}(t,a) := \begin{cases} 1, & 0 \le t \le a \\ 0, & t > a \end{cases}$	$\frac{1-e^{-as}}{s}$	$s > 0$
$e^{a(t-b)} f_{\text{heavy}}(t,b) := e^{a(t-b)}\big(1 - f_{\text{step}}(t,b)\big)$	$\frac{e^{-bs}}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p, \;\; p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

Examples

- Consider the equation

$$y''(t) - y'(t) - 6y(t) = 0, \;\; y(0) = 1, \;\; y'(0) = -1.$$

1. By taking the Laplace transform of both sides we obtain:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2+bs+c}$$

for our equation we have

$$= \frac{y(0)(s-1) + y'(0)}{s^2 - s - 6}$$

for our IC we have

$$= \frac{(s-1)-1}{s^2-s-6}$$
$$= \frac{s-2}{(s-3)(s+2)}$$

2. Next we split it into partial fractions

$$\begin{aligned}\mathcal{L}\{y\}(s) &= \frac{s-2}{(s-3)(s+2)}\\ &= \frac{1/5}{s-3} + \frac{4/5}{s+2}\end{aligned}$$

So we use $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \iff e^{at} = \mathcal{L}\{\frac{1}{s-a}\}^{-1}$

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\Big\{\frac{1/5}{s-3} + \frac{4/5}{s-(-2)}\Big\}(t)\\ &= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}.\end{aligned}$$

3. Indeed, using the method of characteristic equations for second order equations we obtain

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}.$$

Therefore, by using the IC we have

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ -1 = y'(0) = 3c_1 - 2c_2 \end{cases} \Longrightarrow \begin{cases} c_1 = \frac{1}{5} \\ c_2 = \frac{4}{5} \end{cases}.$$

4. So for homogeneous equations it is clearly much faster and less error prone to use the method of characteristics.

- Consider the nonhomogeneous equation

$$y''(t) - 2y'(t) + 2y(t) = e^{-t}, \ \ y(0) = 0, \ \ y'(0) = 1$$

1. First we take the Laplace transform of both sides to obtain:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2 + bs + c}$$

for our equation it becomes

$$\begin{aligned}&= \frac{\frac{1}{s+1} + y'(0) + y(0)(s-2)}{s^2 - 2s + 2}\\ &= \frac{\frac{1}{s+1} + 1}{s^2 - 2s + 2}\\ &= \frac{s+2}{(s+1)(s^2 - 2s + 2)}\end{aligned}$$

by partial fractions we obtain

$$\begin{aligned}&= \frac{1}{5(s+1)} + \frac{8-s}{5(s^2 - 2s + 2)}\\ &= \frac{1}{5(s+1)} + \frac{8-s}{5(s-(1-i))(s-(1+i))}\end{aligned}$$

repeating partial fractions for the last term we have

$$= \frac{1}{5(s+1)} + \frac{\frac{7}{2}i - \frac{1}{2}}{5(s-(1-i))} + \frac{-\frac{7}{2}i - \frac{1}{2}}{5(s-(1+i))}.$$

So by inverting the Laplace transform we have

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\Big\{\frac{1}{5(s+1)}\Big\} + \mathcal{L}^{-1}\Big\{\frac{\frac{7}{2}i - \frac{1}{2}}{5(s-(1-i))}\Big\} + \mathcal{L}^{-1}\Big\{\frac{-\frac{7}{2}i - \frac{1}{2}}{5(s-(1+i))}\Big\} \\
&= \frac{1}{5}e^{-t} + \Big(\frac{7}{10}i - \frac{1}{10}\Big)e^{(1+i)t} + \Big(\frac{-7}{10}i - \frac{1}{10}\Big)e^{(1-i)t}.
\end{aligned}$$

2. Let's check this with the method of undetermined coefficients.
   (a) First we solve the homogeneous problem. The method of characteristics gives:

$$x_h(t) = c_1 e^{(1+i)t} + c_2 e^{(1-i)t}.$$

   (b) We make the ansatz $x_{nh}(t) = ce^{-t}$ (here $s = 0$ because $r_* = -1$ is not a root). Plugging in we have

$$ce^{-t} + 2ce^{-t} + 2ce^{-t} = e^{-t} \implies c + 2c + 2c = 1 \implies c = 1/5,$$

   which is the same nonhomogeneous solution as in the Laplace transform.
   (c) Next we evaluate the coefficients.

