Contents

0.1 Second order equations

The general form of 2^{nd} order equation is

$$y'' = f(t, y, y').$$

We call them **linear non-homogeneous** if the equation can be written in the form

$$y'' + p(t)y' + q(t)y = g(t)$$

and **linear homogeneous** if, in addition to being linear non-homogeneous, $g(t) = 0$

$$y'' + p(t)y' + q(t)y = 0.$$

The method of characteristic equations is for homogeneous equations and the methods of undetermined coefficients and of variation of parameters for homogeneous equations.

0.2 Method 1: Characteristic equation

If the equation is linear homogeneous and further $p(t), q(t)$ are constant, then the equation is referred to as a **constant-coefficients** equation:

$$ay'' + by' + cy = 0$$

and we can apply the method of characteristic equations to solve such an equation. Note that a is assumed to be non-zero since we are working with a *second* order equation.

Method formal steps

1. We assume that the solution is of the form $y(t) = e^{rt}$ (this is called making an ansatz). This gives
$$(ar^2 + br + c)e^{rt} = 0 \Longrightarrow ar^2 + br + c = 0,$$
which equation is called the **characteristic equation.**
2. So to solve the above ODE, it suffices to find the two roots r_1, r_2.
3. Then the general solution is of the form:
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Example-presenting the method

Consider a mass m hanging at rest on the end of a vertical spring of length l, spring constant k and damping constant γ (as depicted in Figure 0.2.1).

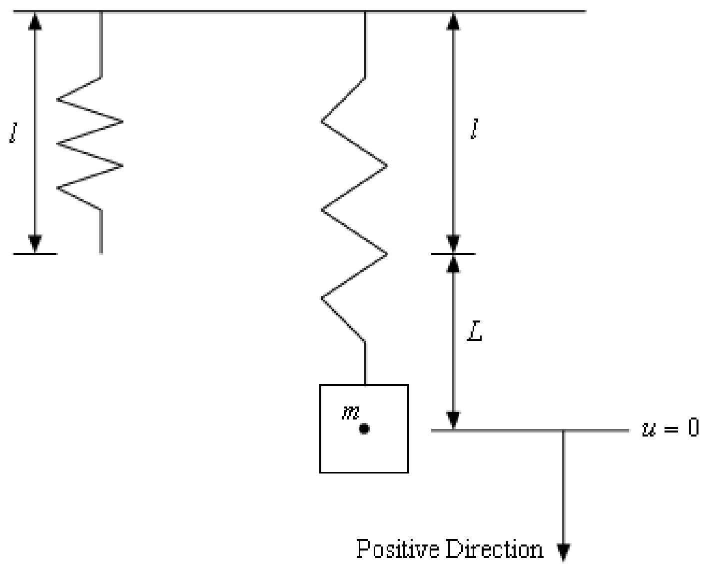


Figure 0.2.1: Spring mass

Let $u(t)$ denote the displacement, in units of feet, from the equilibrium position. Note that since $u(t)$ represents the amount of displacement from the spring's equilibrium position (the position obtained when the downward force of gravity is matched by the will of the spring to not allow the mass to stretch the spring further) then $u(t)$ should increase downward. Then by Newton's Third Law one can obtain the equation
$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$
where $F(t)$ is any external force, which for simplicity we will assume to be zero.

1. First we obtain the characteristic equation:
$$mr^2 + \gamma r + k = 0.$$
2. Suppose that $m = 1\text{lb}, \gamma = 5\text{lb/ft/s}$ and $k = 6\text{lb/ft}$ then we obtain the roots $r_1 = -2$, $r_2 = -3$.

3. Therefore, the general solution will be
$$u(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

4. Further if $u(0) = 0$, $u'(0) = 1$ we obtain $c_1 = 1, c_2 = -1$:
$$u(t) = e^{-2t} - e^{-3t}.$$

Examples

- Consider the IVP
$$4y'' - y = 0,\ y(-2) = 1,\ y'(-2) = -1.$$

  1. We obtain the characteristic equation $4r^2 - 1 = 0 \Rightarrow r = \pm\frac{1}{2}$ and so the general solution will be
  $$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}}.$$
  2. Using the initial conditions we obtain:
  $$1 = c_1 e^{-1} + c_2 e \ \text{ and } \ -1 = \frac{1}{2}(c_1 e^{-1} - c_2 e).$$
  3. Solving these two equations gives: $c_1 = \frac{-1}{2}e, c_2 = \frac{3}{2}e^{-1}$ and so the solution for our IVP is:
  $$y(t) = -\frac{1}{2}e^{1+\frac{t}{2}} + \frac{3}{2}e^{-\frac{t}{2}-1}.$$
  4. Therefore, as $t \to +\infty$ we obtain $y \to -\infty$.

- Consider the IVP
$$y'' + 5y' + 6y = 0,\ y(0) = 2,\ y'(0) = \beta$$

  1. The characteristic equation is $r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3$ and so the general solution will be:
  $$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$
  2. Using the initial conditions we obtain:
  $$2 = c_1 + c_2 \ \text{ and } \ \beta = -2c_1 - 3c_2.$$
  3. Solving these two equations gives: $c_1 = (6 + \beta)$, $c_2 = -(4 + \beta)$ and so the solution for our IVP is:
  $$y(t) = (6 + \beta)e^{-2t} - (4 + \beta)e^{-3t}.$$
  4. Therefore, as $t \to +\infty$ we obtain $y \to 0$.

0.2.1 Wronskian

Now we will show that the general solution of linear homogeneous ode is always of the form:
$$y(t) = c_1 y_1 + c_2 y_2,$$
where the y_i are solutions for the differential equation that satisfy a linear independence condition that is called the **Wronskian**. Then $\{y_1, y_2\}$ will be called the **fundamental solutions** because they can be used to generate all other solutions.

Method formal steps

Consider arbitrary initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

1. Assuming that $y = c_1 y_1 + c_2 y_2$ holds for some choice of c_1, c_2 then we certainly expect that the follow equations will hold:

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0)$$
$$y'_0 = y'(t_0) = c_1 y'_1(t_0) + c_2 y'_2(t_0)$$

   which can be rewritten in matrix form as:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}.$$

   This leads us to studying the matrix

$$W_{matrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{bmatrix}.$$

2. Then we compute the determinant of this matrix, referred to as the *Wronskian*,

$$W = \det(W_{matrix}) = y_1(t_0) y'_2(t_0) - y_2(t_0) y'_1(t_0).$$

3. If it is not zero, then the general solution will be of the form $y = c_1 y_1 + c_2 y_2$ (Although we only found coefficients that allow $c_1 y_1 + c_2 y_2$ to match y when $t = t_0$ it will turn out that both functions agree for all t).

4. If it is zero then these y_1, y_2 will not generate all solutions (In this case it is possible to choose y_0 and y'_0 to make the system have no solutions at t_0. If we can't solve it at t_0 there is no hope for general t).

Example-presenting the method

Going back to the spring example, the characteristic equation is

$$mr^2 + \gamma r + k = 0.$$

Assume that it has two distinct real roots r_1, r_2 and so we can easily check that $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W\left(e^{r_1 t}, e^{r_2 t}, t\right) = e^{(r_1 + r_2)t} \underbrace{(r_2 - r_1)}_{\text{distinct roots}} \neq 0.$$

Therefore, all solutions will be of the form: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for some choice of c_1 and c_2.

General results:

Generalized solution

Suppose that y_1, y_2 are solutions of

$$y'' + p(t)y' + q(t)y = 0.$$

Then the family of solutions

$$y = c_1 y_1 + c_2 y_2$$

for arbitrary c_1, c_2, includes all possible solutions if and only if there is a t_* where the Wronskian of $y_1(t_*), y_2(t_*)$ is not zero.

