$$\mathcal{L}(e^{at})(s) = \int_0^\infty e^{-st} e^{at} dt \qquad s>a \Rightarrow a-s<0$$

$$= \int_0^\infty e^{(a-s)t}$$

$$= \left[\frac{e^{(a-s)t}}{a-s} \right]_0^\infty$$

$$= \frac{-e^0}{a-s} + \cancel{\lim_{t\to+\infty} \frac{e^{(a-s)t}}{a-s}} \to 0$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}(e^{at})(s) = \frac{1}{s-a}.$$

Turns differential equations into algebraic eqs

$$\mathcal{L}(y')(s) = \int_0^\infty e^{-st} y'(t) dt$$

$$= \left[y(t) e^{-st} \right]_0^\infty - \int_0^\infty y(t) e^{-st} dt \cdot (-s)$$

$$= s\mathcal{L}(y) - y(0)$$

$$\mathcal{L}(y'')(s) = s\mathcal{L}(y') - y'(0)$$

$$= s(s\mathcal{L}(y) - y(0)) - y'(0)$$

$$= s^2 \mathcal{L}(y) - sy(0) - y'(0).$$

2nd order

$$y'' + ay' + by = f(t)$$

1) Laplace transf. of both sides.

$$\mathcal{L}(y'' + ay' + by)(s) = \mathcal{L}(f(t))(s) = \hat{f}(s)$$

$$\Rightarrow s^2\hat{y} - y(0)s - y'(0) + a(s\hat{y} - y(0)) + b\hat{y} = \hat{f}$$

Solve for $\hat{y}$

$$\hat{y}(s^2 + as + b) = \hat{f} + y(0)(s+a) + y'(0)$$

$$\Rightarrow \boxed{\hat{y} = \frac{\hat{f} + y(0)(s+a) + y'(0)}{s^2 + as + b}}$$

$$y'' \cancel{+6} - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

$$1) \quad \cancel{\hat{y} = \frac{0 + y(0)(s+6) + y'(0)}{s^2 + 6s + (-6)} \overset{IC}{=} \frac{s+6-1}{s^2+6s-6} = \frac{s+5}{(s-3)(s+2)}}$$

$$\hat{y} = \frac{y(0)(s-1) + y'(0)}{s^2 - s - 6}$$

$$= \frac{s-2}{(s-3)(s+2)}$$

(*) if $\hat{f} = g_1 + g_2$

$f = \mathcal{L}^{-1}(g_1) + \mathcal{L}^{-1}(g_2)$ eg $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$$\hat{y} = \frac{a}{s-3} + \frac{b}{s+2} = \frac{s-2}{(s-3)(s+2)}$$

$$\Rightarrow (a+b)s + 2a - 3b = s - 2$$

$\Rightarrow a+b=1, \quad 2a-3b=-2$

$$\Rightarrow \hat{y} = \frac{1/5}{s-3} + \frac{4/5}{s+2}$$

$$\Rightarrow y = \mathcal{L}^{-1}(\hat{y}) = \mathcal{L}^{-1}\left(\frac{1/5}{s-3} + \frac{4/5}{s+2}\right)$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + \frac{4}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}.$$

$$y'' - y' - 6y = 0$$

$$r^2 - r - 6 = 0 \Rightarrow r = 3, -2$$

$$y = c_1 e^{3t} + c_2 e^{-2t}$$

$$\left.\begin{array}{l} y(0) = 1 \Rightarrow c_1 + c_2 = 1 \\ y'(0) = -1 \Rightarrow 3c_1 - 2c_2 = -1 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = \frac{1}{5} \\ c_2 = \frac{4}{5}. \end{array}$$

$$y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

$$\boxed{\hat{y} = \frac{0 + 0(s-2) + y'(0)}{s^2 - 2s + 2}} = \frac{1}{s^2 - 2s + 2}$$

$$= \frac{1}{(s-(1+i))(s-(1-i))}$$

$$\frac{a}{s-(1+i)} + \frac{b}{s-(1-i)} = 1$$

$$\Rightarrow (a+b)s - a(1-i) - b(1+i) = 1$$

$$\Rightarrow \begin{cases} a = -b \\ b\left((1-i) - (1+i)\right) = 1 \end{cases} \Rightarrow \begin{array}{l} a = \frac{1}{2i} \\ b = \frac{1}{-2i} \end{array}$$

$$\hat{y} = \frac{1}{2i} \frac{1}{s-(1+i)} + \left(\frac{1}{-2i}\right) \frac{1}{s-(1-i)}$$

