ECO 2901
EMPIRICAL INDUSTRIAL ORGANIZATION

Lectures 7 & 8:
Static games of incomplete information with non-equilibrium beliefs

Victor Aguirregabiria (University of Toronto)

February 28, 2019

Lecture 7 & 8: Static games of incomplete information with non-equilibrium beliefs

1. Introduction

Equilibrium beliefs: a common assumption

- In games, the best response of a player depends on her beliefs about the behavior of her opponents.
- The most common assumption is that players' beliefs about other players' actions are in equilibrium, i.e., **they are unbiased expectations of the actual behavior of other players**.
- There are good reasons to impose assumption of equilibrium beliefs:

  (a) This assumption has identification power.
  (b) Counterfactual analysis: model predicts how beliefs change endogenously.

Strategic Uncertainty

- In reality, firms can face substantial uncertainty about other competitors' strategies.

- There are different sources of bias in players beliefs:

  (a) **Limited information / attention***: Some players do not have information about variables that are know to other players.*

  (b) **Bounded rationality***: Limited capacity to process information / compute;*

  (c) **Strategic uncertainty:** *With multiple equilibria, players can have different beliefs about the selected equilibrium. Some players believe that they are playing equilibrium A, other players believe they are playing equilibrium B, . . .*

Strategic Uncertainty and oligopoly competition

- In the context of games of oligopoly competition, Strategic Uncertainty seems particularly plausible, especially when firms are **competing in new markets, or after substantial regulatory changes**.
- Oligopoly games are typically characterized by multiple equilibria, and some equilibria are better for some firms, i.e., firms do not have incentives to coordinate their beliefs.
- Firms are very secretive about their own strategies and face significant uncertainty about the strategies of their competitors.
- Implications for: estimation of strategic interaction parameters; evaluation of a new policy.

2. Model

Discrete Game of Incomplete Information

- N players indexed by i. Each player takes an action $y_i \in \{0, 1, ..., J\}$.
- Payoff function is:

$$\Pi_i = \pi_i(y_i, \mathbf{y}_{-i}, \mathbf{x}) + \varepsilon_i(y_i)$$

  we normalize $\pi_i(0, \mathbf{y}_{-i}, \mathbf{x}) = 0$.

- $\mathbf{y}_{-i} = \{y_k : k \neq i\}$ = Actions of other players.
- $\mathbf{x}$ is a vector of common knowledge state variables.
- $\varepsilon_i = \{\varepsilon_{it}(0), ..., \varepsilon_{it}(J)\}$ is private info of player i and unobservable to researcher. It is i.i.d. over time and players with CDF G.

Bayesian Nash Equilibrium (BNE)

ASSUMPTION 1: *Players' strategy functions depend only on payoff relevant state variables:* $\mathbf{x}$ and ε_i.

ASSUMPTION 2: *Players have beliefs about the behavior (strategies) of other players. Given these beliefs, they maximize expected intertemporal payoffs.*

ASSUMPTION EQUIL: *Players have rational expectations about other players' behavior (strategies).*

Relaxing BNE

- Following the concept of Rationalizability:
- We maintain Assumptions 1 to 2;
- We relax Assumption EQUIL of "equilibrium beliefs".

Strategies, Choice Probabilities, and Beliefs

- Let $\sigma_i(\mathbf{x}, \varepsilon_i)$ be the strategy function for player i.
- $P_i(j \mid \mathbf{x}) \equiv \Pr(\sigma_i(\mathbf{x}, \varepsilon_i) = j \mid \mathbf{x})$. $\quad \mathbf{P}_i(\mathbf{x}) \equiv [P_i(1 \mid \mathbf{x}), \ldots, P_i(J \mid \mathbf{x})]$ is the vector of CCPs of player i.
- Player i's beliefs about the behavior (choice probabilities) of player k is: $\mathbf{B}_{ik}(\mathbf{x}) \equiv [B_{ik}(1 \mid \mathbf{x}), \ldots, B_{ik}(J \mid \mathbf{x})]$.
- Equilibrium (unbiased) beliefs: $\mathbf{B}_{ik}(\mathbf{x}) = \mathbf{P}_k(\mathbf{x})$ for any i, k and $\mathbf{x}$.

