Identification in Models of Oligopoly Entry*

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Abstract

In the empirical study of markets, models of entry are often used to study the nature of firms' profits and the nature of competition between firms. Most of these estimated models have been parametric. In this paper, we review and extend a number of results on the identification of models that are used in the empirical literature. We study simple versions of both static and dynamic entry models. For simple static models, we show how natural shape restrictions can be used to identify competition effects. We consider extensions to models with heterogeneous firms, mixed-strategy equilibria and private information and provide insights that can be used in these settings to conduct inference. In the case of dynamic models, we review existing results on the model with i.i.d. linear errors, and then consider more realistic cases, such as when the distribution of fixed costs is unknown.

1 Introduction

In the empirical study of markets, models of entry are often used to study the nature of firms' profits and the nature of competition between firms. The idea of these models is that firms enter into a market only when they expect to operate profitably, and therefore entry decisions can be used as an indicator of a latent profit function.

The study of entry into oligopoly markets is complicated by strategic interactions between firms. This means that traditional ideas in the econometric literature on discrete choice models have to be modified somewhat to account for these strategic interactions.

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Most of the empirical literature on entry models has focused on ideas for estimating models under various assumptions. This literature has typically relied on strong parametric assumptions and no formal approach to identification has been well examined. In this paper, we review and extend a number of results on the formal identification of models that are used in the empirical literature giving insights about the problem and new avenues to explore.

These identification results and ideas are interesting because they provide intuition on the role of various qualitative assumptions, and of different sources of variation in the data especially when estimating parametric models.

1.1 Why Study Entry Models

There are a number of important economic questions that one can hope to answer by considering models of oligopoly entry.

These economic questions might include:

1. The prediction of equilibrium market structure under alternative hypothetical scenarios,
2. Determining the sources of firm profitability,
3. Identifying the "nature of competition."

The first two questions can sometimes be answered easily from the data, but the "nature of competition" question is more subtle. Researchers have been interested in questions like "how fast do profits decline in the number of firms" and "to what degree do high and low quality firms compete." Also, the question of predicting equilibrium market structure can also be reasonably complicated. Predictions within the range of the observable data can often be made without inferring any "structural" parameters, but sometimes we want predictions about, say, the effect of changes in the number of potential entrants (as might come about as a result of changes in regulatory structure.) These latter predictions often require us to first uncover structural parameters.

1.2 Identification Ideas in Entry Models

We study identification of a set of entry models. These are typically binary (enter or not[1]) choice models with multiple decision makers and strategic interaction.

[1] Even though the results will be tailored for the entry case, the insights carry over to models where the discrete outcome has larger support.

The identification strategy we follow is the following. We consider the economic model of entry mostly based on the fact that a firm enters a market when entry is profitable. Heuristically, given a profit function containing both variable and fixed profits, one can obtain information on the choice probabilities predicted by the model. Hence, the identification question clarifies what can and cannot be learned about the parameters of the profits functions under a set of maintained assumptions, some plausible and others not. This is done by comparing the predicted choice probabilities under the maintained assumptions to the observed choice probabilities (the data).

Naturally, using fewer assumptions (functional forms or stochastic restrictions) helps to ensure credibility and robustness of results. However, using fewer assumptions also puts strain on identification whereby the econometric model becomes ineffectual in that data are not helpful in distinguishing between different models (the mapping from parameters to data distribution is not one-to-one). So, one needs to delicately balance this tension between model assumptions and identification. Generally two approaches can be followed. A top down approach where one starts with a complete parametric econometric model and then try to study its robustness to the various assumptions made. Heuristically, this can be done by studying the sensitivity of the results to "peeling off" some of the parametric assumptions. The second is the bottom up approach where the analyst starts with the weakest assumptions (like monotonicity for example) and gradually moves "up" by adding restrictions. At every step, one examines the prediction of the model or set of models and sees how these change with the addition of further assumptions. This approach generally relies on starting first with the necessary conditions that an economic model imply and seeing what restrictions these conditions impose (if any). In both approaches, one needs to clearly define the object of interest (a parameter of economic interest).

1.3 Outline of the Paper

We begin with a discussion of "static" entry models. These are models of equilibrium market structure where firms make a one-time decision to be in or out of the market. First, we review existing identification results for threshold crossing models. We do this in the context of a simple "monopoly entry" model. We then turn to the the classic "Bresnahan and Reiss" (or BR) ordered entry model, which has much in common with traditional threshold-crossing models. We use the threshold-crossing model results to show that identification of the parameter of interest (the "extent of competition" for example) depends critically on the underlying assumptions and is sometimes not identified at all.

One extension to the Bresnahan and Reiss models is to allow for firm heterogeneity. This makes the econometric model look more like models used in the multi-variate discrete choice literature, but the strategic interactions now prove to be a complicating problem. One problem is the possibility of multiple equilibria and mixed strategies. This complicates the identification problem and in many cases would lead us to consider inference in models that does not point identify the parameter of interest. We also discuss models with incomplete information and highlight the inferential problem and the role different assumptions play.

In the last part of the paper, we consider simple dynamic entry models with i.i.d. errors. We first review the existing identification results, which typically require one to know the true distribution of fixed costs, and then suggest some alternative approaches that provide identification under different assumptions. These alternative approaches help to illustrate the role of different sources of variation in the data.

2 Identification in Static Entry Models with Complete Information

In this section, we study identification and inference strategies in static entry models of complete information. As a review, we begin with existing results for the "monopoly entry" (or more generally, "threshold crossing") model. We then move on to the more interesting oligopoly set-up of the Bresnahan and Reiss model and then consider models with firm heterogeneity. We emphasize the role multiple equilibria play in these models and strategies to examine the practical implication of multiple equilibria on the parameters of interest. We also examine the impact of allowing for mixed strategies.

2.1 The Monopoly Entry, or Threshold Crossing, Model

It is useful to review some existing results in discrete choice identification by considering the simple monopoly entry model, which takes the exact form of the traditional econometric threshold-crossing model. There are no new results in the section, just a review, using the language of entry models, of older results on the threshold-crossing model.

Consider a cross-section of markets, with one potential entrant in each market. Profits in market i are given by

$$\pi(x_i, F_i) = v(x_i) - F_i, \tag{2.1}$$

where v gives the deterministic part of profits, as a function of observable profit-shifters x, and F is the random component of profits, assumed to enter linearly. In many specific entry applications, we will think of v as "variable profits" and F as "fixed costs," although this strict interpretation may not be appropriate in some contexts.

The firm enters whenever $F_i \leq v(x_i)$. For purposes of identification, we treat the entry probabilities, $p(x)$, as being observed without error. The unknown objects of interest are the function $v(x)$ and the distribution of fixed costs, $\Phi(.)$, where we assume in the simplest case, that F is independent of x. We also consider the generalization to a limited dependence of the distribution of F on x. For example, one traditional assumption (Manski (1988)) is to require that F is median independent of x and to treat the unknowns as the function v (which is typically known up to a finite dimensional parameter) and sometimes the (conditional) distribution of F. This allows for example, for heteroskedasticity of unknown form (assumptions on other quantiles of the distribution of F can be handled in the same way).

We begin our analysis with the case where F is independent of x. The first classic problem is suggested by the threshold-crossing condition, $F \leq v(x)$. It is clear that a monotonic transformation of both sides of this equation will not change any entry probability. Thus, without further restrictions, the distribution of F and the function v are identified, at best, only up to a monotonic transformation.