$$\begin{cases} 0 = y(0) = c_1 + c_2 + \frac{1}{5} \\ 1 = y'(0) = (1+i)c_1 + (1-i)c_2 - \frac{1}{5} \end{cases} \implies \begin{cases} c_1 = \frac{7}{10}i - \frac{1}{10} \\ c_2 = -\frac{7}{10}i - \frac{1}{10} \end{cases}.$$

- Consider the equation

$$y''(t) + 4y(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$$

with initial data $y(0) = 1$, $y'(0) = 0$.

1. The Laplace transform of the above step function[1] is

$$\frac{1 - e^{-\pi s}}{s}.$$

2. We take the Laplace transform of both sides:

$$\mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2 + bs + c}$$

for our equation the above becomes

$$\begin{aligned}
\frac{\mathcal{L}\{g\}(s) + ay'(0) + y(0)(as+b)}{as^2 + bs + c} &= \frac{\frac{1-e^{-\pi s}}{s} + y'(0) + sy(0)}{s^2+4} \\
&= \frac{1 - e^{-\pi s} + s^2}{s(s^2+4)} \\
&= \frac{s}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-\pi s}}{s(s^2+4)} \\
&= \frac{s}{s^2+4} + \Big(\frac{1}{4s} - \frac{1}{8(s+2i)} - \frac{1}{8(s-2i)}\Big)(1 - e^{-\pi s}).
\end{aligned}$$

[1]Technically we have defined the Laplace transform only for continuous functions of controlled growth. However, the Laplace transform is extendable to Riemann integrable functions of controlled growth. In particular, for this example, one computes the Laplace transform by splitting the function into the two regions where it is understood. For more information see this exercises at the end of this section.

We will use the following Laplace transforms:

$$
\begin{aligned}
\mathcal{L}\{\cos(2t)\}(s) &= \frac{s}{s^2+4} \\
\mathcal{L}\Big\{\frac{1}{4}\Big\}(s) &= \frac{1}{4s} \\
\mathcal{L}\Big\{\frac{1}{8}e^{-2it}\Big\}(s) &= \frac{1}{8(s+2i)} \\
\mathcal{L}\Big\{\frac{1}{8}e^{2it}\Big\}(s) &= \frac{1}{8(s-2i)} \\
\mathcal{L}\Big\{e^{-\pi s}\Big\}(s) &= \frac{1}{s-(-\pi)} = \frac{1}{s+\pi} \\
\mathcal{L}\Big\{e^{-a(t-b)}\big(1-f_{\text{step}}(t,b)\big)\Big\} &= \frac{e^{-bs}}{s+a}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} + \mathcal{L}^{-1}\Big\{\frac{1}{4s}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s+2i)}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s-2i)}\Big\} \\
&\quad + \mathcal{L}^{-1}\Big\{e^{-\pi s}\frac{1}{4s}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s+2i)}e^{-\pi s}\Big\} + \mathcal{L}^{-1}\Big\{-\frac{1}{8(s-2i)}e^{-\pi s}\Big\} \\
&= \cos(2t) + \frac{1}{4} - \frac{1}{8}e^{-2it} - \frac{1}{8}e^{2it} \\
&\quad + \frac{1}{4}\big(1-f_{\text{step}}(t,\pi)\big) + \frac{1}{8}e^{-2i(t-\pi)}(1-f_{\text{step}}(t,\pi)) + \frac{1}{8}e^{2i(t-\pi)}(1-f_{\text{step}}(t,\pi))
\end{aligned}
$$

2 Laplace transform for systems

Consider the Laplace transform of vectors $\mathcal{L}\{\mathbf{x}\}(s)$ defined componentwise[2]

$$
\mathcal{L}\{\mathbf{x}\}(s) := \begin{pmatrix} \mathcal{L}\{x_1\}(s) \\ \vdots \\ \mathcal{L}\{x_n\}(s) \end{pmatrix}.
$$

Therefore, as with the usual Laplace transform we obtain, by repeatedly using the scalar version of this identity, that:

$$
\mathcal{L}\{\mathbf{x}'\}(s) = s\mathcal{L}\{\mathbf{x}\}(s) - \mathbf{x}(0).
$$

2.1 Method formal steps

Consider the nonhomogeneous system

$$
\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t).
$$

[2]Recall that integration extends to vectors by integration componentwise. Since this transform is defined by integration it too extends to vectors by acting componentwise.