Best Response Functions

- Given her beliefs, $\{\mathbf{B}_{ik}(\mathbf{x}) : k \neq i\}$, player i's best response strategy is:

$$\sigma_i(\mathbf{x}, \varepsilon_i) = \arg \max_{j \in \{0,...,J\}} \ \pi_i^{\mathbf{B}}(j, \mathbf{x}) + \varepsilon_i(j)$$

where $\pi_i^{\mathbf{B}}(j, \mathbf{x})$ is the expected payoff function:

$$\pi_i^{\mathbf{B}}(j, \mathbf{x}) = \sum_{\mathbf{y}_{-i}} \left[\prod_{k \neq i} B_{ik}(y_k | \mathbf{x}) \right] \ \pi_i(j, \mathbf{y}_{-i}, \mathbf{x})$$

- And the best response probability function:

$$P_i(j \mid \mathbf{x}) = \int 1\{\sigma_i(\mathbf{x}, \varepsilon_i) = j\} \ dG(\varepsilon_i)$$

Best Response Functions (2)

- For instance, if $\varepsilon' s$ are i.i.d. extreme value type 1:

$$P_i(j \mid \mathbf{x}) = \frac{\exp\left\{\pi_i^{\mathbf{B}}(j, \mathbf{x})\right\}}{1 + \sum_{j'=1}^{J} \exp\left\{\pi_i^{\mathbf{B}}(j', \mathbf{x})\right\}}$$

- The "primitives" of the model are the payoff functions $\{\pi_i\}$ and the belief functions $\{\mathbf{B}_{ik}\}$.

- The endogenous predictions of the model are the CCP functions $\{P_i\}$.

3. Identification

Data

- We have a random sample of M markets, indexed by m, where we observe

$$\{y_{1m},\ ...,y_{Nm},\ \mathbf{x}_m : m = 1, 2, ..., M\}$$

- Here I assume that the researcher observes all the common knowledge state variables, $\mathbf{x}_m$.

- The analysis can be extended to a version of the model with $\mathbf{x}_m = (\mathbf{z}_m, \omega_m)$ where the researcher observes $\mathbf{z}_m$ but ω_m is unobservable.

Identification problem

- Given this model and data, **is it possible to fully identify all the structural functions of the model**, i.e., players' payoff functions $\pi_i(y_i, \mathbf{y}_{-i}, \mathbf{x})$ and players' belief functions $B_{ik}(j|\mathbf{x})$?

- The answer is NO. This model is seriously under-identified.

  $\#$ of identified CCPs (P_i): $N \; J \; |\mathcal{X}|$
  $\#$ of payoff parameters (π_i): $N \; J \; (J+1)^{N-1} \; |\mathcal{X}|$
  $\#$ of beliefs parameters (B_{ik}): $N(N-1) \; J \; |\mathcal{X}|$

$$\frac{\#\text{Parameters}}{\#\text{Restrictions}} = (J+1)^{N-1} + (N-1) >>> 1$$

Identification problem [2]

- Two remarks about this identification problem.

- **[1]** The model with equilibrium beliefs ($B_{ik} = P_k$) is also under-identified:

  \# of identified CCPs (P_i): $N \; J \; |\mathcal{X}|$

  \# of payoff parameters (π_i): $N \; J \; (J+1)^{N-1} \; |\mathcal{X}|$

$$\frac{\#\text{Parameters}}{\#\text{Restrictions}} = (J+1)^{N-1} >>> 1$$

- The identification of games requires exclusion restrictions. We will consider the identification of the model with biased beliefs under conditions where the model with equilibrium beliefs is identified.

Identification problem [3]

- **[2]** Second, even if the model is not identified, it may be possible to test for the null hypothesis of equilibrium beliefs.

Exclusion Restriction in Payoff functions

- Bajari et al. (JBES, 2010) provide conditions for the nonparametric identification of games under the restrictions of BNE.

- Identification is based on the following exclusion restriction.

- **Exclusion Restriction (ER)**: $\mathbf{x} = (s_1, \ldots, s_N, \mathbf{w})$ such that s_i enters in the payoff function of player i but not in the payoff of the other players.

$$\pi_i \left(y_i, \mathbf{y}_{-i}, \mathbf{x} \right) = \pi_i \left(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w} \right)$$

- Furthermore, the number of points in the support of the space of s_i, $\mathcal{S}$, is $|\mathcal{S}| \geq J + 1$.

Exclusion Restriction in Payoff functions [2]

- Under the assumption ER and with equilibrium beliefs, we have that:
  \# of identified CCPs (P_i): $N\ J\ |\mathcal{S}|^N\ |\mathcal{W}|$
  \# of payoff parameters (π_i): $N\ J\ (J+1)^{N-1}\ |\mathcal{S}|\ |\mathcal{W}|$

$$\frac{\#\text{Parameters}}{\#\text{Restrictions}} = \frac{(J+1)^{N-1}}{|\mathcal{S}|^{N-1}} < 1$$

- such that the Order Condition for identification holds.