Proof. Consider general solution $\varphi(t)$ of the above ODE. We will show that there are constants a, b s.t. $\varphi(t) = ay_1 + by_2$. Let t_* be the time for which $W(y_1, y_2, t_*) \neq 0$ and let $K_0 = \varphi(t_*), K_1 = \varphi'(t_*)$. Then

$$\begin{bmatrix} y_1(t_*) & y_2(t_*) \\ y_1'(t_*) & y_2'(t_*) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$$

has a solution $\binom{a}{b}$ because the matrix is invertible. So if $\zeta(t) := ay_1(t) + by_2(t)$ we have $\zeta(t_*) = K_0$, $\zeta'(t_*) = K_1$. Therefore, the existence and uniqueness theorem for 2nd order ODEs gives us $\varphi(t) = \zeta(t) = ay_1(t) + by_2(t)$ for all t. □

In fact if $W(y_1, y_2, t_*) \neq 0$ for one t_* , then the Wronskian is actually never zero for all t s.t. $W(y_1, y_2, t) \neq 0$. This is proved via Abel's identity:

Proposition 0.2.1 (Abel's identity)**.** *Let y_1, y_2 be solutions to*

$$y'' + p(t)y' + q(t)y = 0$$

then for any t_* we can write

$$W(y_1, y_2, t) = W(y_1, y_2, t_*)exp\left\{ -\int_{t_*}^{t} p(s)ds. \right\}$$

Proof. Next we prove Abel's identity which will imply $W(y_1, y_2, t_*) \neq 0 \Rightarrow W(y_1, y_2, t) \neq 0$. Differentiating the Wronskian we obtain

$$W' = y_1 y_2'' - y_1'' y_2.$$

Plugging in the ODE for y_1'' and y_2'' gives

$$\begin{aligned} W' &= y_1(-py_2' - q(t)y_2) - (-py_1' - q(t)y_1)y_2 \\ &= -p(y_1 y_2' - y_1' y_2) \\ &= -pW. \end{aligned}$$

Therefore, we obtain the first order ODE $W' = -p(t)W$ which is solved by

$$W(t) = W(t_*)exp\left\{ -\int_{t_*}^{t} p(s)ds \right\}.$$

□

One application of this is in disproving that two functions y_1, y_2 are the fundamental solutions for some second order linear non-homogeneous constant coefficient ODE. For example, let $y_1 = 1 - t$, $y_2 = t^3$ then their Wronskian is

$$W(y_1, y_2, t) = t^2(3 - 2t)$$

and so $W(y_1, y_2, 0) = 0$ and $W(y_1, y_2, 1) = 1$. Therefore, these y_1, y_2 cannot be solutions to any such ODE (If such an ODE existed then since the Wronskian is non-zero at $t = 1$ then by Abel's identity the Wronskian is nowhere zero. However, the Wronskian is 0 at $t = 0$).

Examples

- Consider the equation $y'' - 2y' + y = 0$ and functions $y_1 := e^t, y_2 := te^t$

  1. One can easily check that both y_1, y_2 solve the above ODE, so now we will check if they are fundamental solutions.
  2. The Wronskian is
  $$W\big(e^t, te^t, t\big) = e^t\big(e^t + te^t\big) - te^{2t} = e^{2t} \neq 0$$
  3. So indeed a general solution for the above ODE is $y = ae^t + bte^t$.

- Consider the ODE $y'' - y' - 2y = 0$ and functions $y_1 := e^{2t}$, $y_2 := -2e^{2t}$.

  1. One can easily check that both solve the ODE.
  2. Their Wronskian is
  $$W\big(e^{2t}, -2e^{2t}, t\big) = -2e^{4t} - \big(-4e^{2t}e^{2t}\big) = 0$$
  and so they do not form a linearly independent set and in turn a fundamental solution.

0.2.2 Complex roots

In some cases the roots are complex (when $b^2 - 4ac < 0$). For example, suppose that there is no damping in the above spring example ($\gamma = 0$), then the equation will be:

$$mu'' + ku = 0.$$

Therefore, the roots will be $r = \pm\sqrt{-k/m} = \pm i\sqrt{k/m} =: \pm i\omega$, where we define $i := \sqrt{-1}$ called the imaginary unit as well as $\omega = \sqrt{\frac{k}{m}}$. The main result we will need is **Euler's formula**

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

Here we can easily check that $y_1(t) = \cos(\omega t)$ and $y_2(t) = \sin(\omega t)$ are both solutions for this ODE. Now by computing the Wronskian we will check whether all possible solutions are of that form:

$$W(\cos(\omega t), \sin(\omega t), t) = \omega\cos^2(\omega t) + \omega\sin^2(\omega t) = \omega \neq 0.$$

This is to be expected since we don't imagine that the imaginary part of this solution (i.e $\sin(\omega t)$) interferes with the real part of this solution (i.e $\cos(\omega t)$ and so they should be independent. Therefore, all solutions will be of the form: $y = c_1\cos(\omega t) + c_2\sin(\omega t)$, where c_i could be complex constants. Physically this periodicity is expected because there is no external force or damping to remove energy from the spring and so it can keep oscillating forever.

Examples

- Consider the equation $y'' + y = 0,\ y(\pi/3) = 2,\ y'(\pi/3) = -4$

  1. The roots are $r^2 + 1 = 0 \Rightarrow r = \pm i$ and so the general solution is (with $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$)
  $$y(t) = c_1e^{it} + c_2e^{-it} = a_1\cos(t) + a_2\sin(t).$$
  2. Using the initial conditions we obtain:
  $$2 = a_1\frac{1}{2} + a_2\frac{\sqrt{3}}{2} \text{ and } -4 = -a_1\frac{\sqrt{3}}{2} + a_2\frac{1}{2}.$$

3. Solving these two equations gives: $a_1 = (1+2\sqrt{3})$, $a_2 = -(2-\sqrt{3})$ and so the solution for our IVP is:
$$y(t) = (1 + 2\sqrt{3})\cos(t) - (2 - \sqrt{3})\sin(t).$$

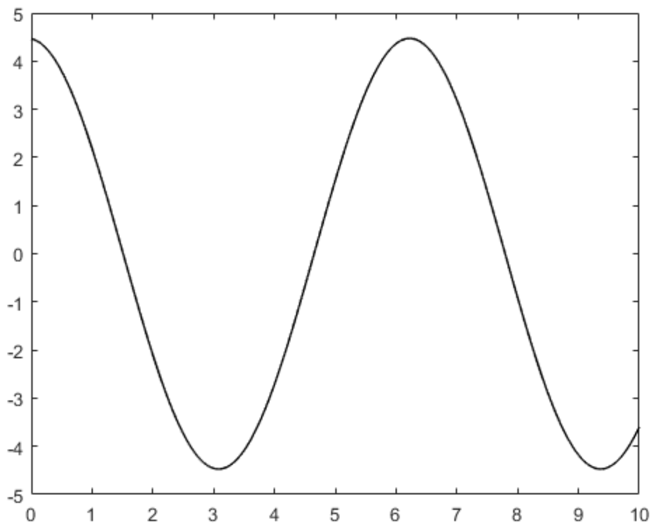


Figure 0.2.2: Spring mass

4. So as $t \to \infty$ the system simply keeps oscillating steadily (depicted in Figure 0.2.2). Physically this is because it is damping free $\gamma = 0$.