1. Taking the Laplace transform of each term in the above equation we have:

$$s\mathcal{L}\{\mathbf{x}\}(s) - \mathbf{x}(0) = \mathbf{A}\mathcal{L}\{\mathbf{x}\}(s) + \mathcal{L}\{\mathbf{g}\}(s).$$

2. For simplicity we assume that $\mathbf{x}(0) = \mathbf{0}_{n\times 1}$.
3. We then obtain the system:

$$(s\mathbf{I}_n - \mathbf{A})\mathcal{L}\{\mathbf{x}\}(s) = \mathcal{L}\{\mathbf{g}\}(s).$$

4. By inverting the matrix, assuming s is not an eigenvalue of $\mathbf{A}$, we obtain:

$$\mathcal{L}\{\mathbf{x}\}(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\mathcal{L}\{\mathbf{g}\}(s).$$

5. Then we do inverse Laplace transform of each component using known Laplace transform relations.
6. The main computational aspect of this is splitting partial fractions to get the known relations. But Heaviside motivated by the same problem when computing the Laplace transform, came up with the cover-up method. In computing the coefficients below

$$\frac{p(s)}{(s-a_1)\cdots(s-a_n)} = \frac{A_1}{s-a_1} + ... + \frac{A_n}{s-a_n},$$

for polynomial $p(s)$, we see by rearranging that:

$$\frac{p(s)}{(s-a_1)\cdots(s-a_{i-1})(s-a_{i+1})\cdots(s-a_n)} = \frac{A_1(s-a_i)}{s-a_1} + ... + A_i + ... + \frac{A_n(s-a_i)}{s-a_n},$$

and by setting $s = a_i$ we obtain the ith coefficient A_i:

$$A_i = \frac{p(a_i)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}.$$

2.2 Examples

- Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

1. We take Laplace transform of both sides

$$(s\mathbf{I} - \mathbf{A})\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \begin{pmatrix} s-2 & 1 \\ 0 & s-1 \end{pmatrix}\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}.$$

2. We simplify

$$\mathcal{L}\Big\{\mathbf{x}\Big\}(s) = \begin{pmatrix} s-2 & 1 \\ 0 & s-1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$= \frac{1}{(s-2)(s-1)} \begin{pmatrix} s-1 & -1 \\ 0 & s-2 \end{pmatrix} \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{2}{(s+1)(s-2)} - \frac{3}{(s-1)(s-2)s^2} \\ \frac{3}{(s-1)s^2} \end{pmatrix}.$$

By partial fractions we get

$$= \begin{pmatrix} \frac{2}{3(s-2)} - \frac{2}{3(s+1)} - \left(\frac{3}{2s^2} - \frac{3}{s-1} + \frac{9}{4}\frac{1}{s} + \frac{3}{4}\frac{1}{s-2}\right) \\ \frac{-3}{s^2} - \frac{3}{s} + \frac{3}{s-1} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-1}{12(s-2)} - \frac{2}{3(s+1)} - \frac{3}{2s^2} + \frac{3}{s-1} - \frac{9}{4}\frac{1}{s} \\ \frac{-3}{s^2} - \frac{3}{s} + \frac{3}{s-1} \end{pmatrix}.$$

3. We use known Laplace transform relations to obtain $\mathbf{x}$ by inverting:

$$\mathbf{x}(t) = \begin{pmatrix} \mathcal{L}^{-1}\left\{\frac{-1}{12(s-2)} - \frac{2}{3(s+1)} - \frac{3}{2s^2} + 3\frac{1}{s-1} - \frac{9}{4}\frac{1}{s}\right\}(t) \\ \mathcal{L}^{-1}\left\{\frac{-3}{s^2} - \frac{3}{s} + \frac{3}{s-1}\right\}(t) \end{pmatrix}.$$

For the first component we have

$$\begin{aligned} x_1(t) &= \frac{-1}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) \\ &+ 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \frac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) \\ &= \frac{-1}{12}e^{2t} - \frac{2}{3}e^{-t} - \frac{3}{2}t + 3e^t - \frac{9}{4}. \end{aligned}$$