- It is possible to show that the Rank Condition for identification holds generically.

Examples of exclusion restrictions

- The exclusion restriction in Assumption ID-2 appears naturally in games of oligopoly competition with predetermined state variables.

- **Entry-exit**: $s_i =$ Incumbent status of firm i at previous period. s_i is payoff relevant for firm i because it determines whether the firm has to pay an entry cost or not to be active at this period. But the payoff of firm i depends on y_{-i} (the firms that are active at current period) and not on s_{-i} (whether the competitors were active or not at previous period).

- **Quality competition:** $s_i =$ Quality of firm i at previous period. s_i is payoff relevant for firm i if there are costs of adjusting/modifying a firm's quality. But the payoff of firm i depends on y_{-i} (the qualities of the competitors at the current period) and not on s_{-i} (the qualities of the competitors at previous period).

Identification with equilibrium: 2 x 2 logit game

- Consider binary choice game with two players and logit private information. W.l.o.g., we omit the common state variable **w**.

- Let $P_1(s_1, s_2) \equiv P_1(1|s_1, s_2)$. We have that:

$$\pi_1^{\mathbf{P}}(1, s_1, s_2) = [1 - P_2(s_1, s_2)] \ \pi_1(1, 0, s_1) + P_2(s_1, s_2) \ \pi_1(1, 1, s_1)$$

- And:

$$P_1(s_1, s_2) = \frac{\exp\left\{\pi_1^{\mathbf{P}}(1, s_1, s_2)\right\}}{1 + \exp\left\{\pi_1^{\mathbf{P}}(1, s_1, s_2)\right\}}$$

  Such that:

$$\ln\left[\frac{P_1(s_1,s_2)}{1-P_1(s_1,s_2)}\right] = \pi_1(1, 0, s_1) + P_2(s_1, s_2) \ \left[\pi_1(1, 1, s_1) - \pi_1(1, 0, s_1)\right]$$

Identification with equilibrium: 2 x 2 logit game [2]

$$\ln\left[\frac{P_1(s_1,s_2)}{1-P_1(s_1,s_2)}\right] = \pi_1(1,0,s_1) + P_2(s_1,s_2)\ \left[\pi_1(1,1,s_1) - \pi_1(1,0,s_1)\right]$$

- Let a and b be two value of s_2 such that $P_2(s_1,a) \neq P_2(s_1,b)$. Then, we can identify $\left[\pi_1(1,1,s_1) - \pi_1(1,0,s_1)\right]$ as:

$$\pi_1(1,1,s_1) - \pi_1(1,0,s_1) = \frac{\ln\left[\frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right] - \ln\left[\frac{P_1(s_1,b)}{1-P_1(s_1,b)}\right]}{P_2(s_1,a) - P_2(s_1,b)}$$

- And then, we $\pi_1(1,0,s_1)$ is identified as:

$$\begin{aligned}\pi_1(1,0,s_1) &= \ln\left[\frac{P_1(s_1,s_2)}{1-P_1(s_1,s_2)}\right] \\ &\quad - P_2(s_1,a)\ \left(\frac{\ln\left[\frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right] - \ln\left[\frac{P_1(s_1,b)}{1-P_1(s_1,b)}\right]}{P_2(s_1,a) - P_2(s_1,b)}\right)\end{aligned}$$

Identification with equilibrium: N x (J+1) game

- This identification result for the 2 x 2 game can be extended to any game with N players and $J + 1$ choice alternatives that satisfies the exclusion restriction above.

Identification.Out-of-equilibrium beliefs

- Still, we have this model is not identified:

  # of identified CCPs (P_i): $N\ J\ |\mathcal{S}|^N\ |\mathcal{W}|$
  # of payoff parameters (π_i): $N\ J\ (J+1)^{N-1}\ |\mathcal{S}|\ |\mathcal{W}|$
  # of beliefs parameters (B_{ik}): $N(N-1)\ J\ |\mathcal{S}|^N\ |\mathcal{W}|$

$$\frac{\#\text{Parameters}}{\#\text{Restrictions}} = \frac{(J+1)^{N-1}}{|\mathcal{S}|^{N-1}} + (N-1) > N-1 \geq 1$$

- However, we can show that in this model the null hypothesis of equilibrium beliefs is testable.