For example, one can perfectly explain the data via a model where $v(x)$ is set equal to $p(x)$ and the distribution of F is uniform on $[0, 1]$. In this case, the entry probability is the probability of the threshold crossing condition $F \leq p(x)$, which is exactly $p(x)$, as required. Thus, with the assumption that F is independent of x, we learn that $v(x)$ is some monotonic transformation of $p(x)$, but we learn nothing more.

Now, knowing $v(x)$ up to a monotonic transformation is still useful. Simply from the data $p(x)$ we can answer some questions about predicted "market structure": i.e. we know the probability that a market is served by a firm. Also, knowing that $p(x)$ is a monotonic transformation of v, we learn the sign of $\partial v/\partial x$ and thereby can answer questions about the "sources of profitability." We can also note that for two different elements of x, say x^1 and x^2, the relative derivatives,

$$\frac{\partial v/\partial x^2}{\partial v/\partial x^1} = \frac{\partial p(x)/\partial x^2}{\partial p(x)/\partial x^1} \tag{2.2}$$

are also invariant with respect to monotonic transformations of v.

One way to proceed is to assume that we already know either $v(x)$ or the distribution of F. We might learn about $v(x)$ from using traditional data on prices, quantities demand, production and so forth. Once v is known, we learn about the distribution of F from $p(x) = \Phi(v(x))$. Alternatively, we might have prior knowledge of the true distribution of F. In practical empirical applications, auxiliary data relevant to the distribution of fixed costs seems much less common than data relevant to v. However, many authors have been willing to simply assume the distribution of $\Phi(F)$, (or that it is known up to a finite dimensional parameter) in which case it is trivial to solve for v.

Matzkin (1992) considers the case of identification of the threshold-crossing models, with i.i.d. errors (i.e. our F), under "qualitative" assumptions on the shape of the deterministic v. In particular, her theorem implies that if $v(x)$ is homogeneous of degree one in some subset of the x's, then both v and the distribution of F can be identified.

Result 1 *(Matzkin[2]) In the model above, if the function $v(x)$ is homogeneous of degree 1 and if there exists an x_0 such that $v_1(x_0) = 1$, then the functions $v(.)$ and $\Phi(.)$ are identified[3].*

To illustrate the Matzkin argument in our context, consider the special case where v is homogeneous of degree one in exactly one variable. In particular, BR note that there are economic assumptions that generate a variable profit function that is proportional to population. In particular, assume that [i] demand is proportional to population and that [ii] marginal cost is constant.[4] The first assumption implies that observed x's account for all the shifts in per-capita demand, so that we can model market demand as per-capita demand times population. The assumption on marginal cost is strong, but considered to be approximately true in many markets (note that there are still economies of scale in the model, generated by the fixed costs.)

Under the assumptions of the last paragraph, the first-order condition for profit maximization does not depend on population and profit in market i and can be written as

$$z_i v(x_i) - F_i, \tag{2.3}$$

where v is per-capita profit and z is observed population. To set the units of profits, normalize $v(x_0) = 1$ for some arbitrary x_0. The probability of entry at z, x_0 is then

$$\begin{aligned} p(z, x_0) &= Pr(F < zv(x_0)) \\ &= Pr(F < z) \\ &= \Phi(z), \end{aligned} \tag{2.4}$$

where $p(.,.)$ is the observed entry probability.

[2]A more precise statement of the result with all the needed assumptions and regularity conditions is given in Theorem 1 of Matzkin (1992) on page 244.

[3]Heuristically, what "identified" means here is that the functions $v(.)$ and $\Phi(.)$ can be recovered uniquely from $p(.)$.

[4]The necessary assumption on demand is that market demand is equal to population times a per-capita demand function that can depend on x's other than population.

If across the range of z, $p(z, x_0)$ takes on every value between zero and one, then (2.4) defines the entire function Φ. We can then trivially find $v(x)$ for other values of x and hence recover the function v on the support of x. This is a simple constructive version of Matzkin's proof, which works for a broader array of cases where the (normalized) v function is known along some ray or curve in the space of exogenous data.

Another example where Matzkin's proof applies is the partially linear model where $v(x) = x^1 + v(x^2)$. Here, v is linear in the variable x^1 and is an unknown function of the remaining variables $v(x^2)$. Some version of this partially linear model is frequently used in applied econometrics, although in the entry case under discussion, this partial linearity is harder to generate from qualitative economics assumptions such as the "proportional to population" model.

One criticism of Matzkin's approach is that it relies on the independence of F and x. Manski (1988) suggests the weaker assumption of median independence but requires that v be known up to a finite dimensional parameter ($v(x) = x\beta$ for example). Note that once we drop the full independence assumption and assume median independence (i.e. $Med(F|x) = 0$), an appropriate qualitative shape restriction (like linearity) identifies v (up to scale), given the appropriate support conditions on x. Moreover, once v is identified[5], the (conditional) distribution of F can also be recovered using for example Horowitz (1992)'s estimator. Recently, under the conditional quantile restriction, Khan (2004) provided a method to get consistent estimates of both v and the distribution of F using sieve methods. A lot less is known about semiparametric models where F and x are allowed to be correlated. One approach is to first write down an economically coherent entry model where this correlation is generated from a well defined optimizing framework and plausible economic restrictions (like the "proportional to population" restriction of BR) are imposed. Then at the inference stage, instead of looking for esoteric statistical assumptions that guarantee point identification, one can study the identified features of the class of models without making these typically non-plausible assumptions. The econometric structure, tightly linked to the economic model, admits consistent sets of parameters that rationalize the observed data and so appropriate inferential methods are required to handle, statistically, the possibility that the parameter of interest is a set.[6]

The monopoly model is, however, not typically of interest in the entry literature, which focuses on potential oligopolies. Questions of identification may become more subtle and new questions start to arise about the nature of competition, the structure of information across firms and the nature of equilibrium. In these more complicated models, we might want to know further information about

[5] One can use Manski's maximum score estimator to get v for example.

[6] See Chernozhukov, Hong, and Tamer (2002) and Andrews, Berry, and Jia (2003) for some recent inference methods in these settings.

the shape of $v(x)$ and/or the distribution of F. We now take these results to one classic model of oligopoly entry, the "Bresnahan and Reiss" model.

2.2 Bresnahan and Reiss Models

We first examine the symmetric entry model of Bresnahan and Reiss (1991b)[7], in which we assume that a cross section of markets is available where in each markets we observe the number of symmetric firms in the market and other market characteristics. This is the standard benchmark model of entry where all firms are similar. One object of interest in this kind of work is the shape of the variable profit function. Specifically, how do variable profits change with the number of firms in the market? This question is related to important questions about the nature of competition. But some researchers express skepticism about what exactly we can learn about the shape of variable profits from entry data alone. A more complete examination of the non-parametric identification of the model might shed light on this debate.

The Bresnahan and Reiss model is a simple extension of the Monopoly model with identical potential entrants. Variable profits v are assumed to decline in the number of firms, y: $v_y(x)$, where x is a vector of profit shifters.

