Outline

1. Locally linear systems

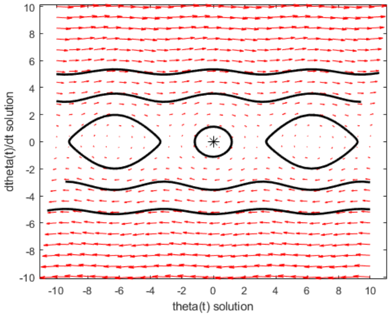

Figure: Phase portrait and solutions for undamped oscillating pendulum.

We will study systems

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where the components of $\mathbf{f}$ are C^1 functions so that we are able to Taylor expand them. The following system

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$$

is called *locally linear* around a critical point $\mathbf{x}_0$ if

$$\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0 \text{ and } as \mathbf{x} \to \mathbf{x}_0.$$

Consider the following oscillating pendulum: a mass m is attached to one end of a rigid, but weightless, rod of length L which hangs from the pivot point.

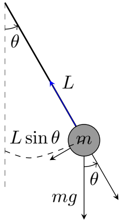

Figure: oscillating pendulum

Presenting the method: oscillating pendulum

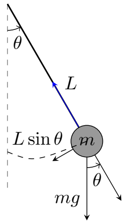

Figure: oscillating pendulum

The gravitational force mg acts downward and the damping force $c|\frac{d\theta}{dt}|$ is always opposite to the direction of motion. A rotational analog of Newton's second law of motion might be written in terms of torques:

$$mg \cdot Lsin(\theta) + \frac{\mathrm{d}\theta}{\mathrm{dt}} \cdot L + m\frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}}L^2 = 0 \Rightarrow \frac{\mathrm{d}^2\theta}{\mathrm{d}^2\mathrm{t}} + \gamma\frac{\mathrm{d}\theta}{\mathrm{dt}} + \omega^2 sin(\theta) = 0.$$

Presenting the method: oscillating pendulum

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This is a nonhomogeneous second order equation, but we can also view it as a system of equations by letting $x := \theta$ and $y := \frac{\mathrm{d}\theta}{\mathrm{dt}}$:

Presenting the method: oscillating pendulum

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$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -\gamma y - \omega^2 sin(x),$$

where γ is called the damping constant and as in the spring problem it is responsible for removing energy. This is an autonomous system.

Presenting the method: oscillating pendulum

(1) we find the critical points:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow y = 0, sin(x) = 0 \Rightarrow (k\pi, 0) \text{ for } k \in \mathbb{Z}.$$

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(2) we set $F = \binom{\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{\mathrm{d}y}{\mathrm{d}t}} = \binom{y}{-\gamma y - \omega^2 sin(x)}$ and 2D-Taylor expand around arbitrary critical point (x_0, y_0):

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(2) we set $F = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{dt}} \\ \frac{\mathrm{d}y}{\mathrm{dt}} \end{pmatrix} = \begin{pmatrix} y \\ -\gamma y - \omega^2 sin(x) \end{pmatrix}$ and 2D-Taylor expand around arbitrary critical point (x_0, y_0):

$$F(x,y) = F(x_0, y_0) + J_F(x_0, y_0) + \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2)$$

$$= \begin{pmatrix} 0 & 1 \\ -\omega^2 cos(x_0) & -\gamma \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + O(\|(x - x_0, y - y_0)\|^2).$$

Presenting the method: oscillating pendulum

(3) The linearization around $(x_0, y_0) = (n \cdot \pi, 0)$ for even integer n (the downward equilibrium position) is:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x - n \cdot \pi \\ y \end{pmatrix} + O(\|(x - n \cdot \pi, y)\|^2).$$

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(4) The eigenvalues of that matrix are:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}.$$

If $\gamma^2 - 4\omega^2 > 0$, then the eigenvalues are real,distinct and negative. Therefore, the critical points will be stable nodes.

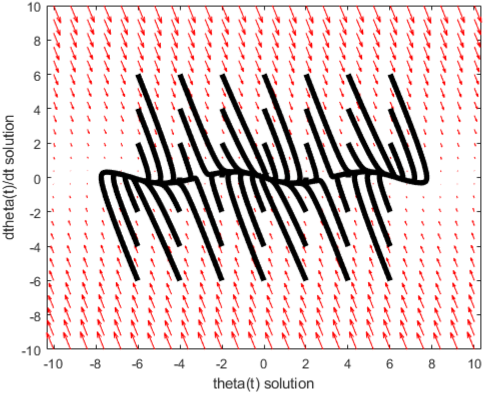

Figure: Stable nodes at even integer n critical points $(n\pi, 0)$ for n=0,2,-2.