- Consider the equation $y'' - 2y' + 5y = 0,\ y(\pi/2) = 0,\ y'(\pi/2) = 2$

1. The roots are $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and so the general solution is (with $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$)
$$y(t) = c_1 e^{t(1+2i)} + c_2 e^{t(1-2i)} = e^t(a_1 \cos(2t) + a_2 \sin(2t)).$$
2. Using the initial conditions we obtain:
$$0 = e^{\frac{\pi}{2}}(a_1 \cdot (-1) + a_2 \cdot 0) \ \text{ and } \ 2 = e^{\frac{\pi}{2}}(a_1 \cdot (-1) + a_2 \cdot (-2)).$$
3. Solving these two equations gives: $a_1 = 0$, $a_2 = -e^{-\pi/2}$ and so the solution for our IVP is:
$$y(t) = -e^{t-\pi/2}\sin(2t).$$

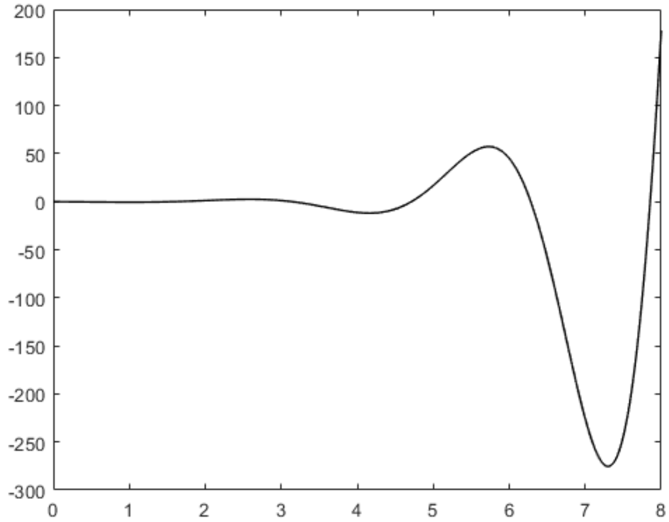


Figure 0.2.3: Spring mass

4. So as $t \to \infty$ the system simply keeps oscillating with increasing amplitude. Physically this is because the damping is negeative $\gamma = -2 < 0$ and so instead of removing energy, it adds.

0.2.3 Repeated roots

In some cases the roots are equal (when $b^2 - 4ac = 0$). For example, suppose that $\gamma^2 \approx 4km$ (called **critically damped**), then the roots will be

$$r_1 = r_2 = -\frac{\gamma}{2m} =: r.$$

This only gives one solution $y_1 = e^{rt}$, but to find the general one we require a second solution y_2 that is linearly independent: $W(y_1, y_2, t_*) \neq 0$ for some t_*. It turns out (proved below) that $y_2(t) := te^{rt}$ is such a function:

$$W(y_1, y_2, t) = e^{rt}\big(e^{rt} + rte^{rt}\big) - re^{rt}te^{rt} = e^{2rt} \neq 0.$$

Example

- Consider the IVP
$$y'' - 2y + 2 = 0,\ y(0) = 1,\ y'(0) = 2$$

  1. The root of the characteristic equation is $r = 1$ and so the two solutions are $y_1 = e^t$, $y_2 = te^t$. Thus, the general solution will be of the form
  $$y = ae^t + bte^t.$$
  2. The initial conditions give $1 = a$, $2 = a + b \Rightarrow a = 1$, $b = 1$ and so the solution satisfying these conditions is
  $$y = e^t + te^t.$$
  3. This solution goes to infinity as $t \to +\infty$.

- Consider the IVP
$$y'' - 6y' + 9y = 0,\ y(0) = 0,\ y'(0) = 2$$

1. The root is $r = 3$ and so the independent solutions are $y_1 = e^{3t}, y_2 = te^{3t}$. Thus, the general solution will be
$$y = ae^{3t} + bte^{3t}.$$
2. The initial conditions give $0 = a$, $2 = 3a + b \Rightarrow a = 0$, $b = 2$ and so the solution satisfying these conditions is
$$y = 2te^{t}.$$
3. This solution goes to infinity as $t \to +\infty$.

General result

Repeated root

If the ODE $ay'' + by' + cy = 0$ has a characteristic equation with repeated root $r := \frac{-b}{2a}$, then its general solution is of the form:
$$y = c_1 e^{rt} + c_2 te^{rt}.$$

Proof. For $y_2 := g(t)y_1$ we will first find which ODE $g(t)$ must satisfy in order that y_2 is a solution of our ODE.
$$\begin{aligned} &a(g(t)y_1)'' + b(g(t)y_1)' + c = 0 \\ \Rightarrow &a\big(g''(t)e^{rt} + 2g'(t)re^{rt}\big) + bg'e^{rt} = 0 \end{aligned}$$
where we used that y_1 satisfies the ODE $ay'' + by" + cy = 0$
$$0 = a(g''(t) + g'(t)(2ar + b)) = ag''(t) + g'(t)\left(2a\frac{-b}{2a} + b\right) = ag''$$
$$\begin{aligned} \Rightarrow &ag'' = 0 \\ \Rightarrow &g = c_1 + c_2 t. \end{aligned}$$
We conclude that
$$y_2(t) = (c_1 + c_2 t)e^{rt} = c_1 e^{rt} + c_2 te^{rt}.$$
Since we are interested in finding an independent solution (so that we can find the general solution) we may as well take $c_1 = 0$ and $c_2 = 1$ since for any $a, b \in \mathbb{R}$ we have
$$ae^{rt} + b\big(c_1 e^{rt} + c_2 te^{rt}\big) = (a + bc_1)e^{rt} + bc_2 te^{rt}.$$
That is, any linear combination of the solutions e^{rt} and $c_1 e^{rt} + c_2 te^{rt}$ can be generated by e^{rt} and te^{rt} by a different set of coefficients. The opposite is also true. We conclude that the candidates for fundamental solutions are e^{rt} and te^{rt}. As shown earlier, by means of a Wronskian computation, these solutions are independent. Thus, the general solution is of the form
$$y = d_1 e^{rt} + d_2 te^{rt}$$
□

0.2.4 Stability

Consider nonhomogeneous equation of the form
$$y'' + ay' + by = c,$$

where a, b, c are constants. If we have a solution y_h for the homogeneous problem, then we can construct a solution for the nonhomogeneous problem:

$$y = y_h + \frac{c}{b}.$$

The solution $y_s := \frac{c}{b}$ is called **globally stable** when for all solutions y we have $y \to y_s$ as $t \to +\infty$, which is equivalent to saying $y_h \to 0$ as $t \to +\infty$.

Method formal steps

1. If the characteristic equation has two real distinct roots r_1, r_2 then the general solution is

$$y_h = c_1 e^{r_1} + c_2 e^{r_2 t}$$

   and so y_s is stable iff $r_1, r_2 < 0$.

2. If the characteristic equation has two complex roots r_1, r_2 then $\left(\text{if we let } \beta = \frac{\sqrt{4b-a^2}}{2})\right)$

$$y_h = e^{-\frac{at}{2}}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

   and so y_s is stable iff $a > 0$.

3. If the characteristic equation has a double root $r = r_1 = r_2$ then

$$y_h = e^{rt}(c_1 + c_2 t)$$

   and so y_s is stable iff $r < 0$.

In fact, as we will prove below it suffices to check whether the coefficients a, b are positive.