Identification.Out-of-equilibrium beliefs: 2x2 logit game

- Now, we have:

$$\pi_1^{\mathbf{B}}(1, s_1, s_2) = [1 - B_{12}(s_1, s_2)] \ \pi_1(1, 0, s_1) + B_{12}(s_1, s_2) \ \pi_1(1, 1, s_1)$$

- And:

$$P_1(s_1, s_2) = \frac{\exp\left\{\pi_1^{\mathbf{B}}(1, s_1, s_2)\right\}}{1 + \exp\left\{\pi_1^{\mathbf{B}}(1, s_1, s_2)\right\}}$$

Such that:

$$\ln\left[\frac{P_1(s_1,s_2)}{1-P_1(s_1,s_2)}\right] = \pi_1(1, 0, s_1) + B_{12}(s_1, s_2) \ [\pi_1(1, 1, s_1) - \pi_1(1, 0, s_1)]$$

Identification: out-of-equilibrium: 2x2 logit game [2]

$$\ln\left[\frac{P_1(s_1,s_2)}{1-P_1(s_1,s_2)}\right] = \pi_1(1,0,s_1) + B_{12}(s_1,s_2)\ \left[\pi_1(1,1,s_1) - \pi_1(1,0,s_1)\right]$$

- Let a, b, and c be three values of s_2 such that $P_2(s_1,a) \neq P_2(s_1,b)$ and $P_2(s_1,a) \neq P_2(s_1,c)$. Then:

$$\ln\left[\frac{P_1(s_1,b)}{1-P_1(s_1,b)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right] = \left[B_{12}(s_1,b) - B_{12}(s_1,a)\right]\left[\pi_1(1,1,s_1) - \right.$$

$$\ln\left[\frac{P_1(s_1,c)}{1-P_1(s_1,c)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right] = \left[B_{12}(s_1,c) - B_{12}(s_1,a)\right]\left[\pi_1(1,1,s_1) - \right.$$

- such that:

$$\frac{\ln\left[\frac{P_1(s_1,b)}{1-P_1(s_1,b)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right]}{\ln\left[\frac{P_1(s_1,c)}{1-P_1(s_1,c)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right]} = \frac{B_{12}(s_1,b) - B_{12}(s_1,a)}{B_{12}(s_1,c) - B_{12}(s_1,a)}$$

Identification: out-of-equilibrium: 2x2 logit game [3]

$$\frac{\ln\left[\frac{P_1(s_1,b)}{1-P_1(s_1,b)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right]}{\ln\left[\frac{P_1(s_1,c)}{1-P_1(s_1,c)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right]} = \frac{B_{12}(s_1, b) - B_{12}(s_1, a)}{B_{12}(s_1, c) - B_{12}(s_1, a)}$$

- This expression shows that the observed behavior of a player (player 1) reveals information about her beliefs. An object that depends only on beliefs.

- The intuition is simple: state variable s_2 is not payoff relevant for player 2. Therefore, the observed dependence of the behavior of player 1 with respect to s_2, in $P_1(s_1, s_2)$, is only because her beliefs $B_{12}(s_1, s_2)$.

- The dependence of $P_1(s_1, s_2)$ w.r.t. s_2 can be represented in a way (equation above) that depends only on beliefs, not on preferences.

Test of Equilibrium Beliefs

- Under the condition of equilibrium beliefs,

$$\frac{B_{12}(s_1, b) - B_{12}(s_1, a)}{B_{12}(s_1, c) - B_{12}(s_1, a)} = \frac{P_2(s_1, b) - P_2(s_1, a)}{P_2(s_1, c) - P_2(s_1, a)}$$

- As shown above, $\dfrac{B_{12}(s_1, b) - B_{12}(s_1, a)}{B_{12}(s_1, c) - B_{12}(s_1, a)}$ can be identified from the behavior of player 1 (from $P_1(s_1, s_2)$).

- And $P_2(s_1, s_2)$ is identified from the behavior of player 2.

- Therefore, we can test (nonparametrically) these equilibrium restrictions.