Suppose that fixed costs, F, are *identical* across firms within a market and are distributed i.i.d. across markets with unknown distribution $\Phi(\cdot)$. If there are y_m firms in market m, then each firm operating in the market earns identical profits of

$$\pi(y_m, x_m, F_m) = v_{y_m}(x_m) - F_m \tag{2.5}$$

Entry is profitable for y firms when $v_y(x_m) \geq F_m$ and is profitable for no more than y firms when $v_y(x_m) < F_m$. The unique Nash equilibrium number of firms is the largest y for which entry is profitable. This implies that the probability distribution over the number of observed firms is given by:

$$\begin{aligned}
\Pr(y = 0|x) &= 1 - \Phi(v_1(x)) && (2.6)\\
\Pr(y = 1|x) &= \Phi(v_1(x)) - \Phi(v_2(x)) \\
\Pr(y = 2|x) &= \Phi(v_2(x)) - \Phi(v_3(x)) \\
&\cdots && (2.7)
\end{aligned}$$

[7]See also presentations of the same model in Bresnahan and Reiss (1987), and Bresnahan and Reiss (1990).

This implies that

$$\begin{aligned}
\Phi(v_1(x)) &= 1 - \Pr(y = 0|x) = G_1(x) \\
\Phi(v_2(x)) &= 1 - \Pr(y = 1|x) - \Pr(y = 0|x) = G_2(x) \\
\Phi(v_3(x)) &= 1 - \Pr(y = 3|x) - \Pr(y = 1|x) - \Pr(y = 0|x) = G_3(x) \\
&\vdots
\end{aligned} \tag{2.8}$$

The data identifies the choice probabilities and hence one can assume that the rhs in (2.8) , the G's, are known. Taking the first equation in (2.8), we see that this is similar to identification in binary choice models.

BR make parametric assumptions on both $v(.)$ and $\Phi(.)$ and estimate the resulting model by MLE. One specific economic question that they want to ask is what is the value of the ratio $\frac{v_2(x)}{v_1(x)}$? This ratio is interesting under various benchmark models. In particular,

1. Given fixed prices, $\frac{v_2(x)}{v_1(x)} = 1/2$, while given
2. Cournot competition: $\frac{v_2(x)}{v_1(x)} \in (0, 1/2)$, and given
3. Homogeneous goods Bertrand: $\frac{v_2(x)}{v_1(x)} = 0$.

One can think of similar ratios for other $y > 2$. These ratios are the object of economic interest for BR.

We can relate the BR model directly to the earlier threshold crossing literature by converting it into a series of related threshold crossing models. These ordered-probit style "entry probabilities" can be re-written as follows:

$$\begin{aligned}
\Pr(y \geq 1|x) &= \Phi(v_1(x)) \\
\Pr(y \geq 2|x) &= \Phi(v_2(x)) \\
&\dots \\
\Pr(y \geq n|x) &= \Phi(v_n(x))
\end{aligned} \tag{2.9}$$

A very important point that follows directly from the previous section is that the ratios of variable profits that are of economic interest are not robust to monotonic transformations of v_y and F and so:

> the economic parameter of interest is not non-parametrically identified in Bresnahan and Reiss models without shape restrictions.

It is easy to show that one can find a monotonic transformation that sets the relevant ratio to anything between 0 and 1. That is,

Result 2 *Let F be independent of x and $v_y(x)$ is unspecified except that it is weakly monotonic in y. Assume that we observe a random sample of iid markets. Then, the ratio $\frac{v_2(x)}{v_1(x)}$ can take any value in $[0, 1]$.*

This seems like a negative result for the Bresnahan and Reiss model. However, the intuition for identification in BR depends very much on the use of the "proportional to population" restriction discussed in the prior subsection. By the same argument as in the prior subsection, restricting variable profits to be proportional to population **will identify** variable profits. Indeed, once we assume the $v_y(x, z) = zv_y(x)$, then for each number of firms n we can identify both v_n and Φ from the threshold condition $\Pr(y \geq n|x) = \Phi(zv_n(x))$. In particular, we have the following result:

Result 3 *Consider the symmetric game in (2.5) above. Then,*

1. *Suppose that there exits (x, x') such that $G_i(x') = G_j(x)$ for $i \neq j$, then $v_i(x') = v_j(x)$. For example, for (x, x') such that $G_1(x') = G_2(x)$ then $v_1(x') = v_2(x)$.*
2. *If $v_y(x) = zv_y(\tilde{x})$, and $x = (z, \tilde{x})$ and $x' = (z', \tilde{x})$ such that $G_2(x') = G_1(x)$, then the ratio of interest is identified:*
$$\frac{v_2(\tilde{x})}{v_1(\tilde{x})} = \frac{z}{z'}$$
3. *If $v_y(x) = zv_y(\tilde{x})$ and $v_1(x_0) = a \equiv 1$, then*
$$\Phi(z) = G_1(z, x_0)$$
and the function $\Phi(.)$ is identified on the support of z. Also, for all y,
$$v_y(\tilde{x}) = \frac{F^{-1}(G_y(z, \tilde{x}))}{z}$$
i.e. the function $v_y(.)$ is identified on the support of x and z.

In part 1 of the result above, and without making any assumptions, we can match the choice probabilities (the G's) from markets with different entrants, to recover regressor values for which the variable profits match. This matching allows one to examine how much one needs to change (or compensate) a regressor so that variable profits in a duopoly market is equal to variable profits in a monopoly market. The second part of the result above shows that the object of interest in BR is nonparametrically point identified in an oligopoly setting with the usual "proportional to population" assumption given variation in z and x. It is interesting to note for example that this ratio of interest is equal to a specific ratio of the populations. In addition, if we normalize profits, we can recover the distribution of fixed costs and the variable per capita profit function.

Actually, with the proportional-to-population assumption, we can even relax the model to allow for a different distribution, $\Phi_y(F)$, of fixed costs for every y. We then convert the model into a set of unrelated threshold-crossing models, one for each value of y. One economic motivation for the distribution of F to shift with the number of firms y would be the possibility of some fixed resource with an upward sloping supply curve. Indeed, BR include a parametric shift in Φ for different y.

An even easier result is possible when there is additional data available that we can use to identify $v_y(x)$ exactly. There is a large literature in empirical industrial organization on how to use observed data on prices, and quantities (and possibly product characteristics and/or direct information on inputs or costs) to identify the oligopoly and (possibly differentiated products) versions of the "supply and demand" models (see as examples Bresnahan (1989) and Berry, Levinsohn, and Pakes (1995).) In these cases, we may be able to learn about $v_y(x)$ without using the information on entry. It is then trivial to use the observed v's to back out the distribution of F from the relationship $\Pr(y \geq n|x) = \Phi(v_n(x))$. This idea is implemented in a parametric context in Reiss and Spiller (1989) and Berry and Waldfogel (1999). Also, recall from the discussion in section (2.1) that even when the "nature of competition" is not defined, other features of v that are subject to monotonic transformations will still be defined.

To summarize, then, in the binary choice symmetric model of entry à la Bresnahan and Reiss, and without assumptions on the shape of the variable profit function or the distribution of fixed costs conditional on x, (point) identification is not possible. However, results from the threshold crossing literature suggest that variable profits are point identified when one imposes shape restrictions like homogeneity. It is thus important to motivate the assumptions one uses. An example of such assumptions is the insight of BR to write the variable profit as per capita profit multiplied by the size of the market. Or, alternatively, in some settings one has additional information on the variable portion of profits (such as a separate data set from which the form of variable profits can be ascertained) that can be used and so the problem becomes one on inference on the distribution of fixed costs.