Example-Presenting the method

- **[CW]** In the price adjustment example (from the section on linear integrating factor and autonomous dynamics), we assumed that the demand and supply are functions of the price alone:

$$D(P) = a - bP \text{ and } S(P) = \alpha + \beta P.$$

  However, buyers may also base their behavior on whether the price is increasing or decreasing. For example, if the price of newer versions of a phone brand have been increasing steadily or in an accelerating manner, they may decide to switch to another brand. So the demand will also be a function of the derivative of the price P' (growth) and the second derivative of the price P'' (steady or accelerating growth).
  To keep things simple we will consider the following updated models:

$$D(P) = a - bP + mP' + nP'' \text{ and } S(P) = \alpha + \beta P + uP' + wP'',$$

  where $a, b, \alpha, \beta > 0$ and m, n, u, w can be any sign. For now we will study it from the buyers perspective and set $u = w = 0$. To obtain an ODE for it, we assume that the market is cleared and thus $D(P) = S(P)$.

$$a - bP + mP' + nP'' = \alpha + \beta P \Rightarrow P'' + \frac{m}{n}P' - \frac{b+\beta}{n}P = \frac{\alpha - a}{n}.$$

1. Once we obtain the solution y_h of the homogeneous problem:

$$P'' + \frac{m}{n}P' - \frac{b+\beta}{n}P = 0$$

then for the above unhomogeneous ode the solution will simply be

$$y := y_h + \left(-\frac{b+\beta}{n}\right)^{-1}(\frac{\alpha - a}{n}) = y_h + \frac{a-\alpha}{b+\beta}.$$

2. The characteristic equation is:

$$r^2 + \frac{m}{n}r - \frac{b+\beta}{n} = 0.$$

Its roots are:

$$r_{1,2} = -\frac{m}{2n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}}.$$

(a) If $\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n} > 0$ then we obtain two distinct real roots r_1, r_2 and the solution will be

$$P = c_1 exp\left\{\left(-\frac{m}{2n} - \frac{1}{2}\sqrt{\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}}\right)t\right\}$$
$$+ c_2 exp\left\{\left(-\frac{m}{2n} + \frac{1}{2}\sqrt{\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}}\right)t\right\} + \frac{a-\alpha}{b+\beta}.$$

Since $\frac{b+\beta}{n} < 0 \Leftrightarrow n < 0$, then if $n < 0$ and $m < 0$ then both roots are negative and thus as $t \to +\infty$ we obtain:

$$P = c_1 e^{r_1 t} + c_1 e^{r_2 t} + \frac{a-\alpha}{b+\beta} \xrightarrow{t \to +\infty} \frac{a-\alpha}{b+\beta}.$$

Intuitively this means that when the demand $D(P)$ depends negatively on P'' ($n < 0$), the buyer will be averse to accelerating prices and so demand will not rise but simply converge to the equilibrium price $\frac{a-\alpha}{b+\beta}$.
For example, suppose $a = 42, b = 1, \alpha = -6, \beta = 1, m = -4, n = -1$, then our ODE will be:

$$P'' + 4P' + 2P = 48$$

and the equilibrium will be $\frac{a-\alpha}{b+\beta} = 24$. Solving this ODE with $P(0) = 1,\ P'(0) = 0$ gives us:

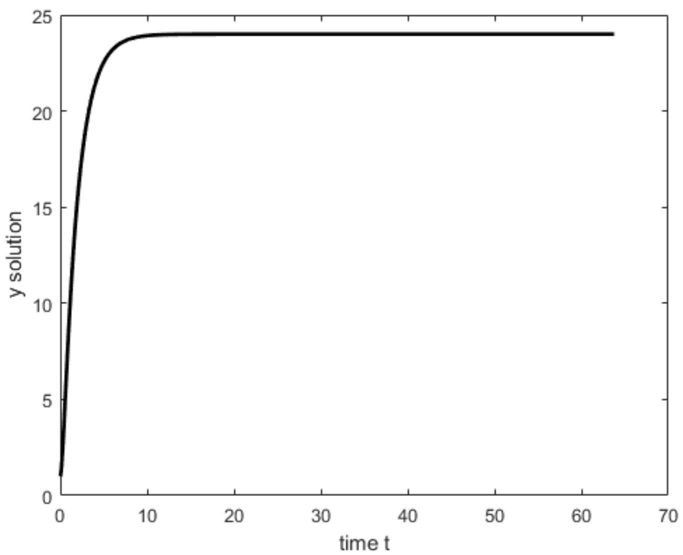


Figure 0.2.4: The solution stabilizes around the equilibrium price $\frac{\alpha-a}{b+\beta} = 24$

This agrees with the result below, namely the coefficients $\frac{m}{n}, -\frac{b+\beta}{n}$ are both positive and so the y_s is globally stable.

(b) If $\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n} < 0$ then we obtain two distinct complex roots r_1, r_2 and the solution will be (with $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$)

$$P = c_1 e^{-\frac{m}{2n}t} exp\left\{-\frac{i}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|})t\right\}$$
$$+ c_2 e^{-\frac{m}{2n}t} exp\left\{\frac{i}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|})t\right\} + \frac{\alpha-a}{b+\beta}$$
$$= e^{-\frac{m}{2n}t}\left[a_1 \cos\left(\frac{t}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|}\right) + a_2 \sin\left(\frac{t}{2}\sqrt{\left|\left(\frac{m}{n}\right)^2 + 4\frac{b+\beta}{n}\right|}\right)\right] + \frac{a-\alpha}{b+\beta}.$$

So this solution will diverge or go to zero depending on whether $\frac{m}{n} > 0$ or $\frac{m}{n} < 0$ respectively. For example, suppose $a = 40, b = 2, m = -2, \alpha = -5, \beta = 3, n = -1$ then our ODE will be:

$$P'' + 2P' + 5P = 45$$

and for $m = 2$

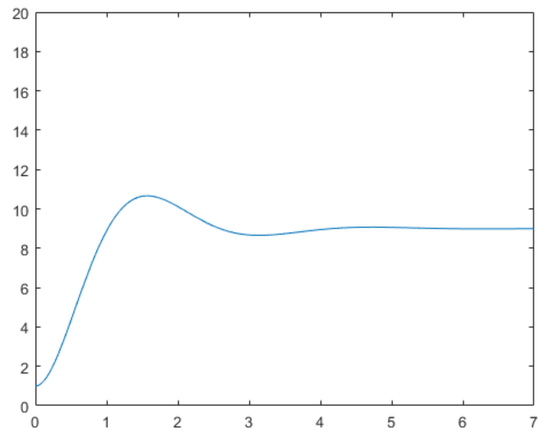
$$P'' - 2P' + 5P = 45.$$

The corresponding solutions will be

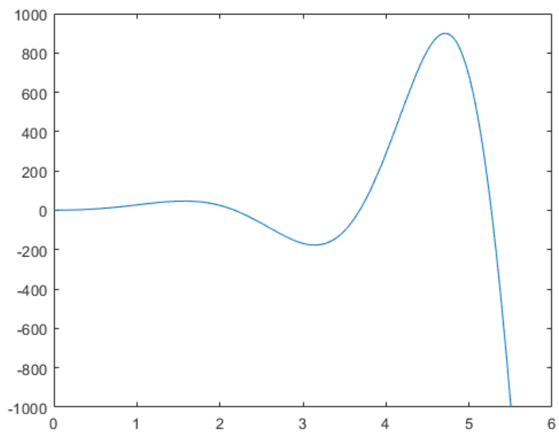
$$P(t) = e^{-t}[a_1 \cos(2t) + a_2 \sin(2t)] + 9$$

and

$$P(t) = e^{t}[a_1 \cos(2t) + a_2 \sin(2t)] + 9.$$



(a) The solution stabilizes around the equilibrium price $\frac{a-\alpha}{b+\beta} = 9$



(b) The solution oscillates with increasing amplitude.

As expected from the result below the first ODE has globally stable solution due to the positivity of the coefficients whereas the second ODE does not.

Intuitively, when $m = -2 < 0$, the demand $D(P)$ will depend negatively on growing price P' and so the price will have to drop to market equilibrium. When $m = 2 > 0$, the buyer will not stop even if the price is growing eg. for vital goods such as bread, and so the price is free to keep growing. But why is it oscillating? This is because the condition $(\frac{m}{n})^2 + 4\frac{b+\beta}{n} < 0$ forces that $n < 0$ and so the buyer will always be averse to accelerating growth in the price, which in turn causes the downturns in price.