For the second component we have

$$\begin{aligned} x_2(t) &= -3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) \\ &= -3t - 3 + 3e^t. \end{aligned}$$

Therefore, together give

$$\mathbf{x}_{nh}(t) = \begin{pmatrix} \frac{-1}{12}e^{2t} - \frac{2}{3}e^{-t} - \frac{3}{2}t + 3e^t - \frac{9}{4} \\ -3t - 3 + 3e^t \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} \frac{-1}{12} \\ 0 \end{pmatrix} - \frac{2}{3}e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - 3\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} + 3e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4. Therefore, the general solution is

$$\mathbf{x}(t) = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} \frac{-1}{12} \\ 0 \end{pmatrix} - \frac{2}{3}e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - 3\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} + 3e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Consider the second order equation

$$w''(t) + w(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

with zero initial data. By setting $x = w$, $y = w'$ we obtain $x' = y$, $y' + x = f_{\text{step}}(t)$ or in system form

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0, & 1 \\ \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f_{\text{step}}(t) \end{pmatrix}.$$

1. We take Laplace transform of both sides

$$(s\mathbf{I}_n - \mathbf{A})\mathcal{L}\{\mathbf{x}\}(s) = \begin{pmatrix} s & -1 \\ \\ 1 & s \end{pmatrix}\mathcal{L}\{\mathbf{x}\}(s) = \begin{pmatrix} 0 \\ \mathcal{L}\{f_{\text{step}}\}(s) \end{pmatrix}.$$

2. The Laplace transform of the RHS is

$$\mathcal{L}\{f_{\text{step}}\}(s) = \int_0^\infty e^{-st} f_{\text{step}}(t)\mathrm{d}t = \int_0^1 e^{-st}\mathrm{d}t = \frac{1-e^{-s}}{s}.$$

3. We simplify

$$\begin{aligned}
\mathcal{L}\{\mathbf{x}\}(s) &= \begin{pmatrix} s & -1 \\ \\ 1 & s \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \frac{1-e^{-s}}{s} \end{pmatrix} \\
&= \frac{1}{s^2+1}\begin{pmatrix} s & 1 \\ \\ -1 & s \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1-e^{-s}}{s} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-e^{-s}}{s(s^2+1)} \\ \frac{1-e^{-s}}{s^2+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-i}{2}\frac{1-e^{-s}}{s+i} - \frac{i}{2}\frac{1-e^{-s}}{s-i} + \frac{1-e^{-s}}{s} \\ \frac{-i}{2}\frac{1-e^{-s}}{s+i} + \frac{i}{2}\frac{1-e^{-s}}{s-i} \end{pmatrix}.
\end{aligned}$$

4. We use known Laplace transform relations to invert

   For the first component we have

$$\begin{aligned}
x_1(t) &= \mathcal{L}^{-1}\Big\{\frac{1-e^{-s}}{s}\Big\}^{-1}(t) + \mathcal{L}^{-1}\Big\{\frac{-i}{2}\frac{1-e^{-s}}{s+i} - \frac{i}{2}\frac{1-e^{-s}}{s-i}\Big\}^{-1}(t) \\
&= f_{step}(t,1) + \frac{-i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s+i}\Big\}(t) - \mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s+i}\Big\}(t)\Big] \\
&+ \frac{-i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s-i}\Big\}^{-1}(t) - \mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s-i}\Big\}^{-1}(t)\Big] \\
&= f_{step}(t,1) + \frac{-i}{2}\Big[e^{-it} - e^{-i(t-(-1))}(1 - f_{step}(t,1))\Big] \\
&+ \frac{i}{2}\Big[e^{it} - e^{i(t+1)}(1 - f_{step}(t,1))\Big].
\end{aligned}$$

For the second component we have

$$x_2(t) = \mathcal{L}^{-1}\Big\{\frac{-i}{2}\frac{1-e^{-s}}{s+i} + \frac{i}{2}\frac{1-e^{-s}}{s-i}\Big\}^{-1}(t)$$
$$= \frac{-i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s+i}\Big\}(t) - \mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s+i}\Big\}(t)\Big] + \frac{i}{2}\Big[\mathcal{L}^{-1}\Big\{\frac{1}{s-i}\Big\}^{-1}(t) - \mathcal{L}^{-1}\Big\{\frac{e^{-s}}{s-i}\Big\}^{-1}(t)\Big]$$
$$= \frac{-i}{2}\Big[e^{-it} - e^{-i(t-(-1))}(1 - f_{step}(t,1))\Big] + \frac{i}{2}\Big[e^{it} - e^{i(t+1)}(1 - f_{step}(t,1))\Big].$$