2.3 Bivariate Game with Heterogenous Profits

We consider a richer version of the BR model by allowing firms' profits to be different, as in the work by Berry (1989), Bresnahan and Reiss (1991a), Berry (1992), and Ciliberto and Tamer (2003).

In particular, consider the simple duopoly model with two firms and heterogenous fixed costs. Let the profit of firm i in market m be

$$\begin{aligned}\pi_{im}(x_{im}, y_{jm}, f_{im}) &= v(y_{jm}, x_{im}) - f_{im} &(2.10)\\ &= v_{0i}(x_{im}) + y_{jm}v_{1i}(x_{im}) - f_{im} &(2.11)\end{aligned}$$

where the second equality is without loss of generality since y_{jm} is a binary variable. The profit of firm j is similar. Again, we assume that a firm enters a market when it is profitable to do so, and hence firm i enters market m, $y_{im} = 1$, if and only if $\pi_{im}(x_{im}, y_{jm}, f_{im}) \geq 0$ (here non-entry profits are set to zero) and we also assume that both functions v_{1i} and v_{1j} are nonpositive. Notice that in the above, the effect of firm i on firm j's profits is different than the effect of firm j on firm i's profits. The objects of interest are the variable profits and the joint distribution of the fixed costs conditional on the regressors, $F_{f_i,f_j|x_i,x_j}$. The correlation among the unobservables is of interest since it allows one to see whether entry occurs because of unobserved profitability that is independent of the competition effect.

We assume that we have a random sample of observations on markets where every observation is an observable realization of an equilibrium game played between i and j. The sampling assumptions allows us to obtain the choice probabilities $\Pr(0,0|x), \Pr(1,0|x), \Pr(0,1|x)$ and $\Pr(1,1|x)$. To analyze identification, as usual, one needs to relate these quantities to their counterparts predicted by the model.

2.4 Pure Strategies

Here, we study inference in the bivariate model above when we allow only pure strategy equilibria. With mixed strategies ruled out, we can focus on the $(0,0)$ and $(1,1)$ outcomes, the only 2 outcomes of the game that are observed only if they are pure strategy equilibria of the game[8]. The predicted choice probabilities for $(0,0)$ and $(1,1)$ are

$$\begin{aligned}&\Pr(0,0|x_{im}, x_{jm}) = \Pr(f_{im} \geq v_{0i}(x_{im}); f_{jm} \geq v_{0j}(x_{jm})|x_{im}, x_{jm})\\ &\Pr(1,1|x_{im}, x_{jm}) = \Pr\left(f_{im} \leq v_{0i}(x_{im}) + v_{1i}(x_{im}); f_{jm} \leq v_{0j}(x_{jm}) + v_{1j}(x_{jm})|x_{im}, x_{jm}\right)\end{aligned} \tag{2.12}$$

The strategy is to use the first equation above to identify the v_0's and the second equation to identify the v_1's (recall that the v_1 functions are assumed to be nonnegative). Without any further

[8] What is meant here is that $(0,0)$ and $(1,1)$ are uniquely predicted as pure strategy equilibria by this model. If one allows for mixing, then these outcomes can show when players mix between entering and not entering. This will be examined in section 2.5 below.

assumptions, the v's and F are not separately identified. One can then use sufficient conditions, mainly ones involving support conditions, to (point) identify the objects of interests. One example of sufficient conditions that involve support conditions is contained in the next result and discussed after the result.

Result 4 *Assume that the random vector* (f_{im}, f_{jm}) *is distributed independently of* (x_{im}, x_{jm}) *with a joint distribution* F. *Assume that only pure strategy equilibria are considered. Suppose that* $v_{0i}(x_{im}) = z_{im}v_0(\tilde{x}_{im})$ *and similarly for* v_{0j}, $v_{0i}(x_0) = v_{0j}(x_0) = 1$ *and that the* $v_1(.)$*'s are non-positive. Moreover, assume that* $z_{im}|z_{jm}, x_{im}, x_{jm}$ *has a distribution with support on* $\mathcal{R}$ *a.e.* z_{jm}, x_{im}, x_{jm} *and similarly for* z_{jm}. *Then,* v_{0i}, v_{0j}, v_{1i}, v_{1j} *and* F *are identified.*

To show the above result, and using the first equation in (2.12), we can recover the joint distribution F from the choice probabilities evaluated at the normalization point x_0. Then, again using the first equation, we can "push" z_{jm} far out so that effectively we can transform the problem into a simple threshold crossing model with known distribution (marginal of F) and solve for v_{0i}. We can then use the second equation in (2.12) to get the v_1's. Notice that the sufficient conditions above impose exclusion restrictions: z_i with wide support is included in the payoff of firm i but not in firm j's, and vice versa for j. Notice also that we used only the two outcomes $(0, 0)$ and $(1, 1)$ to get identification (where we implicitly used the $(1, 1)$ choice probability to get the v_1 functions). These outcomes are observed if and only if they are pure strategy equilibria of the underlying game. This is a result of this being an entry model (negative v_1's) and the restriction to pure strategy equilibria.

Identification in more realistic games with a larger set of players is harder under the same conditions especially if one maintains a rich correlation structure on the joint distribution of the unobservables (like allowing heterogeneity in the marginal distributions). Mazzeo (2002) allowed for types of firms and hence his model is a bivariate version of the BR model above. Berry (1992) introduced a more heterogenous model of entry in airline markets with potential entrants and player specific variables where the profit functions was specified up to a finite dimensional parameter. Berry showed there that the model predicts the *number* of players in a market uniquely which allows one to reduce the identification into one similar to identification in nonlinear parametric method of moments model. Ciliberto and Tamer (2005) extended the Berry model by allowing for heterogeneity and player identities.

2.4.1 Variation in the Number of Potential Entrants

We will now consider another approach to identification in complete information entry games with heterogeneous firms: variation in the number or identity of potential entrants (from regulation or geographic variation.) This idea is related to some intuition on identification found in Berry (1992).

We begin with the relatively simple case of independent draws on F and then consider the more complicated (and interesting) case of correlated fixed costs. Consider the simple case where the profits of firm i in market m, given y total firms in the market, are $v_y(x_m) - F_{im}$, which is a simplification from the earlier examples. (This form can be derived from examples with heterogeneity only in fixed costs.) We continue to assume that F is independent of x.

The basic idea is to return in part to the binary threshold identification literature by breaking the problem into two parts. First, we think of identifying certain threshold crossing probabilities, such as the probability that a given firm would be profitable if that firm were (exogenously) made into a monopoly entrant. We will see that these probabilities can be useful for many exercises, such as predicting market share, or else considering "sources of profitability" (the relative effect of different x's on firm's profits). These can be derived under fairly weak assumptions – for example we don't need "shape" restrictions on the variable profit function. Second, if necessary for the question at hand, the threshold probabilities can be transformed into the underlying variable profit function $v_y(x)$ under the same conditions (e.g. shape restrictions) as discussed in section (2.1). Given an appropriate restriction, we can then discuss questions about the "nature of competition" as in BR.