$$\Rightarrow y = \frac{1}{\ldots} e^{(1+i)t} + \frac{-1}{\ldots} e^{(1-i)t}$$

$$\Rightarrow y = \frac{1}{2i} e^{-(1+i)t} + \frac{-1}{2i} e^{-(1-i)t}.$$

$$\hat{y} = \frac{\hat{f} + 1}{s^2 - 2s + 2}$$

$$f = e^{-t} \Rightarrow \hat{f} = \int_0^\infty e^{-t} e^{-st} dt = \frac{1}{s+1}.$$

$$\hat{y} = \frac{(1/s+1) + 1}{(s-a_1)(s-a_2)} = \frac{s+2}{(s+1)(s-a_1)(s-a_2)}$$

$$\Rightarrow \frac{s+2}{s+1}\left(\frac{1}{2i}\frac{1}{s-a_1} - \frac{1}{2i}\frac{1}{s-a_2}\right)$$

$$\Rightarrow \frac{s+2}{(s+1)(s-(1+i))}, \quad \frac{s+2}{(s+1)(s-(1-i))}$$

$$(a+b)s + -a(1+i) + b = s+2$$

$$\Rightarrow a+b=1, \quad -a(1+i)+b = 2$$

$$\begin{cases} a = 1-b \\ (b-1)(1+i) + b = 2 \end{cases} \Rightarrow \begin{cases} a = 1-b = \frac{-1}{2+i} \\ b = \frac{2+1+i}{2+i} = \frac{3+i}{2+i} \end{cases}$$

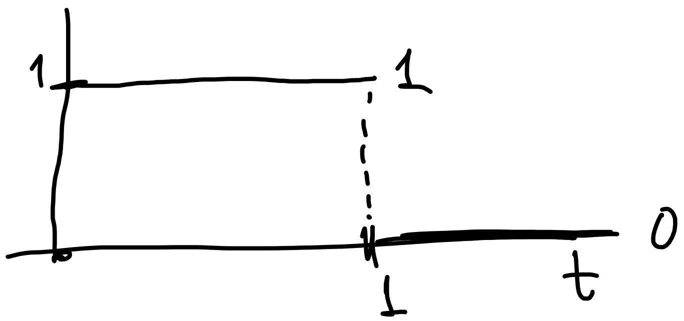
$$2(b-1)(1+i)+b=2 \Rightarrow 2b = \frac{[illegible]}{2+i} = \frac{[illegible]}{2+i}$$

$$\Rightarrow \hat{y} = \frac{a_1}{s+1} + \frac{a_2}{s-(1+i)} + \frac{a_3}{s-(1-i)}$$

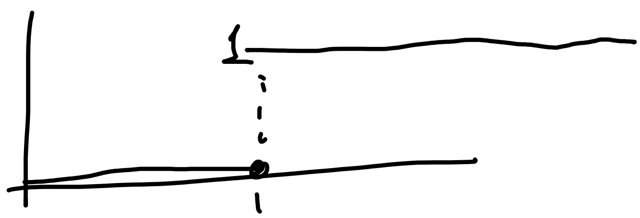
$$\Rightarrow y = a_1 e^{-t} + a_2 e^{-(1+i)t} + e^{-(1-i)t}$$

$$\hat{y} = \frac{\hat{f} + y(0)(s+a) + y'(0)}{s^2+as+b} = \frac{\hat{f}+s}{s^2+1}$$

$$f_{step} = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$



$$f_{Heaviside} = 1 - f_{step}$$



$$\hat{f}_{Stu} = \int_0^\infty f_{stu}(t)\, e^{-st}\, dt = \int_0^1 1 \cdot e^{-st}\, dt + \int_1^\infty 0 \cdot e^{-st}\, dt$$

$$= \int_0^1 e^{-st}\, dt = \frac{1-e^{-s}}{s}$$

[illegible]

$$\mathcal{L}(f_{step}) = \frac{1-e^{-s}}{s} \Leftrightarrow \mathcal{L}^{-1}\left(\frac{1-e^{-s}}{s}\right)$$

$$= f_{step}$$

$$\hat{y} = \frac{\hat{f}_{ste} + s}{s^2+1} = \frac{\frac{1-e^{-s}}{s} + s}{s^2+1}$$

$$= \frac{1-e^{-s}+s^2}{(s^2+1)s}$$

$$\cancel{\underset{s}{=} \frac{1}{(s^2+1)s} - \frac{e^{-s}}{(s^2+1)s}}$$

$$\cancel{\frac{1}{(s^2+1)s} = \frac{a}{s}} \qquad = \frac{1}{s} - \frac{e^{-s}}{(s-i)(s+i)s}$$