Therefore, together we obtain

$$\mathbf{x}_{nh}(t) = f_{step}(t,1)\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\frac{-i}{2}\Big[e^{-it} - e^{-i(t-(-1))}f_{step}(t,1)\Big] + \begin{pmatrix}1\\-1\end{pmatrix}\frac{i}{2}\Big[e^{it} - e^{i(t+1)}f_{step}(t,1)\Big].$$

5. The general solution will be:

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{it}\begin{pmatrix}i\\1\end{pmatrix} + c_1 e^{-it}\begin{pmatrix}-i\\1\end{pmatrix}\\ &+ f_{step}(t,1)\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\frac{-i}{2}\Big[e^{-it} - e^{-i(t-(-1))}(1 - f_{step}(t,1))\Big]\\ &+ \begin{pmatrix}1\\-1\end{pmatrix}\frac{i}{2}\Big[e^{it} - e^{i(t+1)}(1 - f_{step}(t,1))\Big].\end{aligned}$$

6. For comparison we also compute the solution of the second order equation:

$$w''(t) + w(t) = \begin{cases}1, & 0 \le t \le 1\\ 0, & t > 1\end{cases}.$$

7. By taking the Laplace transform of both sides we obtain

$$s^2\mathcal{L}\Big\{w\Big\} - sw(0) - w'(0) + \mathcal{L}\Big\{w\Big\} = \mathcal{L}\Big\{f_{step}(\cdot,1)\Big\}(s)$$

using that $w(0) = w'(0) = 0$ we obtain

$$\begin{aligned}\mathcal{L}\Big\{w\Big\} &= \frac{\mathcal{L}\Big\{f_{step}(\cdot,1)\Big\}(s)}{s^2+1}\\ &= \frac{1-e^{-s}}{s(s^2+1)}\\ &= \frac{1-e^{-s}}{s(s^2+1)}\\ &= \frac{1-e^{-s}}{s} + \frac{-i}{2}\frac{1-e^{-s}}{s+i} - \frac{i}{2}\frac{1-e^{-s}}{s-i}.\end{aligned}$$

Therefore, by inverting we obtain

$$w(t) = f_{step}(t,1) + \frac{-i}{2}\Big[e^{-it} - e^{-i(t+1)}(1 - f_{step}(t,1))\Big] + \frac{i}{2}\Big[e^{it} - e^{i(t+1)}(1 - f_{step}(t,1))\Big].$$

8. This is indeed the solution we obtained for the first component $x_1(t) := w(t)$.

3 Properties of Laplace Transform

We begin by demonstrating the following commonly used identities for the Laplace transform:

Proposition 1. *The Laplace transform satisfies the following identities:*

1. *If $f, g \in C([0,\infty), e^{at})$ then for $b, c \in \mathbb{R}$ $\mathcal{L}\{bf + cg\}(s) = b\mathcal{L}\{f\}(s) + c\}(s)$ for $s > a$.*
2. *If $f \in C([0,\infty), e^{at})$ and $b > -a$ then $e^{bt} \cdot f \in C([0,\infty), e^{(a+b)t})$ and $\mathcal{L}\{e^{bt}f\}(s) = \mathcal{L}\{f\}(s-b)$ for $s > a+b$.*
3. *Suppose $f \in C([0,\infty), e^{at})$ for $a \neq 0$ and define $F : [0,\infty) \to \mathbb{R}$ by $F(t) = \int_0^t f(s)\mathrm{d}s$. Then $F \in C([0,\infty), e^{at})$ and for $s > a$ we have $\mathcal{L}\{F\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$.*
4. *Suppose $F : [0,\infty) \to \mathbb{R}$ is defined by $F(t) = \int_0^t f(s)\mathrm{d}s$ for $f \in C([0,\infty), e^{at})$ and we assume that $F \in C([0,\infty), e^{at})$ then for $s > a$ we have $\mathcal{L}\{F\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$.*
5. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{a\}(s) = \frac{a}{s}$ for $s > 0$.*
6. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2+a^2}$ for $s > 0$.*
7. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}$ for $s > 0$.*
8. *For $a \in \mathbb{R}$ we have $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$ for $s > a$.*
9. *For $a, b \in \mathbb{R}$ we have $\mathcal{L}\{\sin(at)e^{bt}\}(s) = \frac{a}{(s-b)^2+a^2}$ for $s > b$.*
10. *For $a, b \in \mathbb{R}$ we have $\mathcal{L}\{\cos(at)e^{bt}\}(s) = \frac{s-b}{(s-b)^2+a^2}$ for $s > b$.*