Define, to begin, the probability that a firm would be profitable as a monopolist as

$$\mu(x) \equiv \Pr(F_{im} < v_1(x)) \tag{2.13}$$

and the probability that a firm would be profitable as a duopolist as

$$\delta(x) = \Pr(F_{im} < v_2(x)) \tag{2.14}$$

Note that these are not equilibrium probabilities, but rather just particular transformation of the model primitives. For example, $\delta(x)$ is the probability that a firm would be profitable if it was placed exogenously into a market with one rival firm already present.[9]

In the monopoly case of section (2.1), as in the BR case, $\mu(x)$ is just the probability of entry and so is directly observed in the data. In the case of heterogeneous firms, however, the threshold

[9]Note that μ and δ could be derived under somewhat weaker assumptions – for example that profits are a non-linear (but monotonically increasing) function of the independent error, F.

probabilities must be derived from the observed probabilities of different market structure. The derivation is simple when the fixed costs F are independent across firms within a market, which is not a particularly interesting case but provides a good starting point.

In the two-firm case with independent fixed cost, we have for $x = (x_1, x_2)$, the conditions

$$Pr(0,0\ |x) = (1-\mu(x_1))(1-\mu(x_2)) \tag{2.15}$$
$$Pr(1,1\ |x) = \delta(x_1)\delta(x_2), \tag{2.16}$$

and these equations can be used to express μ and δ as unique functions of the data probabilities. In particular, with $x_1 = x_2 \equiv x$ we have two equations in two unknowns and a unique solution for $\mu(x)$ and $\delta(x)$. This is a point-wise (in x) identification argument and, once μ and δ are identified for multiple x's, the model has testable restrictions for values of x_1 not equal to x_2.

These probabilities are useful on their own account. For example, they can be used to solve for counter-factual predictions of market structure: if one of the potential firms goes bankrupt, what is the probability that a market will be served by the remaining firm? The answer is $\mu(x_i)$. Also, as in (2.1), we can identify the sign of the effect of a specific x on profits and we can identify the relative (but not absolute) effects of those x's.

Interestingly, the questions in Berry (1992) concern the sources of profitability and so they might have been answered without solving for the underlying v functions. However, if (as in BR), one has questions about the "nature of competition", then one needs to transform the μ's and δ's into the underlying v's using, e.g., the shape restrictions discussed in section (2.1). This approach is relatively easy to extend to more than 2 potential firms.

The advantages of solving for the threshold crossing-probabilities include:

1. weak assumptions on underlying profit function
2. easy to state restrictions (polynomials)
3. the ability to make equilibrium predictions about market structure
4. the ability to learn about "sources of profitability", and
5. the ability to learn, using stronger restrictions, about v and the distribution of F.

Disadvantages of the approach in this section include:

1. not making use of stronger restrictions from the underlying profit model, and

2. limited modeling of correlation (but see below)

Adding Market-Level Correlation

So far, this subsection has not considered the realistic case of within-market correlation in the unobserved fixed costs (or, more generally, the unobserved profit shocks) of the firms. In particular, one thinks that some markets are more profitable (have lower fixed costs) than others. Say that ϵ_m is a market-level scalar random variable that shifts the distribution of fixed costs for all firms in the market.

The probability of being profitable as a monopolist is now $\mu(x, \epsilon)$ and the observed probabilities of market outcomes take forms like

$$Pr(0,0 \mid x) = \int (1 - \mu(x_1, \epsilon))(1 - \mu(x_2, \epsilon))d\Gamma(\epsilon). \tag{2.17}$$

where $\Gamma(.)$ is the distribution of market level unobservables which are assumed to be independent of x. For discrete distributions of ϵ, we can try to solve for the threshold crossing probabilities $\mu(x, \epsilon)$, but one will typically need some additional variation in the data. As an example, assume that the market level shifter takes on two values, 0 and 1, with $Pr(\epsilon = 0) = \lambda$.

In the two potential entrant case, the probability of no firms is now:[10]

$$\begin{aligned} Pr(0,0 \mid x) &= \lambda(1 - \mu(x_1, 0))(1 - \mu(x_2, 0)) + && (2.18) \\ & \quad (1 - \lambda)(1 - \mu(x_1, 1))(1 - \mu(x_2, 1)) && (2.19) \end{aligned}$$

Holding the number of potential entrants fixed, it is clear that there are not enough restrictions to get the monopoly threshold crossing probabilities from this equation alone.

However, we get a further source of identification if there is variation in the number of potential entrants, as in Berry (1992).[11] In that paper, it was assumed (consistent with data), that the only potential entrants into an airline city-pair were those with some service out of at least one of the endpoints of the city-pair. Another example would be competition between retail chains that do not operate in all regions of the country. Or, consider a dataset that contains market outcomes before and after the entry or exit of a major rival.

To illustrate the possibilities, take even simple case with $x = x_1 = x_2$ and let the number of

[10]Note that the term $\mu(x, 0)$ has the interpretation of being the probability that a firm is profitable [i] as a monopolist [ii] when overall market profitability is "low".

[11]Another useful restriction comes from markets that share a value for x_1 but have a different x_2 (and vice-versa).

potential entrants be K. With K = 1,

$$Pr(0\ |x) \quad = \quad \lambda(1-\mu(x,0)) + (1-\lambda)(1-\mu(x,1)),$$

whereas with K = 2,

$$Pr(0,0\ |x) \quad = \quad \lambda(1-\mu(x,0))^2 + (1-\lambda)(1-\mu(x,1))^2,$$

and so forth for arbitrary K:

$$Pr(0,0,\ldots\ |x) \quad = \quad \lambda(1-\mu(x,0)^K + (1-\lambda)(1-\mu(x,1))^K.$$

Assume now that we observe market outcomes in markets with varying numbers of potential entrants, from $K=1$ to $K=\bar{K}$. Then we have $\bar{K}$ polynomial equations in 3 unknowns: $\mu(x,0)$, $\mu(x,1)$ and the "correlation" term, λ. Hence the problem becomes one of finding solutions to (nonlinear) equations in 3 unknowns (that are probabilities). As $\bar{K}$ increases further, we will find multiple solutions for the unknowns only in degenerate cases and so we can likely solve uniquely for both the μ's and the discrete distribution of ϵ.

Once again, "shape restrictions" can be used to uncover variable profits, and the appropriate marginal distributions of F. In this example, we would be solving for the marginal distribution of F conditional on the appropriate value of the market-level shock ϵ. Since we also know the discrete distribution of ϵ, we also uncover the unconditional distribution of F.

In this subsection, expanding on the suggestion of Berry (1992), we have sketched the possibilities of using the variation in number of entrants as a source of identification. We have also suggested an identification strategy that is different from most of the existing literature. First, we try to identify the probability that firms would be profitable if placed (exogenously) into various market structure. In the case of discrete market-level shocks, this involves a solution to a set of simple polynomial equations. Then, if necessary, these threshold probabilities can be transformed into the underlying variable profit function and fixed-cost distribution.

2.5 Multiple Equilibria and Mixed Strategies

Inference in static discrete games with complete information is complicated due to the presence of multiple equilibria and equilibria in mixed strategies. In the game of the previous section, one is able to derive a set of sufficient conditions for point identification without dealing with multiplicity.

The reason for this is that the game, under pure strategies, predicts equilibria that are unique in the *number* of players (for example, identification was based on the choice probabilities of the $(0,0)$ and $(1,1)$, outcomes which are equivalent to no firms and two firms in the market while the third outcome is the union of $(1,0)$ and $(0,1)$ and represents the one firm equilibrium). Berry (1992) studied a more general class of entry games with more players in which the number of firms in a market is unique. It is worth noting that in these games without mixed strategies, one need not specify (or model) any *equilibrium selection rules* since the game is transformed into one that predicts a unique feature (the number of players in the market).