(c) If $(\frac{m}{n})^2 + 4\frac{b+\beta}{n} = 0$ then we obtain one double root $r = r_1 = r_2$ and the solution will be

$$P = c_1 e^{-\frac{m}{2n}t} + c_2 t e^{-\frac{m}{2n}t} + \frac{a - \alpha}{b + \beta}.$$

Similarly, depending on the sign of m, the solutions will either diverge $(m > 0)$ or converge to market equilibrium price $(m < 0)$.

General solution for non-homogeneous and Stability for second order

Stability criterion for second order nonhomogeneous ODEs

Consider non-homogeneous equation

$$y'' + ay' + by = f(t).$$

Then the solution is of the form $y = y_h + y_s$, where y_h solves the homogeneous problem and y_s is any solution of the nonhomogeneous problem. We have that

$$\lim_{t\to\infty} y = y_s \text{ iff } a > 0, b > 0.$$

In other words, y_s is globally stable iff $a > 0, b > 0$ iff the real parts of the roots of the characteristic equation are both negative.

Proof. As with constant $f(t)$, we again obtain that the generalized solution is of the form

$$y = c_1 y_1 + c_2 y_2 + y_s =: y_h + y_s.$$

So by studying when $y_h \to 0$, we can identify when y_s is the globally stable solution.

1. If the characteristic equation has two real distinct roots r_1, r_2 then

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

and so y_s is stable iff $r_1, r_2 < 0$. The roots are

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

We have $r_1 < 0 \Leftrightarrow a > 0, b > 0$ and $r_2 < 0 \Leftrightarrow a > 0$. So to have both conditions we must require $a > 0$ and $b > 0$.

2. If the characteristic equation has two complex roots r_1, r_2 then $\left(\text{for } \beta = \frac{\sqrt{4b - a^2}}{2}\right)$

$$y_h = e^{-\frac{at}{2}}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

and so y_s is stable iff $a > 0$. The condition $b > 0$ follows from $a^2 < 4b$.

3. If the characteristic equation has a double root $r = r_1 = r_2$ then

$$y_h = e^{rt}(c_1 + c_2 t)$$

and so y_s is stable iff $r = \frac{-a}{2} < 0 \Rightarrow a > 0$. The condition $b > 0$ follows from $a^2 = 4b$. □

Examples

- Solve the IVP and determine long term behaviour

$$y'' + y = 9,\ \ y(\pi/3) = 2,\ \ y'(\pi/3) = -4$$

  1. As showed in the complex roots section the solution to the homogeneous problem is:

$$y_h(t) = \left(1 + 2\sqrt{3}\right)\cos(t) - \left(2 - \sqrt{3}\right)\sin(t).$$

2. So the solution to our problem is:
$$y = y_h + 9.$$
3. However, y_h will keep oscillating steadily around the constant solution 9.

- Solve the IVP and determine long term behaviour
$$y'' + 5y' + 6y = 3, y(0) = 2, y'(0) = 1$$
1. The solution to the homogeneous problem is:
$$y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$
2. The general solution to our problem is:
$$y = y_h + \frac{3}{6} = y_h + \frac{1}{2}.$$
3. So the particular one for our IC is:
$$y = \frac{11}{2} e^{-2t} - 8e^{-3t} + \frac{1}{2}$$
4. Therefore, y will converge to the constant solution $y_s \equiv \frac{1}{2}$.

0.3 Method 2: Undetermined coefficients

We will now consider non-homogeneous equations with constant coefficients of the form
$$ay'' + by' + cy = f(t).$$
By managing to find a particular solution y_{nh}, then we can generate every other one. Let v be any another solution, then
$$a(v - y_{nh})'' + b(v - y_{nh})' + c(v - y_{nh}) = f(t) - f(t) = 0.$$
Therefore, by finding the fundamental set of solutions y_1, y_2 for the homogeneous problem we have
$$v - y_{nh} = c_1 y_1 + c_2 y_2 \Rightarrow v = y_{nh} + c_1 y_1 + c_2 y_2.$$
So we managed to generate any solution starting from y_{nh}, y_1, y_2. Here we will find y_{nh} for $f(t)$ of the following possible forms:
$$f_1(t) := Ct^m e^{r_* t},\ f_2(t) := Ct^m e^{\alpha} \cos(\beta t),\ f_3(t) := Ct^m e^{\alpha t} \sin(\beta t).$$
In fact once we obtain solutions $y_i, i = 1, 2, 3$ for them, we also obtain solutions for their sums. For example, consider the equation
$$ay'' + by' + cy = t^m e^{r_* t} + \sin(\beta t).$$
Observe that if y_1, y_2 solve
$$ay'' + by' + cy = t^m e^{r_* t}$$
$$ay'' by' + cy = \sin(\beta t)$$
respectively then their sum, $y_1 + y_2$, solves
$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = t^m e^{r_* t} + \sin(\beta t).$$

Method formal steps

1. If $f = Ct^m e^{r_* t}$ then we make the ansatz (assume the solution to be of the form)

$$y_{nh}(t) = t^s(a_0 + a_1 t + ... + a_m t^m)e^{r_* t}.$$

   Now the way we pick the exponent s, depends on whether or not r_* is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof 0.3below.

   (a) If r_* is not a root, then we set $s := 0$.

   (b) If r_* is a simple root, then we set $s := 1$.

   (c) If r_* is a double root, then we set $s := 2$.

2. If $f = Ct^m e^{\alpha} \cos(\beta t)$ or $ct^m e^{\alpha t} \sin(t)$ then we make the ansatz

$$y_{nh}(t) = t^s e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m) \cos(\beta t) + (b_0 + a_1 t + ... + b_m t^m) \sin(\beta t)].$$

   Now the way we pick the exponent s, depends on whether or not $\alpha + i\beta$ is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.

   (a) If $\alpha + i\beta$ is not a root, then we set $s := 0$.

   (b) If $\alpha + i\beta$ is a root, then we set $s := 1$.

Example-presenting the method

Resuming the spring example, let $u(t)$ denote the displacement from the equilibrium position. Then by Newton's Third Law one can obtain the equation

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where $F(t)$ is any external force. Above we assumed that $F(t) = 0$, and now we will take it to be any of the above mentioned functions. For example, consider the equation

$$y'' + 3y' + 2y = \sin(t).$$

Here we are shaking the spring system periodically in time.

1. First we are in the second case and so we make the ansatz

$$y_{nh}(t) = t^s e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m) \cos(\beta t) + (b_0 + a_1 t + ... + b_m t^m) \sin(\beta t)]$$

   which simplifies because $m = 0, \alpha = 0$ and $\beta = 1$:

$$y_{nh}(t) = t^s(a_0 \cos(t) + b_0 \sin(t)).$$

2. Next we pick s, depending on whether $a + i\beta = i$ is a root of our ODE's characteristic equation:

$$r^2 + 3r + 2 = 0.$$

3. Its roots are $r_1 = -2$, $r_2 = -1$ and so we set $s = 0$ and have

$$y_{nh}(t) = a_0 \cos(t) + b_0 \sin(t).$$

4. Plugging into our ODE we obtain

$$y'' + 3y' + 2y = -(a_0 \cos(t) + b_0 \sin(t)) + 3(-a_0 \sin(t) + b_0 \cos(t)) + 2(a_0 \cos(t) + b_0 \sin(t))$$

$$= (a_0 + 3b_0)\cos(t) + (-3a_0 + b_0)\sin(t)$$

and so to have this be equal to $\sin(t)$ we require

$$\begin{cases} a_0 + 3b_0 & = 0 \\ -3a_0 + b_0 & = 1 \end{cases} \implies a_0 = -0.3,\ b_0 = 0.1.$$

5. So the solution will be
$$y_{nh}(t) = -0.3\cos(t) + 0.1\sin(t).$$

6. Therefore, the general solution will be:
$$y = y_{nh} + c_1 e^{-2t} + c_2 e^{-t}.$$

7. But why is it periodic given that damping is involved ($\gamma \neq 0$)? The sinusoidal external force keeps pumping energy into the system.