Proof.

1. Suppose $f, g \in C([0,\infty), e^{at})$ and $b, c \in \mathbb{R}$. Observe that

$$|bf(t) + cg(t)| \leq |b| \cdot |f(t)| + |c| \cdot |g(t)| \leq |b| \cdot C_f e^{at} + |c| C_g e^{at} = (|b|C_f + |c|C_g)e^{at}$$

for $t \geq 0$ where C_f and C_g are non-negative constants. We conclude that $bf + cg \in C([0,\infty), e^{at})$. Thus, the Laplace transform of $bf + cg$, f, and g are all defined for $s > a$. Computing gives, for $s > a$, that:

$$\begin{aligned}\mathcal{L}\{bf + cg\}(s) &= \int_0^\infty e^{-st}(bf(s) + cg(s))\mathrm{d}s \\ &= b\int_0^\infty e^{-st} f(s)\mathrm{d}s + c\int_0^\infty e^{-st} g(s)\mathrm{d}s \\ &= b\mathcal{L}\{f\} + c\mathcal{L}\{g\}.\end{aligned}$$

2. Suppose $f \in C([0,\infty), e^{at})$ and $b > -a$. Then for $t \geq 0$ we have:

$$|f(t)e^{bt}| = e^{bt}|f(t)| \leq C_f e^{bt} \cdot e^{at} = C_f e^{(a+b)t}.$$

Thus, $e^{bt}f \in C([0,\infty), e^{(a+b)t})$ which means the Laplace transform of $e^{bt}f$ is defined for $s > a+b$. Observe that, for $s > a+b$

$$\mathcal{L}\{e^{bt}f\}(s) = \int_0^\infty e^{-st}e^{bt}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-b)t}f(t)\mathrm{d}t = \mathcal{L}\{f\}(s-b).$$

3. Suppose $f \in C([0,\infty), e^{at})$ and $F(t) = \int_0^t f(s)\mathrm{d}s$ for $t \geq 0$. Then for $t \geq 0$ we have, if $a \neq 0$

$$|F(t)| = \left|\int_0^t f(s)\mathrm{d}s\right| \leq \int_0^t |f(s)|\mathrm{d}s \leq C_f \int_0^t e^{as}\mathrm{d}s = C_f \cdot \frac{e^{e^{at}} - 1}{a} \leq \frac{C_f}{a} \cdot e^{at}.$$

Thus, for $a \neq 0$ the Laplace transform is defined for f for $s > a$. In particular, by integrating by parts, which is permitted since f is continuous, we get:

$$\mathcal{L}\{F\}(s) = \int_0^\infty e^{-st}F(t)\mathrm{d}t = \frac{-e^{-st}F(t)}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = \frac{1}{s}\mathcal{L}\{f\}(s).$$

4. By assumption the Laplace transform of both f and F is defined for $s > a$. Thus, for $s > a$ we have, by integrating by parts.

$$\mathcal{L}\{F\}(s) = \int_0^\infty e^{-st}F(t)\mathrm{d}t = \frac{-e^{-st}F(t)}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = \frac{1}{s}\mathcal{L}\{f\}(s).$$

5. Suppose $a \in \mathbb{R}$. Then the constant function defined by $f(t) = a$ for $t \geq 0$ is bounded and hence $f \in C([0,\infty), e^{0\cdot t})$. Thus, the Laplace transform is defined for $s > 0$. Computing this we obtain, for $s > 0$:

$$\mathcal{L}\{a\}(s) = \int_0^\infty e^{-st}a\mathrm{d}t = a\int_0^\infty e^{-st}\mathrm{d}t = \frac{a}{s}$$