Test of Equilibrium Beliefs (3)

- We can test the null hypothesis of equilibrium beliefs using a **Likelihood Ratio Test** in a **nonparametric multinomial model** for the CCPs P_1 and P_1.
- The log-likelihood function of this multinomial model is:

$$\begin{aligned}\ell\left(\mathbf{P}_1, \mathbf{P}_2\right) &= \sum_{m=1}^{M} y_{1m} \ln P_1(s_{1m}, s_{2m}) + (1 - y_{1m}) \ln \left[1 - P_1(s_{1m}, s_{2m}\right. \\ &+ \sum_{m=1}^{M} y_{2m} \ln P_2(s_{1m}, s_{2m}) + (1 - y_{2m}) \ln \left[1 - P_2(s_{1m}, s_{2m}\right.\end{aligned}$$

- And the restrictions are, for $i \neq j \in \{1, 2\}$ and any triple (a, b, c):

$$\frac{\ln\left[\frac{P_1(s_1,b)}{1-P_1(s_1,b)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right]}{\ln\left[\frac{P_1(s_1,c)}{1-P_1(s_1,c)} \Big/ \frac{P_1(s_1,a)}{1-P_1(s_1,a)}\right]} = \; = \frac{P_j(s_i, b) - P_j(s_i, a)}{P_j(s_i, c) - P_j(s_i, a)} = 0$$

Test of Equilibrium Beliefs (4)

- Let $\left(\widehat{\mathbf{P}}_1^u, \widehat{\mathbf{P}}_2^u\right)$ and $\left(\widehat{\mathbf{P}}_1^c, \widehat{\mathbf{P}}_2^c\right)$ be the unconstrained and the constrained MLEs of $(\mathbf{P}_1, \mathbf{P}_2)$.
- The test statistic is the Likelihood Ratio:

$$LR = 2\left[\ell\left(\widehat{\mathbf{P}}_1^u, \widehat{\mathbf{P}}_2^u\right) - \ell\left(\widehat{\mathbf{P}}_1^c, \widehat{\mathbf{P}}_2^c\right)\right]$$

Full Identification of the Model

- The full model is not point identified without additional assumptions.
- Suppose that the researcher is willing to assume that beliefs are unbiased (in equilibrium) at two values of s_2, say A and B:

$$B_{12}(s_1, A) = P_2(s_1, A) \quad \text{and} \quad B_{12}(s_1, B) = P_2(s_1, B)$$

- Then, these additional restrictions imply the full identification of $B_{12}(s_1, s_2)$ at any state (s_1, s_2):

$$\begin{aligned} B_{12}(s_1, s_2) &= P_2(s_1, A) \\ &+ [P_2(s_1, B) - P_2(s_1, A)] \frac{\ln\left[\frac{P_1(s_1,s_2)}{1-P_1(s_1,s_2)} / \frac{P_1(s_1,A)}{1-P_1(s_1,A)}\right]}{\ln\left[\frac{P_1(s_1,B)}{1-P_1(s_1,B)} / \frac{P_1(s_1,A)}{1-P_1(s_1,A)}\right]} \end{aligned}$$

- And given these beliefs and the exclusion restriction, the payoff functions are also identified.

How to choose points where to impose unbiased beliefs?

- *(a) Applying the test of equilibrium beliefs.*
- *(b) Testing for the monotonicity of beliefs and using this restriction.*
- *(c) Minimization of the player's beliefs bias.*
- *(d) Most visited states.*

Identification under Rationalizability

- Aradillas-Lopez and Tamer (JBES, 2008) study the identification of payoffs and beliefs when we relax the assumption of BNE and replace it with Rationalizability.

- The only restriction on beliefs comes from the **assumption of common knowledge rationality:**
  (i) every player is rational in the sense that she maximizes her own expected payoff given her beliefs;
  (ii) every player knows that the other players know that she knows ... that all the players are rational.

- They show that, without further restrictions [no exclusion restrictions], choice data cannot point identify preferences and beliefs but there is set identification of some parameters in the payoff function.

- This approach does not try to account bounded rationality or limited information, but it can account for strategic uncertainty due to

Identification under Rationalizability: 2x2 logit game

- The restrictions of rationlizability can be obtained as a sequence of bounds on the CCPs. For stages $K = 1, 2, ...$

$$L_1^{(K)}(\mathbf{x}) \leq \ln\left[\frac{P_1(\mathbf{x})}{1 - P_1(\mathbf{x})}\right] \leq U_1^{(K)}(\mathbf{x})$$

- We now derive the structure of the sequence of bounds $L_1^{(K)}(\mathbf{x})$ and $U_1^{(K)}(\mathbf{x})$.