Presenting the method: oscillating pendulum

If $\gamma^2 - 4\omega^2 = 0$, then the eigenvalues are repeated, real and negative. Therefore, the critical points will be stable nodes.

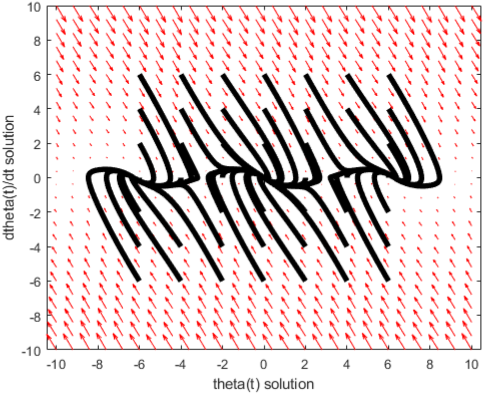

Figure: Stable nodes at even integer n critical points $(n\pi, 0)$.

Presenting the method: oscillating pendulum

If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part. Therefore, the critical points will be stable spiral sinks.

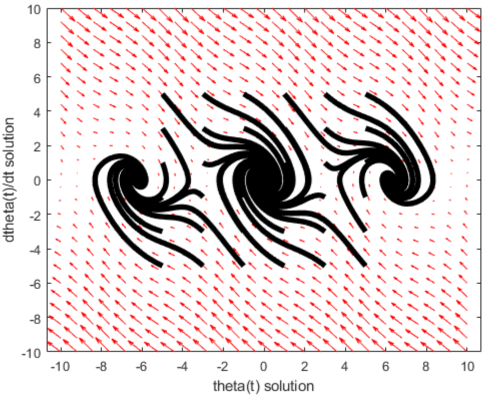

Figure: Stable spiral sinks at even integer n critical points $(n\pi, 0)$.

Presenting the method: oscillating pendulum

The linearization around $(x_0, y_0) = (n \cdot \pi, 0)$ for odd integer n is:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x - n \cdot \pi \\ y \end{pmatrix} + O(\|(x - n \cdot \pi, y)\|^2).$$

Presenting the method: oscillating pendulum

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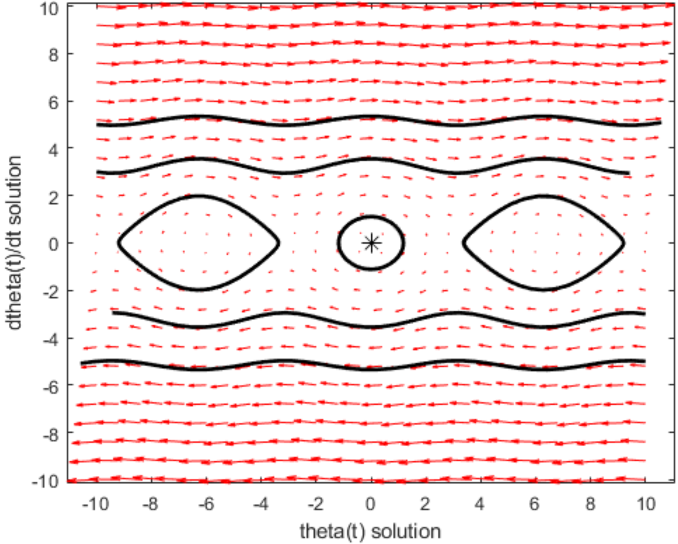
The eigenvalues of that matrix are:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}.$$

Therefore, it has one negative eigenvalue $\lambda_1 < 0$ and one positive eigenvalue $\lambda_2 > 0$, and so the critical points will be unstable saddle points.

Presenting the method: oscillating pendulum



In class example

Consider the system (Duffing's equation)

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = -x + \frac{x^3}{6}.$$

- Find critical points and linearize.
- identify stability behaviour.

In class example

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{dt}} = y, \frac{\mathrm{d}y}{\mathrm{dt}} = x + 2x^3.$$

- Find critical points and linearize.
- identify stability behaviour.
- find implicit solution.

The End