Uniqueness in the number of firms across equilibria does not always hold. Ciliberto and Tamer (2003) extend the Berry setup to cover entry games with identity (or type) specific effects. For example, the effects on American's profits from having Southwest enter is different than if United enters. This heterogeneity in the effects leads the game to have multiple equilibria where different equilibria can differ in the number of firms, even when one is restricted to pure strategies. A simple two types example of such a game is illustrated in the appendix. Another example where uniqueness across equilibria does not hold is in games where one allows for mixed strategy equilibria. In the simple bivariate game studied in section 2.3, if we allow for mixed strategies, then $(0,0)$ is a potentially observable outcome of the mixed strategy equilibrium in which the players mix between entering a market and not since it lies on the support of the mixing distribution. Hence, when one observes $(0,0)$, it is not clear whether this was because it is the unique pure strategy equilibrium or it was a result of the mixing. So, for some values of the unobservables, the econometric model can predict $(0,0)$, $(1,1)$, $(1,0)$ and $(0,1)$ as potentially observables. Recently also, Bajari, Hong and Ryan (2005) provide sufficient conditions based on exclusion restrictions that allow for identification in the presence of mixed strategies.
Below, we first consider a simple game that involves multiple equilibria and mixing and study its observable implication. This will illustrate in a clear way the insights needed to analyze these games and illustrates, in a simple example, that the data might contain information about selection. Then, we go back to the bivariate game of the previous section and study its empirical content in the presence of mixing and multiplicity.

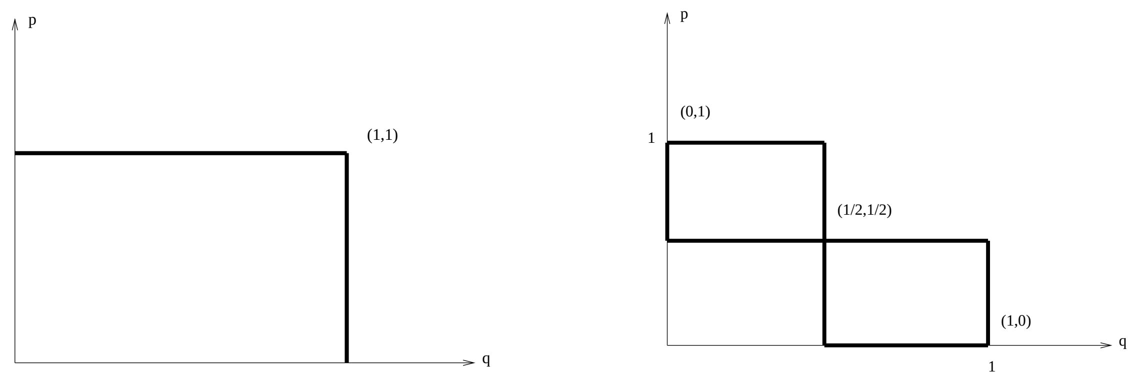
Inference in a Game with Mixed Strategies and Multiple Equilibria

Consider the following simple 2×2 game

	0	1
0	0 , 0	0 , a
1	a , 0	a - 2 , a - 2

where a is a binary random variable that takes the value of 3 with probability l and 1 with probability $1 - l$. The realization of a is not known to the econometrician but is observed by both players. This is a symmetric game with no exogenous regressors and no heterogeneity. The question that we want to ask is what is the observable implication of this game? The "data" we observe is a sample of observable outcomes each of which corresponds to a draw of the random variable a. Given a random sample assumptions (say we observe an *iid* sample of N markets), we are able to consistently estimate the choice probabilities $P(0,0), P(0,1), P(1,0)$, and $P(1,1)$ as N increases. Next, we derive the choice probabilities implied by the model (i.e., as a function of the structural parameters of the model). When, $a = 3$, $(1,1)$ is the unique pure strategy equilibrium of the game (see the lhs display in figure 1). On the other hand, when $a = 1$, the game has multiple equilibria. There are two pure strategy equilibria, $(1,0)$ and $(0,1)$ and one mixed strategy equilibrium $p = \frac{1}{2}$ where p is the mixing probabilities for both players (See rhs of figure 1). Assuming only pure

Figure 1: Equilibria for the case when $a = 3$ (left) and when $a = 1$ (right)



strategies are allowed, the choice probabilities derived by the model are

Predicted probabilities without mixed strategies

$$
\begin{aligned}
\Pr(0,0) &= 0 \\
\Pr(1,1) &= \Pr(1,1|a=1)\Pr(a=1) + \Pr(1,1|a=3)\Pr(a=3) \\
&= \Pr(1,1|a=3)\Pr(a=3) = l \\
\Pr(0,1) &= \Pr(0,1|a=1)\Pr(a=1) + \Pr(0,1|a=3)\Pr(a=3) \\
&= \Pr(0,1|a=1)\Pr(a=1) = \Pr(0,1|a=1)(1-l) \\
\Pr(1,0) &= \Pr(1,0|a=1)\Pr(a=1) + \Pr(1,0|a=3)\Pr(a=3) \\
&= \Pr(1,0|a=1)\Pr(a=1) = \Pr(1,0|a=1)(1-l)
\end{aligned}
$$

We have $\Pr(1,1|a=3) = 1$ since $(1,1)$ is the unique equilibrium when $a=3$ and $\Pr(1,1|a=1) = 0$ since when $a=1$ and without allowing for mixed strategies, $(1,1)$ is not a *potentially observable outcome of the game*. So the predicted choice probability by the model above for the $(1,1)$ outcome is $\Pr(a=3) = l$ (and hence l is point identified by looking at $(1,1)$ markets). On the other hand, the predicted choice probability for the $(1,0)$ outcome is $\Pr(0,1) = \Pr(0,1|a=1)\Pr(a=1) = \Pr(0,1|a=1)(1-l)$ where here $\Pr(0,1|a=1)$ is the equilibrium selection rule. This selection rule is also identified (since l is identified from the $(1,1)$ outcomes). So, as we can see in this simple game the parameters of the game (l and the selection rule function $P(0,1|a)$) are point identified[12].

When we allow for mixed strategies, the predicted choice probabilities will change. When $a=3$, the outcome $(1,1)$ is the unique pure strategy equilibrium of the game and hence is the unique observable outcome. On the other hand when $a=1$, the *potentially observable outcomes* are all 4: $(1,0)$, $(0,1)$, $(0,0)$ and $(1,1)$. In particular, $\Pr(0,0|a=1) \neq 0$ since $(0,0)$ will appear as the *observable outcome* of the game if players coordinate on the mixed strategy equilibrium in which $(0,0)$ will appear with probability $\frac{1}{4}$. So the predicted choice probabilities with mixed strategies are:

$$
\begin{aligned}
\Pr(0,0) &= \Pr(0,0|a=1)\Pr(a=1) \\
\Pr(1,1) &= \Pr(1,1|a=3)\Pr(a=3) + \Pr(1,1|a=1)\Pr(a=1) = l + \Pr(1,1|a=1)\Pr(a=1) \\
\Pr(1,0) &= \Pr(1,0|a=1)\Pr(a=1) + \Pr(1,0|a=3)\Pr(a=3) = \Pr(1,0|a=1)\Pr(a=1)
\end{aligned}
$$

Conditional on $a=1$, let the random variable d denote the equilibrium selection mechanism: $d=1$ if $(1,0)$ equilibrium is selected with probability P_1 , $d=2$ if the $(0,1)$ equilibrium is chosen with probability P_2, and $d=3$ if the mixed strategy equilibrium $(\frac{1}{2},\frac{1}{2})$ is chosen with probability

[12] The selection rule depends on unobservables. But since the support of a has only two points, the probabilities are tractable.