- First, for general beliefs, we have that:

$$\ln\left[\frac{P_1(\mathbf{x})}{1-P_1(\mathbf{x})}\right] = \alpha_1(\mathbf{x}) + B_{12}(\mathbf{x})\ \beta_1(\mathbf{x})$$

with $\alpha_1(\mathbf{x}) \equiv \pi_1(1, 0, \mathbf{x})$ and $\beta_1(\mathbf{x}) \equiv \pi_1(1, 1, \mathbf{x}) - \pi_1(1, 0, \mathbf{x})$.

Rationalizability: 2x2 logit game [2]

- Suppose that $\beta_1(\mathbf{x}) \leq 0$, e.g., entry game.

- Player 1 is rational such that $\min\{B_{12}(\mathbf{x})\} = 0$ and $\max\{B_{12}(\mathbf{x})\} = 1$ such that:

$$\alpha_1(\mathbf{x}) + \beta_1(\mathbf{x}) \leq \ln\left[\frac{P_1(\mathbf{x})}{1 - P_1(\mathbf{x})}\right] \leq \alpha_1(\mathbf{x})$$

that is, $L_1^{(1)}(\mathbf{x}) = \alpha_1(\mathbf{x}) + \beta_1(\mathbf{x})$ and $U_1^{(1)}(\mathbf{x}) = \alpha_1(\mathbf{x})$.

Rationalizability: 2x2 logit game [3]

- Player 1 knows that player 2 is rational. She knows that $\alpha_2(\mathbf{x}) + \beta_2(\mathbf{x}) \leq \ln\left[\frac{P_2(\mathbf{x})}{1-P_2(\mathbf{x})}\right] \leq \alpha_2(\mathbf{x})$. Therefore,

$$\Lambda\left(\alpha_2(\mathbf{x}) + \beta_2(\mathbf{x})\right) \leq B_{12}(\mathbf{x}) \leq \Lambda\left(\alpha_2(\mathbf{x})\right)$$

such that

$$\begin{aligned} \alpha_1(\mathbf{x}) + \beta_1(\mathbf{x})\ \Lambda\left(\alpha_2(\mathbf{x})\right) &\leq \ln\left[\frac{P_1(\mathbf{x})}{1-P_1(\mathbf{x})}\right] \\ &\leq \alpha_1(\mathbf{x}) + \beta_1(\mathbf{x})\ \Lambda\left(\alpha_2(\mathbf{x}) + \beta_2(\mathbf{x})\right) \end{aligned}$$

that is, $L_1^{(2)}(\mathbf{x}) = \alpha_1(\mathbf{x}) + \beta_1(\mathbf{x})\ \Lambda\left(\alpha_2(\mathbf{x})\right)$ and $U_1^{(2)}(\mathbf{x}) = \alpha_1(\mathbf{x}) + \beta_1(\mathbf{x})\ \Lambda\left(\alpha_2(\mathbf{x}) + \beta_2(\mathbf{x})\right)$.

Rationalizability: 2x2 logit game [3]

- This idea can be applied recursively to obtain $L_1^{(\infty)}(\mathbf{x})$ and $U_1^{(\infty)}(\mathbf{x})$ such that:

$$L_1^{(\infty)}(\mathbf{x}) \leq \ln\left[\frac{P_1(\mathbf{x})}{1 - P_1(\mathbf{x})}\right] \leq U_1^{(\infty)}(\mathbf{x})$$

- The set $[L_1^{(\infty)}(\mathbf{x}),\ U_1^{(\infty)}(\mathbf{x})] \times [L_2^{(\infty)}(\mathbf{x}),\ U_2^{(\infty)}(\mathbf{x})]$ has the following properties:

- [1] If, for $\mathbf{x}$, the model has a unique BNE, then this set contains a single point that corresponds to the BNE.

- [2] If, for $\mathbf{x}$, the model has multiple BNE, then this set is a the **convex hull of all the BNE the model has**.

Rationalizability: 2x2 logit game [4]

- That is, the set contains all the outcomes where each player has beliefs that are consistent with an equilibrium but that does not correspond to the equilibrium that other players believe they are playing. It also contains other outcomes.

- Given these inequalities (for any level K), the model implies that the vector of payoff parameters $[\alpha_1(\mathbf{x}), \beta_1(\mathbf{x}), \alpha_2(\mathbf{x}), \beta_2(\mathbf{x})]$ is set identified.

Identification of Cognitive Hierarchy model

- Goldfarb and Yang (2009).
- Goldfarb and Xiao (2011).

Combining choice data with firms' costs data

- Hortaçsu and Puller (2008)
- Hortaçsu et al. (2017)

Beliefs data

- DellaVigna (2009). Manski (2018).