General result:

Method of Undetermined coefficients

Consider equations
$$ay'' + by' + cy = f(t),$$
where $f(t)$ has the following possible forms:

$$f_1(t) := Ct^m e^{r_* t}, f_2(t) := Ct^m e^{\alpha} \cos(\beta t), f_3(t) := Ct^m e^{\alpha t} \sin(\beta t).$$

Then their corresponding solutions are of the form:

- If $f = Ct^m e^{r_* t}$ then we make the ansatz (assume the solution to be of the form)
$$y_{nh}(t) = t^s(a_0 + a_1 t + ... + a_m t^m)e^{r_* t}.$$
Now the way we pick the exponent s, depends on whether or not r_* is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.
  1. If r_* is not a root, then we set $s := 0$.
  2. If r_* is a simple root, then we set $s := 1$.
  3. If r_* is a double root, then we set $s := 2$.

- If $f = ct^m e^{\alpha} \cos(\beta t)$ or $ct^m e^{\alpha t} \sin(\beta t)$ then we make the ansatz
$$y_{nh}(t) = t^s e^{\alpha t}[(a_0 + a_1 t + ... + a_m t^m)\cos(\beta t) + (b_0 + a_1 t + ... + b_m t^m)\sin(\beta t)].$$
Now the way we pick the exponent s, depends on whether or not $\alpha + i\beta$ is a root of the characteristic equation of our ODE. The reason for this can be seen in the proof below.
  1. If $\alpha + i\beta$ is not a root, then we set $s := 0$.
  2. If $\alpha + i\beta$ is a root, then we set $s := 1$.

Proof. First we will work with

$$ay'' + by' + cy = Ct^m e^{r_* t}.$$

We assume the solution is of the form:

$$y_{nh}(t) = (a_0 + a_1 t + ... + a_n t^n)e^{rt}.$$

for some yet undetermined n. Then plugging it into our ODE we obtain:

$$\begin{aligned} ay''_{nh} + by'_{nh} + cy_{nh} &= a_n\big(ar^2 + br + c\big)t^n e^{rt} + \big(a_n n(2ar + b) + a_{n-1}\big(ar^2 + br + c\big)\big)t^{n-1}e^{rt} \\ &+ \big[a_n n(n-1)a + a_{n-1}(n-1)(2ar + b) + a_{n-2}\big(ar^2 + br + c\big)\big]t^{n-2}e^{rt} \\ &+ \text{lower order terms} \end{aligned}$$

Case 1: If r is not a root of the characteristic equation $ar^2 + br + c$, then the leading term $t^n e^{rt}$ remains and so to obtain $t^m e^{rt}$ we must set $n := m$ giving:

$$y_{nh}(t) = (a_0 + a_1 t + ... + a_n t^m)e^{rt}.$$

Case 2: If r is a simple root, then $ar^2 + br + cy = 0$ and we are left with $t^{n-1}e^{rt}$ being the leading order term and so we set $n - 1 := m$ giving:

$$y_{nh}(t) = \big(a_0 + a_1 t + ... + a_n t^{m+1}\big)e^{rt}.$$

Moreover, since r is a root, then $y_0 := a_0 e^{rt}$ will solve the homogeneous equation $ay'' + by' + cy = 0$ and so we can ignore it (due to additivity of solutions of the homogeneous equation). Thus,

$$y_{nh}(t) = \big(a_1 t + ... + a_n t^{m+1}\big)e^{rt} = t(a_1 + ... + a_n t^m)e^{rt}.$$

Case 3: If r is a double root, then $ar^2 + br + cy = 0$, $2ar + b = 0$ and we are left with $t^{n-2}e^{rt}$ being the leading term and so we set $n - 2 := m$ giving:

$$y_{nh} = (a_0 + a_1 t + ... + a_n t^{m+2})e^{rt}.$$

Moreover, since r is a repeated root, then e^{rt}, te^{rt} are both solutions of the homogeneous equation $ay'' + by' + cy = 0$ and so we can ignore. Thus,

$$y_{nh} = (a_2 t^2 + ... + a_n t^{m+2})e^{rt} = t^2(a_2 + ... + a_n t^m)e^{rt}.$$

Next we will work with

$$ay'' + by' + cy = Ct^m e^{\alpha t}\sin(\beta t) = \frac{1}{2i}Ct^m e^{\alpha t + i\beta t} - \frac{1}{2i}Ct^m e^{\alpha t - i\beta t}.$$

where we used that

$$\sin(\beta t) = \frac{e^{i\beta t} - e^{-i\beta t}}{2i}.$$

Therefore, from the previous we make the guess

$$\begin{aligned} y_{nh} &= (a_0 + a_1 t + ... + a_n t^n)e^{(\alpha + i\beta)t} + (b_0 + b_1 t + ... + b_n t^n)e^{(\alpha - i\beta)t}. \\ &= e^{\alpha t}(c_0 + c_1 t + ... + c_n t^n)\cos(\beta t) + (d_0 + d_1 t + ... + d_n t^n)e^{\alpha t}\sin(\beta t). \end{aligned}$$

So as above we check whether $r_* = \alpha + i\beta$ is a root and the same analysis shows the result. Note that $\alpha + i\beta$ cannot occur as a double root since the characteristic equation, $ar^2 + br + c$, has real coefficients. In fact, if $\alpha + i\beta$ is a root then the other root must be $\alpha - i\beta$.

By representing $\cos(\beta t)$ as $\frac{e^{i\beta t} + e^{-i\beta t}}{2}$ and applying similar logic to the $\sin(\beta t)$ case we complete the proof. □

Examples

- Consider the spring system governed by

$$y'' + 2y' - 3y = 3te^t.$$

Find the solution and its asymptotic behaviour.

1. For this equation we have $m = 1$, $r_* = 1$ and so our guess is:

$$y = t^s(a_0 + a_1 t)e^t.$$

2. To decide on the value of s, we have to check whether 1 is a root and what kind. The characteristic equation is

$$r^2 + 2r - 3 = 0 \implies r = -3,\, 1.$$

3. Therefore, we set $s = 1$ and our guess is:

$$y_{nh} = t(a_0 + a_1 t)e^t.$$

4. Next we determine a_i by plugging them into the equation and equating to $3te^t$:

$$\begin{aligned} 3te^t = y'' + 2y' - 3y &= \big(t(a_0 + a_1 t)e^{r_* t}\big)'' + 2\big(t(a_0 + a_1 t)e^{r_* t}\big)' - 3\big(t(a_0 + a_1 t)e^{r_* t}\big) \\ &\implies 2e^t(2a_0 + 4a_1 t + a_1) = 3te^t. \end{aligned}$$

This implies the following equation

$$\begin{aligned} &2(2a_0 + 4a_1 t + a_1) = 3t \\ \implies &a_1 = 3/8 \text{ and } 4a_0 + 2a_1 = 0 \\ \implies &a_1 = 3/8 \text{ and } a_0 = -\frac{3}{16}. \end{aligned}$$

Therefore, the general solution is

$$y = c_1 e^{-3t} + c_2 e^t + y_{nh} = c_1 e^{-3t} + c_2 e^t + t\left(-\frac{3}{16} + \frac{3}{8}t\right)e^t.$$

5. Therefore, as $t \to +\infty$, we have $y(t) \to +\infty$. Physically this means that the mass will get displaced towards the positive direction because of the external force $3te^t$.

- Consider the spring system governed by

$$y'' + 2y' - 3y = 2te^t \sin(t).$$

Determine what form the solution will take.