$P_3 = 1 - P_1 - P_2$. Hence,

Predicted choice probabilities with mixed strategies

$$\begin{aligned}
\Pr(0,0|a=1) &= \sum_{i=1,2,3} \Pr(0,0|a=1,d=i)\Pr(d=i|a=1) = \Pr(0,0|a=1,d=3)P_3 = \frac{1}{4}P_3 \\
\Pr(1,0|a=1) &= \sum_{i=1,2,3} \Pr(1,0|a=1,d=i)\Pr(d=i|a=1) \\
&= \Pr(1,0|a=1,d=3)P_3 + \Pr(1,0|a=1,d=1)P_1 = \frac{1}{4}P_3 + P_1 \\
\Pr(1,1|a=1) &= \sum_{i=1,2,3} \Pr(1,1|a=1,d=i)\Pr(d=i|a=1) \\
&= \Pr(1,1|a=1,d=3)P_3 = \frac{1}{4}P_3
\end{aligned}$$

This implies that

$$\begin{aligned}
\Pr(0,0) &= \frac{1}{4}P_3(1-l) \\
\Pr(1,0) &= (\frac{1}{4}P_3 + P_1)(1-l) \\
\Pr(0,1) &= (\frac{1}{4}P_3 + P_2)(1-l) \\
\Pr(1,1) &= l + \frac{1}{4}P_3(1-l) \\
P_1 + P_2 + P_3 &= 1 \\
0 \le l &\le 1
\end{aligned}$$

The parameters of interest in this game are (P_1, P_2, P_3, l). The identification question in this game reduces to one of studying the set of solutions to the above system of equalities/inequalities. The game (or parameters) is point identified if there exists a unique solution, and is set identified if the game restricts the parameters to a non-singleton set of parameters (and of course in case of no solutions, then the game is misspecified).

A simple insight that emerges from this example is the importance of pinpointing the set of *potentially observable outcomes* in a game. Heuristically, given a value for the exogenous variables, the set of potentially observable outcomes or POO is the set of outcomes that can be observed when the game is played. For example, when $a = 1$, the set of observable outcomes is $\{(0,0),(1,1),(1,0),(0,1)\}$. The POO consists of the pure strategy equilibria (which can be mapped one to one to a set of outcomes) and the set of outcomes that are on the support of the mixed strategy distribution.

The equilibrium selection mechanism plays a key role in forming the predicted choice probabilities. Usually, the economist has no information about equilibrium selection or is not willing

Table 1: Bivariate Entry Game

	0	1
0	0,0	0, $x_2\beta_2 - \epsilon_2$
1	$x_1\beta_1 - \epsilon_1$,0	$x_1\beta_1 + \Delta_1 - \epsilon_1$, $x_2\beta_2 + \Delta_2 - \epsilon_2$

to make assumptions on equilibrium selection. The reason is that it is not unusual that different equilibria can be chosen in otherwise "similar" markets. So, the key *practical* question becomes: what can we learn about the features of the model (like the variable profit functions and the joint distribution of unobserved fixed costs) without imposing any restrictions on equilibrium selection? in other words, is using an ad-hoc selection rule practically important? Another important issue is whether the model contains information about the selection mechanism. The simple game above shows that it does. So, the data does have identifying power regarding the selection mechanism (the data generally point identifies the selection rule in the simple game above).

2.5.1 Bivariate Entry Game: Inference, Multiplicity, and Mixed Strategies

Reconsider the bivariate entry game studied in section (2.3) above except that here and for simplicity we parametrize the profit functions as in Table 1. This game is similar to the one studied in Tamer (2003).

Again, we assume that Δ_1 and Δ_2 are negative. So, for $\epsilon_1 \in [x_1\beta_1 + \Delta_1, x_1\beta]$ and $\epsilon_2 \in [x_2\beta_2 + \Delta_2, x_2\beta]$ the game has multiple equilibria: $(0, 1)$ and $(1, 0)$ are pure strategy equilibria and there is a mixed strategy equilibrium. To study identification in this game in the presence of mixed strategies (and multiplicity), we follow the insights given in the simple game above[13]. Notice that, $(1, 1), (0, 0), (1, 0)$ and $(0, 1)$ are potentially observed outcomes in the case where the mixed strategy is an equilibrium. So, again we assume that we have a random sample of markets and that the vector (ϵ_1, ϵ_2) is independent of $x = (x_1, x_2)$ and has a known (up to some parameter) joint density with support on the real plane[14]. We are interested in inference on the parameter vector $\theta = (\beta_1, \beta_2, \Delta_1, \Delta_2, \gamma)$ where γ is the parameter of the joint distribution of the ϵ's. Define the following sets. First, let

$$A^{x,\theta}_{(p,q)} = \{(\epsilon_1, \epsilon_2) : (p, q) \text{ is the unique equilibrium } \}$$

[13]Tamer (2003) studied identification in this game without allowing for mixed strategies. The inequalities based approach used in Ciliberto and Tamer (2005) can easily be extended to allow for mixing. Recently also, Bajari, Hong and Ryan (2005) introduce a model where mixed strategies are explicitly accounted for.

[14]Identification in this game with unknown F is complicated and will not be dealt with here.

In the case of multiplicity, define

$$M_C^{x,\theta} = \{(\epsilon_1, \epsilon_2) : C \text{ is a set of multiple equilibria } \}$$

where either set A or M can be empty. Similar definitions can be written for any discrete game in general, but we keep this example specific definitions for simplicity. For example, sets A are regions for the epsilon's where the outcome $(p, q) \in [0,1] \times [0,1]$ is the unique equilibrium for a given value of x and θ. This equilibrium can either be in pure or in mixed strategies. For example $A_{(1,1)}^{x,\theta}$ is the set of epsilons where $(1,1)$ is the unique pure strategy equilibrium. This region is $(\epsilon_1, \epsilon_2) \in [-\infty, x_1\beta_1 + \Delta_1] \times [-\infty, x_2\beta_2 + \Delta_2]$ for the game above. On the other hand, sets M are regions for the epsilon's where the game predicts multiple equilibria (some of which can be in mixed strategies). In the above game for example, $M_{\{(1,0),(0,1),(p,q)\}}^{x,\theta} = \{(\epsilon_1, \epsilon_2) \in [x_1\beta_1 + \Delta_1, x_1\beta] \times [x_2\beta_2 + \Delta_2, x_2\beta]\}$. For a given x and θ, the game predicts the following choice probabilities:

