$$X' = \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix} X + \begin{pmatrix}1\\ -1\end{pmatrix} e^t$$

1) $\left(-1, \begin{pmatrix}1\\3\end{pmatrix}\right)$ and $\left(1, \begin{pmatrix}1\\1\end{pmatrix}\right)$

2) $X = Ty \Rightarrow y = T^{-1}ATy + T^{-1}g(t)$

$$\Rightarrow y_1 = -y_1 + \left(T^{-1}\begin{pmatrix}1\\-1\end{pmatrix}e^t\right)_1$$

$$y_2 = y_2 + \left(T^{-1}\begin{pmatrix}1\\-1\end{pmatrix}e^t\right)_2$$

3) $T^{-1} = \begin{pmatrix}1 & 1\\ 3 & 1\end{pmatrix}^{-1} = \frac{1}{\det(A)}\begin{pmatrix}1 & -1\\ -3 & 1\end{pmatrix} = \frac{1}{-2}\begin{pmatrix}1 & -1\\ -3 & 1\end{pmatrix}$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{\det(A)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

$$T^{-1}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{-2}\begin{pmatrix}1 & -1\\ -3 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}2\\ -3\cdot 1 - 1\end{pmatrix} = \begin{pmatrix}-1\\ 2\end{pmatrix}$$

4) $y_1 = -y_1 - e^t$

$y_2 = y_2 + 2e^t$

5) $y_1 = e^{-t}\left(\int_0^t e^{+s}(-e^s)\,ds + c_1\right)$

$$y_2 = e^{+t}\left(\int_0^t e^{-s}(2e^s)\,ds + c_2\right)$$

$$y_1 = e^{-t}\left(\int_0^t e^{2s}\,ds + c_1\right)$$

$$= e^{-t}\left(\frac{e^{2t}-1}{2} + c_1\right)$$ we can set $c_1 = \frac{1}{2}$

$$y_2 = e^{t}\left(\int_0^t 2\,ds + c_2\right)$$

$c_2 = 0$

b/c we only need a particular non-homogenem s/n $\underline{V(t)}$

$$= e^{t}(2t + c_2)$$

$$\Rightarrow \quad y_1 = -e^{t}/2$$

$$y_2 = e^{t}\,2t$$

5) $\underline{x = Ty} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 2t \end{pmatrix} e^{t} \Rightarrow$

$$\underline{V(t)} = x = \begin{pmatrix} t-1 \\ t-3 \end{pmatrix} e^{t}/2$$

6) $X_{gen} = c_1 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} t-1 \\ t-3 \end{pmatrix} e^{t}/2$

---

1) $x' = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 6 \end{pmatrix} e^{t}$

2) $T = (\xi_1\ \xi_2) = (ev1\ ev2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

3) $T^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{1}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

4) $x = Ty \Rightarrow (Ty)' = ATy + g(t)$

$$\Rightarrow y' = \underline{T^{-1}AT}\,y + \underline{T^{-1}g(t)}$$

$$= \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} + \begin{pmatrix}h_1\\h_2\end{pmatrix}$$

$$\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = \begin{pmatrix}-2 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}e^t$$

$$\Rightarrow y_1' = -2y_1 + e^t$$
$$y_2' = -y_2$$

$$\Rightarrow y_1 = e^{-2t}\left(\int_0^t e^{2s}e^{s}\,ds + c_1\right)$$
$$y_2 = e^{-t} + c_2$$

$$\Rightarrow y_1 = e^{-2t}\left(\frac{e^{3t}-1}{3} + c_1\right) = e^{t}/3$$
$$y_2 = e^{-t} + c_2 \qquad \underline{c_1 = \frac{1}{3},\ c_2 = 0}$$

$$y = \begin{pmatrix}y_1\\y_2\end{pmatrix} = e^t\begin{pmatrix}1/3\\0\end{pmatrix} + e^{-t}\begin{pmatrix}0\\1\end{pmatrix}$$

6) $v(t) = x_{hh}(t) = Ty = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}e^t/3\\e^{-t}\end{pmatrix}$.

$$= \begin{pmatrix}1/3\\0\end{pmatrix}e^t + \begin{pmatrix}0\\1\end{pmatrix}e^{-t}$$