$$\frac{1}{(s-i)s} = \frac{a}{s} + \frac{b}{s-i} \Rightarrow (a+b)s - ia = 1$$

$$\Rightarrow a = -\frac{1}{i} = i$$

$$\boxed{\frac{1}{i} = -i}$$

$$a = -b$$

$$\frac{1}{(s-i)(s+i)}, \quad \frac{1}{s(s+i)} \qquad \Rightarrow \begin{cases} a = i \\ b = -i \end{cases}$$

$$\Rightarrow \frac{1}{(s-i)(s+i)} = \frac{a}{s-i} + \frac{b}{s+i}$$

$$\Rightarrow (a+b)=0, \quad (a-b)i=1$$

$$\Rightarrow \begin{cases} a=-b, & \Rightarrow b = \frac{1}{2} i \\ a = \frac{1}{2i} = \frac{-1}{2} i \end{cases}$$

$$\frac{1}{s(s+i)} = \frac{a}{s} + \frac{b}{s+i}$$

$$\Rightarrow (a+b)=0, \quad ia=1, \quad b=i$$

$$Y = \frac{1}{s} - e^{-s}\left(\frac{1}{(s+i)} \cdot \left(i + \frac{i}{2}\right) + \frac{1}{s-i}\left(-\frac{1}{2i} + i\right) + \frac{1}{s}\underline{(-i+i)} \right)$$

$$= \frac{1}{s} + a_1 \frac{e^{-s}}{s+i} + a_2 \frac{e^{-s}}{s-i} + a_3 \frac{e^{-s}}{s}$$

$$\mathcal{L}\left(e^{-a(t-b)} f_{Heavs}(t,b) \right) = \frac{e^{-bs}}{s+a}$$

$$\mathcal{L}(a) = \int_0^\infty e^{-st} a \, dt = a \int_0^\infty e^{-st} dt = a \cdot \frac{1}{s}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(a_1 e^{-s}\frac{1}{s+i}\right) + \mathcal{L}^{-1}\left(a_2 e^{-s}\frac{1}{s-i}\right)$$

$$+ \mathcal{L}^{-1}\left(a_3 e^{-s}\frac{1}{s}\right)$$

$$= 1 + a_1 \cdot e^{-i(t-\cancel{(-1)}1)} f_h(t,1)$$

$$+ a_2 e^{i(t-1)} \quad (a = -i, b = 1)$$

$$\cdot f_{heavy}(t,1)$$

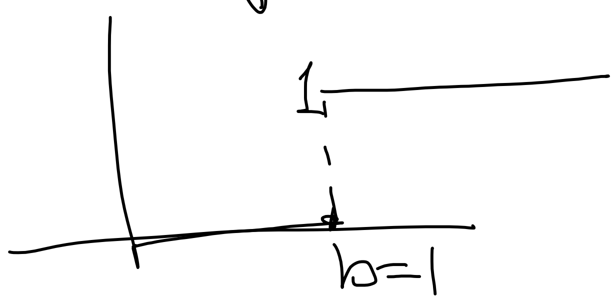
$$+ a_3 e^{0\cdot(t-1)} \quad (a = 0, b = 1$$

$$\cdot f_h(t,1)$$

$$y = 1 + a_1 e^{-i(t-1)} f_h(t,1) + a_2 e^{i(t-1)} f_h(t,1)$$

$$+ a_3 f_h(t,1)$$

$$f_h(t,b) = \begin{cases} 0 & 0 \le t \le b \\ 1 & t > b \end{cases}$$



$$= \begin{cases} 1, & 0 \le t \le 1, \end{cases}$$

$$y'' - y = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases} = f, \quad y(0) = 0, \quad y'(0) = 0$$

1) $$\hat{y} = \frac{\hat{f} + 0 + 0}{s^2 - 1}$$

2) $$\hat{f} = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^1 e^{-st} = \frac{e^{-s}}{s}.$$

3) $$\hat{y} = \frac{e^{-s}}{(s-1)(s+1)\cdot s}$$

$$= e^{-s}\frac{a}{s-1} + e^{-s}\frac{b}{s+1} + e^{-s}\frac{c}{s}$$

$$L\left(e^{-a(t-b)} f_h(t,b)\right) = \frac{e^{-bs}}{s+a}$$

$$y = L^{-1}(\hat{y}) = L^{-1}\left(\frac{e^{-s}a}{s-1}\right) + L^{-1}\left(\frac{e^{-s}b}{s+1}\right) + L^{-1}\left(e^{-s}\frac{c}{s}\right)$$

