Outline

1. nonhomogeneous linear first ord
2. Autonomous systems

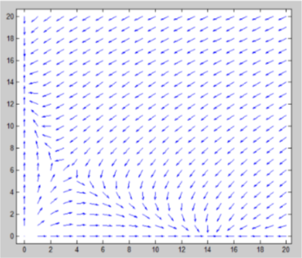

Figure: competing species separatrix

Consider nonhomogeneous linear first order systems:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a vector of continuous functions and $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with eigenvalues $\{\lambda_i\}_{i=1,\dots,n}$. The latter assumption means that if $\mathbf{T}$ has the eigenvectors of $\mathbf{A}$ as columns, then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$ is a diagonal matrix.

Using diagonalization

Plugging in $\mathbf{x} = \mathbf{T}\mathbf{y}$ for some yet unknown $\mathbf{y}$ we obtain

$$\begin{aligned} \mathbf{T}\mathbf{y}' &= \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t) \\ \implies \mathbf{y}' &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t). \end{aligned}$$

As a result, we decoupled the system. From this decoupled system we obtain the first order equations:

$$y_i' = \lambda_i y_i(t) + (\mathbf{T}^{-1}\mathbf{g}(t))_i \quad \text{for } i = 1, ..., n.$$

For $h_i(t) := (\mathbf{T}^{-1}\mathbf{g}(t))_i$ we have (by the method of integrating factors)

$$y_i(t) = e^{\lambda_i t}\left[\int_0^t e^{-\lambda_i s} h_i(s) ds + c_i\right].$$

Therefore, we found the solution $\mathbf{x} = \mathbf{T}\mathbf{y}$.

Presenting the method

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

The eigenpairs (λ, v) are $(-1, \binom{1}{3})$ and $(1, \binom{1}{1})$.

The inverse is $\frac{-1}{2} \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$. That gives $T^{-1}\binom{1}{-1}e^t = \binom{-1}{2}e^t$.

Presenting the method

$$\mathbf{y} = e^t \begin{pmatrix} -1/2 \\ t/2 \end{pmatrix}.$$

Thus,

$$\mathbf{v}(t) = \frac{e^t}{2} \begin{pmatrix} t-1 \\ t-3 \end{pmatrix}.$$

In-class example

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$$

The eigenpairs (λ, v) are $(-2, \binom{1}{0})$ and $(-1, \binom{1}{1})$.

The inverse of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

You should get $T^{-1}\binom{1}{0}e^t = \binom{1}{0}e^t$.

Presenting the method

$$\mathbf{y} = e^t \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}.$$

Thus,

$$\mathbf{v}(t) = e^t \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}.$$

In-class example

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$$

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You should get $T^{-1}\binom{1}{0}e^t = \binom{1}{0}e^t$.

In-class example

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} e^t$$

The eigenpairs (λ, v) are $(3, \binom{1}{2})$ and $(-1, \binom{-1}{2})$.

The inverse of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

You should get $T^{-1}\binom{4}{-2}e^t = \binom{3}{1/2}e^t$. The v(t) should be $e^t\binom{1/2}{-4}$.

As in the 1D case we will study the following system:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = F(x, y), \frac{\mathrm{d}y}{\mathrm{dt}} = G(x, y),$$

where F, G are continuously differentiable functions. Here again we might not be able to obtain explicit solutions, but we can provide a qualitative analysis.

1. First, we find the critical points by setting

$$F(x, y) = 0 \text{ and } G(x, y) = 0.$$

2. Sometimes we can even solve such systems by taking their ratio:

$$\frac{\mathrm{d}y}{\mathrm{dx}} = \frac{\frac{\mathrm{d}y}{\mathrm{dt}}}{\frac{\mathrm{d}x}{\mathrm{dt}}} = \frac{G(x, y)}{F(x, y)}.$$

This ratio depends only on x and y (and not t), so methods from the first order section could be used.

The gravitational force mg acts downward and the damping force $c|\frac{d\theta}{dt}|$ is always opposite to the direction of motion. A rotational analog of Newton's second law of motion might be written in terms of torques:

$$mg \cdot Lsin(\theta) + \frac{\mathrm{d}\theta}{\mathrm{dt}} \cdot L + m\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}}L^2 = 0 \Rightarrow \frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \gamma\frac{\mathrm{d}\theta}{\mathrm{dt}} + \omega^2 sin(\theta) = 0.$$

This is a nonhomogeneous second order equation, but we can also view it as a system of equations by letting $x := \theta$ and $y := \frac{\mathrm{d}\theta}{\mathrm{dt}}$:

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \omega^2 sin(x),$$

where γ is called the damping constant and as in the spring problem it is responsible for removing energy. This is an autonomous system.

In class example

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 2y, \frac{\mathrm{d}y}{\mathrm{dt}} = -8x.$$

In class example

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \omega^2 sin(x),$$

Consider the system (Duffing's equation)

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -x + \frac{x^3}{6}.$$

The End