7) $X_{gen} = c_1 e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1/3\\0\end{pmatrix}e^{t} + \begin{pmatrix}0\\1\end{pmatrix}e^{t}$

$$x' = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}x + \begin{pmatrix}4\\-2\end{pmatrix}e^t$$

1) $T = (ev_1 \;\; ev_2) = \begin{pmatrix}1 & -1\\2 & 2\end{pmatrix}$

2) $T^{-1}\begin{pmatrix}4\\-2\end{pmatrix}e^t = \begin{pmatrix}3/2\\-\frac{5}{2}\end{pmatrix}e^t$

$$\frac{1}{4}\begin{pmatrix}2 & 1\\-2 & 1\end{pmatrix}\begin{pmatrix}4\\-2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}6\\-10\end{pmatrix} = \begin{pmatrix}3/2\\-5/2\end{pmatrix}$$

3) $y_1' = 3y_1 + \frac{3}{2}e^t$

$y_2' = -y_2 - \frac{5}{2}e^t$

4) $y_1 = -e^{2t}/2 \;, \; c_1 = -\frac{1}{2}$

$y_2 = e^{t}/2 \;, \; c_2 = \frac{1}{2}$

5) $X_{gen} = c_1 e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}-1\\2\end{pmatrix} + Ty$

$$Ty = \begin{pmatrix}1 & -1\\2 & 2\end{pmatrix}\left(e^{2t}\begin{pmatrix}-\frac{1}{2}\\0\end{pmatrix} + e^{t}\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right)$$

$= e^{2t}\begin{pmatrix}-\frac{1}{2} \\ \ldots\end{pmatrix} + e^{t}\begin{pmatrix}-\frac{1}{2} \\ \ldots\end{pmatrix}$

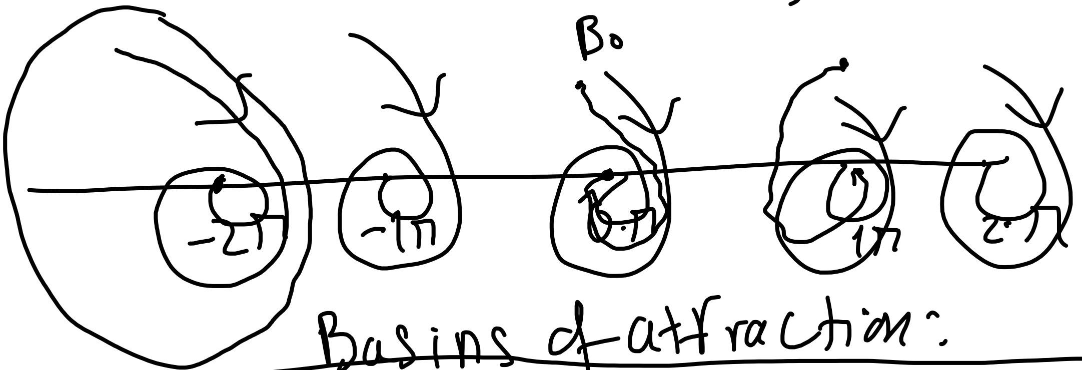
$$= e^{2t}\begin{pmatrix}-\frac{1}{2}\\-1\end{pmatrix} + e^{t}\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}$$

$$\frac{dx}{dt} = y \quad , \quad \frac{dy}{dt} = -\gamma y - \omega^2 \sin(x)$$

1) Find critical pts $\frac{dx}{dt} = 0 = \frac{dy}{dt}$

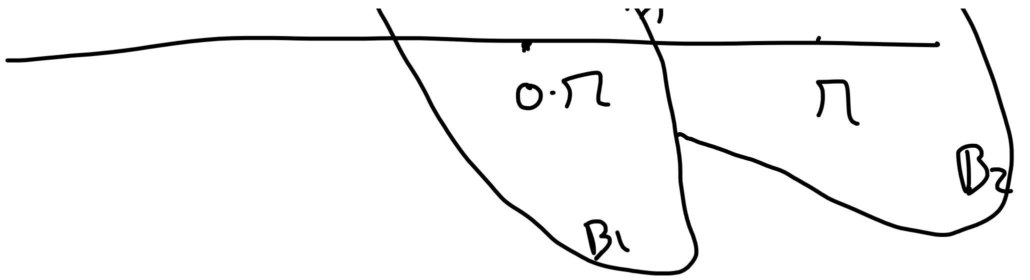
$$\Rightarrow \begin{cases} y = 0 \\ -\gamma y - \omega^2 \sin(x) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ \sin(x) = 0 \end{cases}$$

critical pts are $= (k\cdot\pi, 0) \quad k \in \mathbb{Z}$.