$$= a\, e^{-(-1)(t-1)} f_h(t,1) + b\, e^{-1(t-1)} f_h(t,1)$$

$-0\cdot(t-1)$

$$= a\, e^{\ldots} \quad f_h(t,1) + b\, e^{\ldots} \quad f_h(t,1)$$
$$+ c\, e^{-0\cdot(t-1)} f_h(t,1)$$

$$= a\, e^{(t-1)} f_h(t,1) + b\, e^{-(t-1)} f_h(t,1) + c\, f_h(t,1)$$

(i) $y'' - y = 1$ for $0 \le t \le 1$

(ii) $y'' - y = 0$ for $t > 1$

1) $r^2 - 1 = 0 \Rightarrow r = \pm 1$

for (i) we have $y = c_1 e^{-t} + c_2 e^{t} + \frac{1}{-1}$

(ii) we have $y = c_1 e^{-t} + c_2 e^{t}$.

From Lapl. Tran.

$$y = c_1 e^{-t} f_h(t,1) + c_2 e^{t} f_h(t,1) + a\, f_h(t,1)$$

$$x' = Ax + g$$

$$\Rightarrow L(x') = A L(x) + L(g)$$

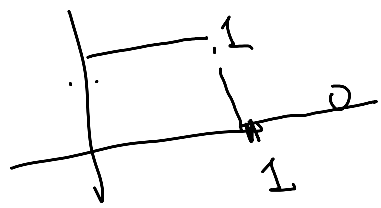
$$\Rightarrow L(x') = A L(x) + L(g)$$

$$\Rightarrow s L(x) - x(0) = A L(x) + L(g)$$

$$\Rightarrow L(x)\,(s \cdot I - A) = L(g)$$

$$\Rightarrow L(x) = \underline{(sI - A)^{-1} L(g)}$$

$$x' = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} f_s \\ t \end{pmatrix}, \quad f_s(t,1) = \begin{cases} 1, \text{ at } t \le 1 \\ 0, \ t > 1 \end{cases}$$



1) Laplace transform of g

$$L(g) = \begin{pmatrix} L(f_s) \\ L(t) \end{pmatrix} = \begin{pmatrix} \frac{1 - e^{-s}}{s} \\ \frac{1}{s^2} \end{pmatrix}.$$

2) $(s \cdot I - A)^{-1} L(g)$

$$= \left(\begin{pmatrix} s-2 & -1 \\ 0 & s-1 \end{pmatrix} \right)^{-1} L(g)$$

$$= \frac{1}{(s-2)(s-1)} \begin{pmatrix} s-1 & 1 \\ 0 & s-2 \end{pmatrix} \begin{pmatrix} \frac{1-e^{-s}}{s} \\ \frac{1}{s^2} \end{pmatrix}$$

$$= \frac{1}{(s-2)(s-1)} \begin{pmatrix} s-1 & 1 \\ 0 & s-2 \end{pmatrix} \begin{pmatrix} \frac{\cdots}{s} \\ \frac{1}{s^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-e^{-s}}{(s-2)s} + \frac{1}{s^2(s-2)(s-1)} \\ \frac{1}{s^2(s-1)} \end{pmatrix}$$

3) Heaviside cover-up method

$$\frac{P(s)}{(s-a_1)\cdots(s-a_n)} = \frac{A_1}{(s-a_1)} + \cdots + \frac{A_n}{(s-a_n)}$$

$$\frac{P(s)}{(s-a_2)\cdots(s-a_n)} = A_1 + \frac{A_2(s-a_1)}{(s-a_2)} + \cdots + \frac{A_n(s-a_1)}{(s-a_n)}$$

$s = a_1$

$$A_1 = \frac{P(a_1)}{(a_1-a_2)\cdots(a_1-a_n)}$$

$$\frac{1}{s^2(s-2)(s-1)} = \frac{1}{s} \frac{1}{s(s-2)(s-1)}$$

$$= \frac{1}{s}\Big(\frac{1}{s(-2)(-1)} + \frac{1}{(s-2)2\cdot 1} + \frac{1}{s-1\,(1)(-1)} \Big)$$

$$= \frac{1}{2s^2} + \frac{1}{2\cdot s(s-2)} + \frac{-1}{s(s-1)}$$

$$= \frac{1}{2s^2} + \frac{1}{2s(-2)} + \frac{1}{(s-2)4} + \frac{1}{s} + \frac{-1}{s-1}$$

$$= \frac{1}{2s^2} + \frac{+3}{4s} + \frac{1}{4(s-2)} - \frac{1}{s-1}$$

$$\mathcal{L}(x_1) = \frac{1-e^{-s}}{-2s} + \frac{1-e^{-s}}{2(s-2)} - \frac{1}{2s^2} - \frac{3}{4s} - \frac{1}{4(s-2)}$$

$$+ \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left(\frac{1-e^{-s}}{s}\right) = f_S(t,1)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s-2}\right) = e^{2(t-1)} f_H(t,1)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$$

$$x' = \begin{pmatrix} -1 & -1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} f_{step} \\ e^t \end{pmatrix}$$

$$x' = \begin{pmatrix} -1 & -1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} f_{step} \\ e^{t} \end{pmatrix}$$