1. For this equation we have $m = 1$ and $\alpha = \beta = 1$, so our ansatz will be

$$y_{nh} = t^s e^t[(a_0 + a_1 t)\cos(t) + (b_0 + b_1 t)\sin(t)].$$

2. To decide on s, we have to check whether $1+i$ is a root for our characteristic equation:

$$r^2 + 2r - 3 = 0 \implies r = -3, 1.$$

3. So we put $s = 0$:

$$y_{nh} = e^t[(a_0 + a_1 t)\cos(t) + (b_0 + b_1 t)\sin(t)].$$

4. Therefore, the general solution is

$$y = c_1 e^{-3t} + c_2 e^t + y_{nh} = c_1 e^{-3t} + c_2 e^t + e^t[(a_0 + a_1 t)\cos(t) + (b_0 + b_1 t)\sin(t)].$$

0.4 Method 3: Variation of parameters

We will now consider non-homogeneous equations with coefficients of the form

$$ay'' + by' + cy = f(t),$$

where $f(t)$ is any continuous function and a, b, c are also functions with $a(t) \neq 0$.

Method formal steps

1. First, we obtain two linearly independent solutions y_1, y_2 for the homogeneous problem

$$ay'' + by' + cy = 0.$$

2. Second, we make a guess

$$y_g = v_1(t)y_1 + v_2(t)y_2$$

and plug it into our ODE. This will gives one equation for v_1, v_2.

3. Third; we have two unknowns, so we will need one more equation in order to solve for both. So we impose another condition for v_1, v_2 to obtain another equation:

$$v_1' y_1 + v_2' y_2 = 0.$$

This equation is helpful because it simplifies the first equation (proved in detail below)

$$y_g' = y_1' v_1 + y_2' v_2 + 0 \Rightarrow ay_g'' + by_g' + cy_g = a(y_1' v_1' + y_2' v_2').$$

4. Therefore, we can obtain v_1, v_2 from the system

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ y_1' v_1' + y_2' v_2' = \frac{f}{a} \end{cases}$$

Example-presenting the method

Returning to the spring example, suppose that it is damping free $\gamma = 0$ and the exernal force is $f(t) = \tan(t)$:

$$y'' + y = \tan(t).$$

1. First we find independent solutions for the homogeneous problem:

$$y'' + y = 0.$$

2. One can easily check that $\cos(t)$, $\sin(t)$ are solutions for it and computing their Wronskian gives:

$$W(\cos(t), \sin(t), t) = \cos^2(t) + \sin^2(t) = 1 \neq 0.$$

3. Therefore, we make a guess

$$y_g = v_1 \cos(t) + v_2 \sin(t).$$

4. Using our system of equations

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ y_1' v_1' + y_2' v_2' = \frac{f}{a} \end{cases}$$

we obtain

$$\begin{cases} v_1' \cos(t) + v_2' \sin(t) = 0 \\ -\cos(t) v_1' + \cos(t) v_2' = \tan(t) \end{cases}$$
$$\Longrightarrow \begin{cases} v_1' = -\tan(t)\sin(t) \\ v_2' = \tan(t)\cos(t) = \sin(t) \end{cases}$$

Therefore, by integrating we obtain

$$\begin{aligned} v_1 &= -\int \tan(t)\sin(t)\mathrm{d}t = -\int \frac{\sin^2(t)}{\cos(t)}\mathrm{d}t \\ &= \int \left(\cos(t) - \frac{1}{\cos(t)}\right)\mathrm{d}t \\ &= \sin(t) - \ln\left|\frac{1+\sin(t)}{\cos(t)}\right| + c_1 \\ &\text{and} \\ v_2 &= \int \sin(t)\mathrm{d}t = -\cos(t) + c_2 \end{aligned}$$

For simplicity we take $c_1 = c_2 = 0$ and we get:

$$\begin{aligned} y_g &= \left(\sin(t) - \ln\left|\frac{1+\sin(t)}{\cos(t)}\right|\right)\cos(t) - \cos(t)\sin(t) \\ &= \cos(t)\ln\left|\frac{\cos(t)}{1+\sin(t)}\right|. \end{aligned}$$

General result:

The equations $y'' + p(t)y' + q(t)y = g(t)$ for continuous p, q, g have solutions of the form

$$Y = y_1 v_1 + y_2 v_2,$$

where y_1, y_2 are fundamental solutions for the homogeneous problem $y'' + p(t)y' + q(t)y = 0$ and

$$v_1 := -\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2, s)}\mathrm{d}s \text{ and } v_2 := \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2, s)}\mathrm{d}s.$$

Proof. We start by making the guess

$$y_g := y_1 v_1 + y_2 v_2.$$

We have

$$y_g' = v_1'y_1 + v_2'y_2 + v_1y_1' + v_2y_2'.$$

So we note that if we set

$$v_1'y_1 + v_2'y_2 = 0$$

then the second derivative will not contain any v_1'', v_2'' terms:

$$\begin{aligned} y_g' &= 0 + v_1y_1' + v_2y_2' \\ \implies y_g'' &= v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''. \end{aligned}$$

Therefore, the ODE for y_g becomes

$$\begin{aligned} y_g'' + py_g' + qy_g &= v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' + p(v_1y_1' + v_2y_2') \\ &+ q(y_1v_1 + y_2v_2) \\ &= v_1(y_1'' + p(t)y_1' + q(t)y_1) + v_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &+ v_1'y_1' + v_2'y_2' \\ &= 0 + v_1'y_1' + v_2'y_2' \end{aligned}$$

because y_1, y_2 are solutions to the homogeneous problem. Therefore, for y_g to be a solution we need

$$v_1'y_1' + v_2'y_2' = g(t).$$

Our second equation was

$$v_1'y_1 + v_2'y_2 = 0.$$

Together they give

$$v_1' = \frac{-y_2g}{W(y_1,y_2,t)} \text{ and } v_2' = \frac{y_1g}{W(y_1,y_2,t)}$$
$$v_1 := C_1 - \int_{t_0}^t \frac{y_2(s)g(s)}{W(y_1,y_2,s)}\mathrm{d}s \text{ and } v_2 := C_2 + \int_{t_0}^t \frac{y_1(s)g(s)}{W(y_1,y_2,s)}\mathrm{d}s.$$

Observe that the constants of integration can be ignored since including them leads to

$$v_1(t)y_1 + v_2y_2 = \left(-\int_{t_0}^t \frac{y_2(s)g(s)}{W(y_1,y_2,s)}\mathrm{d}s\right)y_1 + \left(\int_{t_0}^t \frac{y_1(s)g(s)}{W(y_1,y_2,s)}\mathrm{d}s\right)y_2 + \underbrace{C_1y_1 + C_2y_2}_{\text{solution to homogeneous}}.$$

□

Examples

- Consider the equation

  $$ty'' - (1+t)y' + y = t^2e^{2t}$$

  with given fundamental solutions $y_1 = 1+t, y_2 = e^t$ for the homogeneous problem $ty'' - (1+t)y' + y = 0$.