Basins of attraction:

For each critical point P_k $\exists$ a domain $B_k \subseteq$ phase portrait s.t. $X(0) \in B_k$ then $X(t) \rightarrow P_k$.



3) $\frac{dy}{dx} = \frac{-\gamma y - w^2 \sin x}{y} = -\gamma - w^2 \frac{\sin x}{y}$

$\gamma = 0 \overset{\text{separable}}{\Rightarrow} \int y\,dy = \int -w^2 \sin x\,dx$

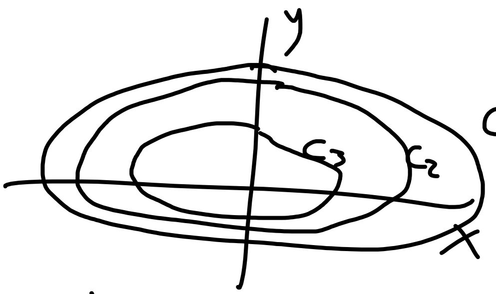
$\Rightarrow y^2 = w^2 \cos(x) + C$

$y\,dy + (\gamma \cdot y + w^2 \sin x)\,dx = 0$

$N = y \quad , \; M = \gamma y + w^2 \sin x \qquad \frac{M_y - N_x}{N} = \frac{\gamma - 0}{y}$

$\frac{dx}{dt} = 2y \quad , \; \frac{dy}{dt} = -8x$

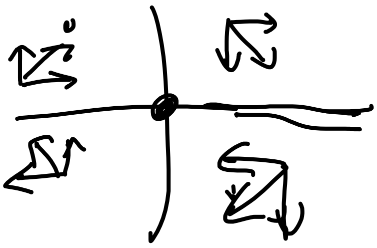
$\frac{dy}{dx} = \frac{-8x}{2y} \Rightarrow y^2 = -4x^2 + C$



$y^2 + 4x^2 = \text{constant}$

<u>nullcline analysis</u>

$\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \Rightarrow y = 0$ (x-axis)
$x = 0$ (y-axis)



1) $x' > 0, y' > 0$

$\Rightarrow 2y > 0, -8x > 0$

$\Rightarrow y > 0, x < 0$

3) $x' < 0, y' > 0$

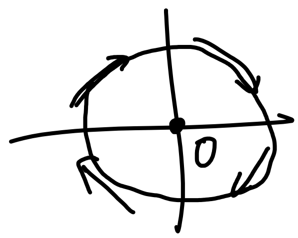
$\Rightarrow y < 0, x < 0$

2) $x' > 0, y' < 0$

$\Rightarrow y > 0, x > 0$

4) $x' < 0, y' < 0$

$\Rightarrow y < 0, x > 0$



$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

we consider function $F = \begin{pmatrix} f \\ g \end{pmatrix}$

By 2d-Taylor expanding around (x_0, y_0) we have

$$F(x,y) = F(x_0,y_0) + J_F(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + O\left(\left\|\begin{matrix} x-x_0 \\ y-y_0 \end{matrix}\right\|^2\right)$$

$y - y_0$

when

$$J_F = \begin{pmatrix} \partial_x F_1 & \partial_y F_1 \\ \partial_x F_2 & \partial_y F_2 \end{pmatrix} = \begin{pmatrix} \partial x f & \partial y f \\ \partial x g & \partial y g \end{pmatrix}$$

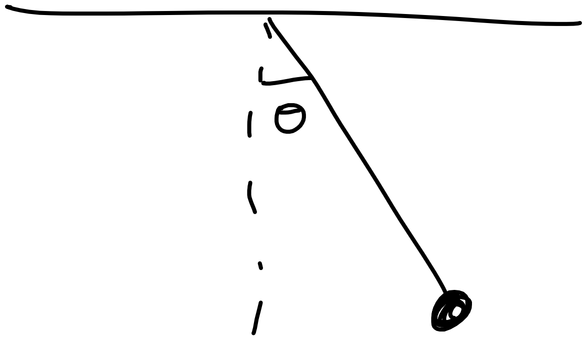
$F(x_0, y_0) = 0$ b/c (x_0, y_0) is a critical point

b/c $F(x_0, y_0) = \begin{pmatrix} x' \\ y' \end{pmatrix}(x_0, y_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