1) $L(f_s) = \dfrac{1-e^{-s}}{s}$, $L(e^{-t}) = \dfrac{1}{s-1}$

2) $(sI-A)^{-1} = \left(\begin{pmatrix} s+1 & 1 \\ 0 & s+3 \end{pmatrix} \right)^{-1}$

$$= \frac{1}{(s+1)(s+3)} \begin{pmatrix} s+3 & -1 \\ 0 & s+1 \end{pmatrix}$$

3) $L(x) = (sI-A)^{-1} L(y) = \dfrac{1}{(s+1)(s+3)} \begin{pmatrix} s+3 & -1 \\ 0 & s+1 \end{pmatrix}$

$$\cdot \begin{pmatrix} \frac{1-e^{-s}}{s} \\ \frac{1}{s-1} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1-e^{-s}}{s(s+1)} - \dfrac{1}{(s+1)(s-1)(s+3)} \\ \dfrac{1}{(s-1)(s+3)} \end{pmatrix}$$

4) $\dfrac{1}{(s+1)(s-1)(s+3)} = \dfrac{1}{(s+1)(-2)\cdot(2)}$

$$+ \frac{1}{(s-1)(2\cdot 4)}$$

$$+ \frac{1}{s+3\,(-2)(\cdot 1)}$$

$$s+3 \quad (-2)(\cdot 1)$$

$$\frac{1}{s(s+1)} = \frac{1}{s+1} + \frac{1}{s(-1)}$$

$$\mathcal{L}(X_1) = \frac{1-e^{-s}}{s+1} + \frac{1-e^{-s}}{-s} + \frac{1}{(s+1)(+4)} + \frac{-1}{(s-1)\cdot 8}$$

$$+ \frac{1}{(s+3)(+2)}$$

$$x_1 = \mathcal{L}^{-1}\left(\frac{1-e^{-s}}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1-e^{-s}}{-s}\right) + \frac{1}{4}e^{-t}$$

$$+ e^{t}\frac{1}{-8} + \frac{1}{2}\cdot e^{-3t}$$

$$= e^{-t} - e^{-(t+1)} f_n(t,1) \cancel{+} f_s(t,1)$$

$$+ \ldots$$

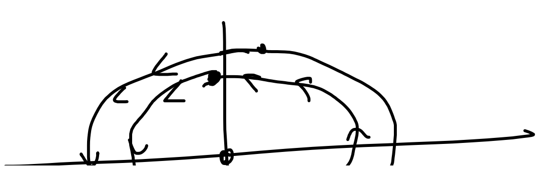
Damping-free pendul

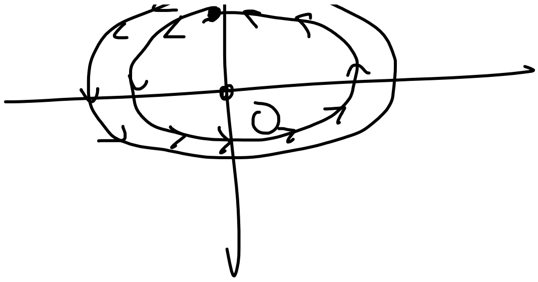
$$\dot{x} = y \quad , \quad \dot{y} = -\frac{g}{L}\sin(x)$$

We look at the Hamiltonian

$$H(x,y) = mgL(1-\cos(x)) + \frac{1}{2}mL^2y^2$$

$$\frac{dH}{dt} = \ldots = 0 \Rightarrow H(x,y) = C$$

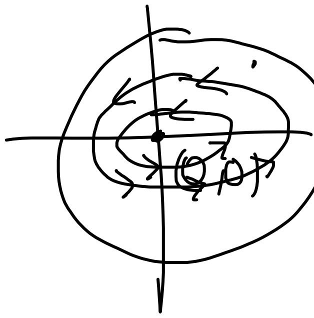




$$\dot{x} = y, \quad \dot{y} = -\gamma y - \frac{g}{l} \sin x$$

$$\frac{dH}{dt} = -\gamma y^2 < 0 \Rightarrow (0,0) \text{ is}$$

Lyapunov stable point



if $|x(0) - x_0| < \delta$

$\Rightarrow |x(t) - x_0| < \varepsilon$

$\forall t > 0$