- We have the system

$$\begin{aligned} &\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ y_1'v_1' + y_2'v_2' = \frac{f}{a} \end{cases} \\ \implies &\begin{cases} v_1'(1+t) + v_2'e^t = 0 \\ v_1' + e^tv_2' = te^{2t} \end{cases} \\ \implies &v_1' = -e^{2t} \text{ and } v_2' = (1+t)e^t \end{aligned}$$

$$\implies v_1 = -\frac{1}{2}e^{2t} \text{ and } v_2 = te^t.$$

Therefore, the solution for the nonhomogeneous problem will be

$$y = v_1y_1 + v_2y_2 = \left(-\frac{1}{2}e^{2t}\right)(1+t) + \left(te^t\right)e^t = \frac{1}{2}(t-1)e^{2t}.$$

- Consider the equation

  $$x^2y'' - 3xy' + 4y = x^2\ln(x)$$

  with given fundamental solutions $y_1 = x^2$, $y_2 = x^2\ln(x)$ for the homogeneous problem $x^2y'' - 3xy' + 4y = 0$.
  We have the system

  $$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ y_1'v_1' + y_2'v_2' = \frac{f}{a} \end{cases}$$
  $$\implies \begin{cases} v_1'x^2 + v_2'x^2\ln(x) = 0 \\ 2xv_1' + (2x\ln(x) + x)v_2' = \ln(x) \end{cases}$$
  $$\implies v_1' = -\ln^2(x)/x \text{ and } v_2' = \ln(x)/x$$
  $$\implies v_1 = -\frac{\ln^3(x)}{3} \text{ and } v_2 = \frac{\ln^2(x)}{2}.$$

  Therefore, the solution for the nonhomogeneous problem will be

  $$y = v_1y_1 + v_2y_2 = -\frac{\ln^3(x)}{3}x^2 + \frac{\ln^2(x)}{2}x^2\ln(x).$$

0.5 Method 4: Reduction of order

For homogeneous equations of the form

$$y'' + p(t)y' + q(t)y = 0 \tag{0.5.1}$$

if we have one solution y_1, we can obtain a second one by setting

$$y_2 := v(t)y_1$$

and identifying an ODE for v. Plugging in y_2 into our ODE we obtain

$$y_1v'' + (2y_1' + py_1)v' = 0$$

where we have used that y_1 satisfies (0.5.1).

Example-presenting the method

Consider the equation

$$x^2y'' - 5xy' + 9y = 0 \implies y'' - 5x^{-1}y' + 9x^{-2} = 0.$$

1. One solution is $y_1 = x^3$. One would guess this solution by observing that this equation preserves the "order" of monomials. That is y'' decreases the power of x by 2 but in the equation y'' is multiplied by x^2. The same phenomenon occurs for xy'. As a result, we obtain a characteristic equation if we guess $y = x^r$ for an r to be determined.

2. Assuming a second solution of the form $y_2 = v(x)x^3$ we obtain

$$0 = y_1 v'' + (2y_1' + py_1)v' = x^3 v'' + \left(6x^2 - 5x^{-1}x^3\right)v' = x^2(xv'' + v') \Rightarrow xv'' + v' = 0.$$

3. This gives us

$$v(x) = c\ln(x).$$

So the general solution will be

$$y = ax^3 + b\ln(x)x^3.$$

0.6 Nonlinear into second order

0.6.1 Riccati

The non-linear Riccati equation can always be reduced to a second order linear ordinary differential equation (ODE): If y satisfies

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2$$

then, wherever q_2 is never zero and differentiable,

$$v = yq_2$$

satisfies a Riccati equation of the form

$$v' = v^2 + R(x)v + S(x),$$

where

$$S = q_2 q_0 \text{ and } R = q_1 + \frac{q_2'}{q_2}$$

because

$$\begin{aligned} v' &= (yq_2)' = y'q_2 + yq_2' \\ &= \left(q_0 + q_1 y + q_2 y^2\right)q_2 + yq_2' \\ &= q_0 q_2 + \left(q_1 + \frac{q_2'}{q_2}\right)q_2 y + (q_2 y)^2 \\ &= q_0 q_2 + \left(q_1 + \frac{q_2'}{q_2}\right)v + v^2. \end{aligned}$$

Substituting

$$v = -\frac{u'}{u},$$

it follows that u satisfies the linear 2nd order ODE

$$u'' + R(x)u' - S(x)u = 0$$

since

$$\begin{aligned} u'' &= (uv)' = u'v + uv' \\ &= u'v + u\left(v^2 + R(x)v + S(x)\right) \\ &= u'\left(\frac{-u'}{u}\right) + u\left(\left(\frac{-u'}{u}\right)^2 + R(x)\left(\frac{-u'}{u}\right) + S(x)\right) \end{aligned}$$

$$= -\frac{(u')^2}{u} + \frac{(u')^2}{u} - R(x)u' + S(x)u$$
$$= -R(x)u' + S(x)u$$

and hence

$$u'' + Ru' - Su = 0.$$

A solution of this equation will lead to a solution

$$y = \frac{-u'}{q_2 u}$$

of the original Riccati equation.

0.7 Problems

- **Real distinct roots**
  1. Find the solution, do a rough sketch, and describe its asymptotic behaviour
     (a) $y'' + y' - 2y = 0,\ y(0) = 1,\ y'(0) = 1,$
     (b) $y'' + 3y' = 0,\ y(0) = -2,\ y'(0) = 3.$
  2. Solve
     $$y'' - y' - 2y = 0, y(0) = a,\ y'(0) = 2$$
     and determine for which a, the solution goes to zero as $t \to +\infty$.
- **Wronskian**
  1. Consider the equation $y'' - y' - 2y = 0$
     (a) Show that $y_1(t) := e^{-t}$, $y_2(t) := e^{2t}$ form a set of fundamental solutions.
     (b) Show that each of $y_3(t) := -2e^{2t}$, $y_4(t) := y_1(t) + 2y_2(t)$, and $y_5(t) := 2y_1(t) - 2y_3(t)$ are solutions to the above ode.
     (c) Which of the following pairs give rise to a fundamental pair of solutions:
     $$\{y_1, y_3\}, \{y_2, y_3\}, \{y_1, y_4\}, \{y_4, y_5\}$$
- **Complex roots**
  1. Imagine a spring satisfying the following equations. Find the solution, do a rough sketch, and describe its asymptotic behaviour (steady/growing/decaying oscillation). Finally, explain the asymptotic behaviour based on the coefficients (see notes on damping effect).
     (a) $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 1,$
     (b) (*)$y'' + 2y' + 2y = 0,\ y(\pi/4) = 2,\ y'(\pi/4) = -2.$
- **Repeated roots**
  1. Find the solution, do a rough sketch, and describe its asymptotic behaviour
     (a) $9y'' - 12y' + 4y = 0,\ y(0) = 2,\ y'(0) = -1,$
     (b) $y'' + 4y' + 4y = 0,\ y(-1) = 2,\ y'(-1) = 1.$
  2. Consider the problem
     $$y'' - y' + \frac{y}{4} = 0,\ y(0) = 2,\ y'(0) = b$$
     Find the solution and determine for which b, the solution remains positive for all $t > 0$.

- **Demand and Supply Problems**

  Let the demand and supply functions be, respectively,

  $$D(P) = 9 - P + P' + 3P'' \text{ and } S(P) = -1 + 4P + 2P' + 5P''$$

  with $P(0) = 4,\ P'(0) = 4$.

  1. Derive the price ODE (, and find the price solution.
  2. Does it have a globally stable solution as $t \to +\infty$? What does the stability result tell you?

- **Method of undetermined coefficients**

  Find the general form of the solution (with abstract coefficients) and use the stability result to determine whether they will have a globally stable solution.

  1. $y'' - 2y' - 3y = 3e^{2t}$,
  2. $y'' - y' - 2y = -2t + 4t^2$,
  3. $y'' + 2y' = 3 + 4\sin(2t)$

- Find the solution of the given IVP:

  $$y'' + y' - 2y = 2t,\ y(0) = 0,\ y'(0) = 1.$$

- **Variation of parameters**

  Below you are given the fundamental solutions y_1, y_2 of the homogeneous problem. Use them to find a solution of the nonhomogeneous one.

  1. $t^2y'' - 2y = 3t^2 - 1$ with $y_1 = t^2,\ y_2 = t^{-1}$,
  2. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$ with $y_1 = t,\ y_2 = te^t$.