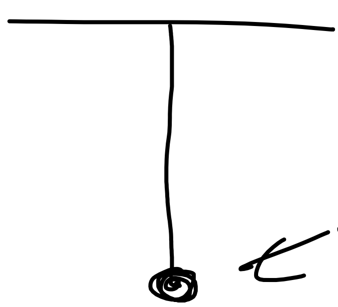
Locally we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = F(x,y) = \underbrace{\begin{pmatrix} \partial x f & \partial y f \\ \partial x g & \partial y g \end{pmatrix}\Bigg|_{(x,y) = (x_0, y_0)}}_{\text{constant coefficient matrix}} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \varepsilon.$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} - A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon$$

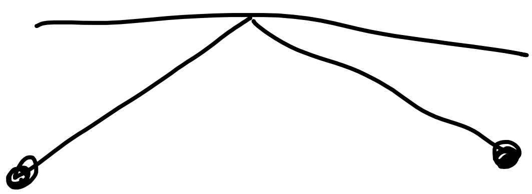


Even $2k\pi$



Stable

odd $k\pi$



unstable

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} y \\ -\gamma y - \omega^2 \sin x \end{pmatrix} =: F(x,y)$$

1) Critical points are $(k\pi, 0)$ $k \in \mathbb{Z}$ $\left(F(k\pi,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$.

2) Taylor expand around arbitrary critical pt (x_0, y_0)

$$F(x,y) = \underbrace{F(x_0,y_0)}_{0} + J_F(x_0,y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|x - x_0, y - y_0\|^2)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = F(x,y) = J_F(k\pi, 0)\begin{pmatrix} x - k\pi \\ y \end{pmatrix} + \text{Correct.}$$

$$J_F(x_0,y_0) = \begin{pmatrix} \partial_x F_1 & \partial_y F_1 \\ \partial_x F_2 & \partial_y F_2 \end{pmatrix}\Bigg|_{(x_0,y_0)} = \begin{pmatrix} \underline{\partial_x(y)} & \underline{\partial_y(y)} \\ \underline{\partial_x(-\gamma y - \omega^2 \sin x)} & \cdot \end{pmatrix}$$

$\partial_y(\underline{-\gamma y - \omega^2 \sin x})$

$$= \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos(x) & -\gamma \end{pmatrix}\Bigg|_{(k\pi, 0)}$$

$\underline{\cos(2k\pi) = 1}$ $\quad$ $\underline{\cos((2k+1)\pi) = -1}$

<u>even k ($\theta = 0$ position)</u>

$$J_F(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}$$

1) Eigen λ eigen

$$\lambda = \frac{Tr(A)}{2} \pm \frac{1}{2}\sqrt{Tr^2(A) - 4 \cdot det(A)}$$
$$= \frac{-\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4\omega^2}}{2}$$

$(C \neq 0)$
$$\xi_1 = \begin{pmatrix} \lambda_1 - d \\ c \end{pmatrix} = \begin{pmatrix} \lambda_1 - \gamma \\ -\omega^2 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \lambda_2 - \gamma \\ -\omega^2 \end{pmatrix}$$

$$X_{1man} = e^{-\gamma/2 t}\left[c_1 e^{\sqrt{\gamma^2 - 4\omega^2} t} \xi_1 + c_2 e^{-\sqrt{\gamma^2 - 4\omega} t} \xi_2\right]$$

$$\lambda = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - 4\omega^2} < \underline{0} \text{ and } \gamma^2 - 4\omega^2 > 0$$

$$\iff \sqrt{\gamma^2 - 4\omega^2} < \gamma \iff \gamma^2 - 4\omega^2 < \gamma^2 \ \checkmark$$

so the behavior's stable

So now we split cases over the sign of $\gamma^2 - 4\omega^2$

In case $\gamma^2 - 4\omega^2 < 0$

$$X = e^{-\gamma/2 t}\left(c_1 e^{\underline{|\gamma^2 - 4\omega^2|^{\frac{1}{2}} i t}} \xi_1 + c_2 e^{-i|\gamma^2 - 4\omega^2|^{\frac{1}{2}} t} \xi_2\right)$$