6. For $a \in \mathbb{R}$ we have $\sin(at) \in C([0,\infty), e^{0\cdot t})$ since this function is bounded. Thus, the Laplace transform is defined for $s > 0$. Computing the transform we get, for $s > 0$:

$$\begin{aligned}\mathcal{L}\{\sin(at)\}(s) &= \int_0^\infty e^{-st}\sin(at)\mathrm{d}t = -\frac{e^{-st}\sin(at)}{s}\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\mathrm{d}t \\ &= -\frac{ae^{-st}\cos(at)}{s^2}\Big|_0^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin(at)\mathrm{d}t \\ &= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}(s).\end{aligned}$$

Thus, we obtain, for $s > 0$

$$\left(\frac{s^2+a^2}{s^2}\right)\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2}$$

and so

$$\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2+a^2}$$

7. Observe that, for $a \neq 0$, $\cos(at) = 1 - a\int_0^t \sin(as)\mathrm{d}s$ and that $\cos(at)$, $\sin(at)$, and the constant function -1 are all bounded functions. In particular, we see that

$$\int_0^t \sin(as)\mathrm{d}s = \frac{1-\cos(at)}{a}$$

is bounded for $t \geq 0$. Thus, the Laplace transform of all functions involved is defined for $s > 0$. Applying properties 1, 4, and 6 we obtain for $s > 0$

$$\mathcal{L}\{\cos(at)\}(s) = \frac{1}{s} - \frac{a}{s}\cdot\frac{a}{s^2+a^2} = \frac{1}{s}\cdot\frac{s^2+a^2-a^2}{s^2+a^2} = \frac{s}{s^2+a^2}.$$

8. By properties 2 and 5 we have
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$
for $s > a$ since 1 is bounded.

9. By properties 2 and 6 we have
$$\mathcal{L}\{\sin(at)e^{bt}\} = \frac{a}{(s-b)^2 + a^2}$$
for $s > b$ since $\sin(at)$ is bounded.

10. By properties 2 and 7 we have
$$\mathcal{L}\{\cos(at)e^{bt}\} = \frac{s-b}{(s-b)^2 + a^2}$$
for $s > b$ since $\sin(at)$ is bounded. □

The spaces $C([0,\infty), e^{at})$ for $a \geq 0$, while large enough to deal with simple functions we encounter in the wild, are not large enough to deal with some of the obstacles we may run into. In particular, these spaces are not suited to dealing with "modestly" growing functions like $x \mapsto x^n$ for $n \in \mathbb{N}$ which grows slower at infinity then any function of the form e^{at} for $a > 0$ but is not bounded. To get around this obstacle we define the spaces $L^p((0,\infty), e^{-at})$ consisting of functions, f, such that $\int_0^\infty e^{-at}|f(t)|^p \mathrm{d}t < \infty$ for $a \geq 0$ and $1 \leq p < \infty$. Observe that such functions have laplace transform defined for $s > a$ and if $a = 0$ then $L^p((0,\infty), e^{-at}) = L^p((0,\infty))$. We will, in particular, consider the case $p = 1$ as this allows an immediate extension to the Laplace transform. With this new definition we will demonstrate some properties of the extended Laplace transform. We will also show that the properties demonstrated in proposition 1 remain true for the extended Laplace transform.

Proposition 2. *The generalized Laplace transform satisfies the following identities:*

1. *Suppose $f \in C([0,\infty), e^{-at})$. Then $f \in L^1((0,\infty), e^{-at})$ and so the laplace transform is defined, by the same formula, for $s > a$.*

2. *If $p \geq 0$ then $f(s) = s^p$ is an element of $L^1((0,\infty))$ and satisfies $\mathcal{L}\{f\}(s) = \frac{\Gamma(p+1)}{s^{p+1}}$ for $s > 0$.*

Proof. 1. Observe that for $s > a$ we have

□

Theorem 3. *(Lerch's theorem) Suppose $f_1, f_2 \in L^p((0,\infty), e^{-at})$ and $\mathcal{L}\{f_1\}(s) = \mathcal{L}\{f_2\}(s)$ for all $s > a$. Then $f_1 = f_2$ almost everywhere on $(0,\infty)$.*

4 Problems