$x = e^{\cdots}(c_1 e \quad \cdots \quad s_1 + c_2 \cdots$

$= e^{-\gamma/2 t}\left(a_1 \cos\left(\sqrt{\gamma^2-4w^2}\,t\right)v_1 + \sin\left(\sqrt{\gamma^2-4w^2}\,t\right)v_2\right)$

b/c $\gamma > 0$ we get spiral sink.

what if $\gamma = 0$?

odd $k \cdot \pi$

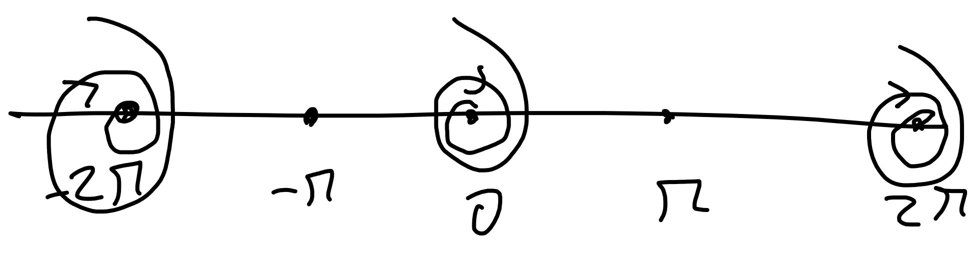
$J_F = \begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix} \quad \lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2+4w^2}$

$\lambda_1 < 0 \quad , \lambda_2 = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2+4w^2} > 0$

$\iff \sqrt{\gamma^2+4w^2} > \gamma$

$\iff \gamma^2+4w^2 > \gamma^2$ ✓

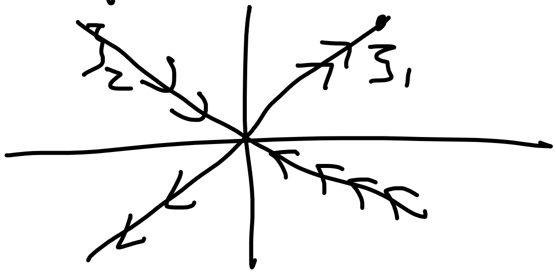
$\gamma^2 - 4w^2 < 0$

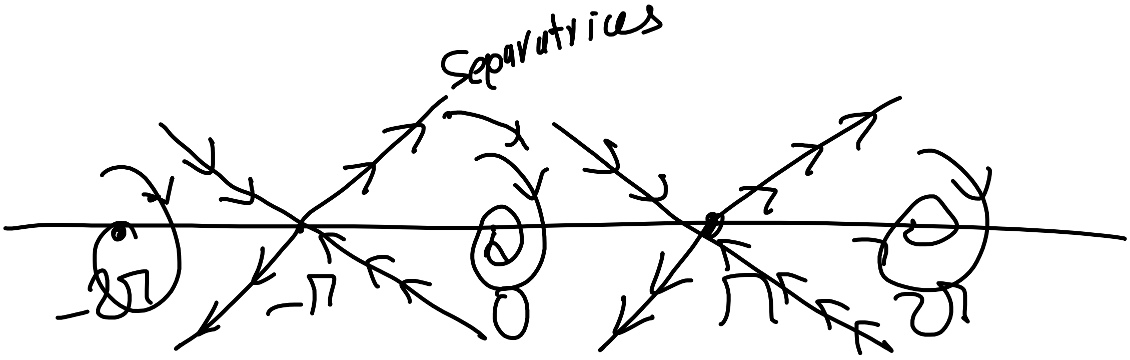


$\xi_1 = \begin{pmatrix} \lambda_1 + \gamma \\ w^2 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} \lambda_2 + \gamma \\ w^2 \end{pmatrix}$

Take $\gamma \ll \varepsilon$ so then $\lambda_1 + \gamma > 0 \quad \lambda_2 + \gamma < 0$

Take $\gamma \ll \varepsilon$ so the [illegible]





$$J_F = \begin{pmatrix} \partial_x y & \partial_y y \\ \partial_x\left(-x+\frac{x^3}{6}\right) & \partial_y\left(-x+\frac{x^3}{6}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1+\frac{x^2}{2} & 0 \end{pmatrix}$$