$$\begin{aligned}
\mathcal{P}(x;\theta,\pi) &= \begin{pmatrix} \Pr((0,0)|x,\theta,\pi) \\ \Pr((1,1)|x,\theta,\pi) \\ \Pr((1,0)|x,\theta,\pi) \\ \Pr((0,1)|x,\theta,\pi) \end{pmatrix} \\
&= \begin{pmatrix} \int \sum_{i\in\mathcal{C}} \Pr((0,0)|x,\theta,\pi,\epsilon,d=i) \Pr(d=i|\epsilon,x) dF_\epsilon \\ \int \sum_{i\in\mathcal{C}} \Pr((1,1)|x,\theta,\pi,\epsilon,d=i) \Pr(d=i|\epsilon,x) dF_\epsilon \\ \int \sum_{i\in\mathcal{C}} \Pr((1,0)|x,\theta,\pi,\epsilon,d=i) \Pr(d=i|\epsilon,x) dF_\epsilon \\ \int \sum_{i\in\mathcal{C}} \Pr((0,1)|x,\theta,\pi,\epsilon,d=i) \Pr(d=i|\epsilon,x) dF_\epsilon \end{pmatrix} \\
&= \begin{pmatrix} \Pr(A_{(0,0)}^{x,\theta}) + \int_{M_{(0,0)}^{x,\theta}} (1 - \frac{x_1\beta_1-\epsilon_1}{-\Delta_1})(1 - \frac{x_2\beta_2-\epsilon_2}{-\Delta_2}) \Pr(d=3|x,\epsilon_1\epsilon_2) dF \\ \Pr(A_{(1,1)}^{x,\theta}) + \int_{M_C^{x,\theta}} \frac{x_1\beta_1-\epsilon_1}{-\Delta_1} \frac{x_2\beta_2-\epsilon_2}{-\Delta_2} \Pr(d=3|x,\epsilon_1\epsilon_2) dF \\ \Pr(A_{(1,0)}) + \int_{M_C^{x,\theta}} \Pr(d=1|x,\epsilon_1,\epsilon_2) dF + \int_{M_C^{x,\theta}} \frac{x_1\beta_1-\epsilon_1}{-\Delta_1} (1 - \frac{x_2\beta_2-\epsilon_2}{-\Delta_2}) \Pr(d=3|x,\epsilon_1\epsilon_2) dF \\ \Pr(A_{(0,1)}) + \int_{M_C^{x,\theta}} \Pr(d=2|x,\epsilon_1,\epsilon_2) dF + \int_{M_C^{x,\theta}} (1 - \frac{x_1\beta_1-\epsilon_1}{-\Delta_1}) \frac{x_2\beta_2-\epsilon_2}{-\Delta_2} \Pr(d=3|x,\epsilon_1\epsilon_2) dF \end{pmatrix}
\end{aligned} \tag{2.20}$$

where $C = \{(0,1),(1,0),(p,q)\}$ and $\pi = (\Pr(d=i|x,\epsilon_1,\epsilon_2), i = 1,2,3)$, $\sum_i \Pr(d=i|x,\epsilon_1,\epsilon_2) = 1$ is the equilibrium selection mechanism in the region of multiplicity. Notice, this equilibrium selection mechanism is left unspecified and can depend on market unobservables. The outcome $d = 1$ corresponds to "selecting" $(1,0)$, $d = 2$ corresponds to selecting $(0,1)$ and $d = 3$ corresponds to selecting the mixed strategy equilibrium. Examining the probability of $(0,1)$ we find that

$$\Pr((1,0)|x,\theta,\pi) = \underbrace{\Pr(A^{x,\theta}_{(1,0)})}_{(1)} + \underbrace{\int_{M^{x,\theta}_C} \Pr(d=1|x,\epsilon_1,\epsilon_2)dF}_{(2)}$$

$$+ \underbrace{\int_{M^{x,\theta}_C} \overbrace{\frac{x_1\beta_1-\epsilon_1}{-\Delta_1}(1-\frac{x_2\beta_2-\epsilon_2}{-\Delta_2})}^{(3a)} \Pr(d=3|x,\epsilon_1\epsilon_2)dF}_{(3)}$$

The first term (1) is the probability of the region $A_{(1,0)}$ under F, i.e., $(1) = \Pr(\epsilon_1 \leq x_1\beta_1; \epsilon_2 \geq x_2\beta_2) + \Pr(\epsilon_1 \leq x_1\beta_1 + \Delta_1; x_2\beta_2 + \Delta_2 \leq \epsilon_2 \leq x_2\beta_2)$. This is the probability mass of the region where $(1,0)$ is the *unique* equilibrium of game. Next, the region $M^{x,\theta}_C$ is the region of multiplicity and this model it is the "square" where $x_1\beta_1 + \Delta_1 \leq \epsilon_1 \leq x_1\beta_1$ and $x_2\beta_2 + \Delta_2 \leq \epsilon_2 \leq x_2\beta_2$. So, in (2) we calculate the probability that the outcome $(1,0)$ is selected (i.e. $d=1$) which is the weighted probability of selection integrated against F in the region of multiplicity. The third term (3) takes account of mixed strategies. If mixed strategies are allowed, all 4 outcomes are potentially observed. In particular, conditional on $d=3$ (mixed strategy equilibrium), (3a) gives the (mixing) probability of observing $(1,0)$.

As we can see, observing event $(1,0)$ is a consequence of several distinct possibilities (three here): first, either that $(1,0)$ is the unique equilibrium of the game (term (1)), $(1,0)$ is an equilibrium of a game with multiplicity where: it is a pure strategy equilibrium (term (2)), or it is on the support of a mixed strategy equilibrium (term (3)). So, without making equilibrium selection assumptions, the presence of multiple equilibria complicates the inferential problem by introducing nuisance parameters (selection probabilities). Moreover, with mixed strategies the model does not necessarily predict multiple equilibria that are unique in the number. So, here, the observable implication of multiple equilibria is that $(1,0)$ or $(1,1)$ can show up and hence these observables do not involve the same number of firms entering.

Common equilibrium selection assumptions are the requirements that the distribution of d conditional on x be independent of (ϵ_1, ϵ_2) and sometimes also the x's, i.e. one uses the same selection mechanisms in similar markets. One then usually parametrizes the probability (as in using a multinomial logit) and uses maximum likelihood for inference. See for example Bjorn and Vuong (1985) and more recently Bajari, Hong, and Ryan (2005).

Identification In this setup, identification is defined as follows. The *sharp* identified set Θ_I is

$$\Theta_I = \left\{\theta : \exists\pi \;\text{ such that }\; \Pr\left(\mathcal{P}(x) = \mathcal{P}(x;\theta,\pi)\right) = 1\right\} \tag{2.21}$$

The set Θ_I is the sharp identified set, i.e., the set of parameters θ that are consistent with the data and the model. Heuristically, a $\theta \in \Theta_I$ if and only if there exists a (proper) selection mechanism $\pi = (P(d = i|x, \epsilon_1, \epsilon_2),\ i = 1, 2, 3)$ such that the induced probability distribution $\mathcal{P}(x; \theta, \pi)$ matches the choice probabilities $\mathcal{P}(x)$ for all x almost everywhere. So, the presence of multiple equilibria introduces nuisance parameters that are not specified and hence makes it harder to identify the parameter θ. In the case of point identification, Θ_I reduces to a singleton (this set is non-empty otherwise the model would be misspecified). When one models equilibrium selection (by specifying a consistent π), the identification problem reduces to one in a modified multivariate discrete choice model where sometimes, one is able to provide sufficient point identification conditions (usually involving large support). Under these conditions, the model identifies a unique parameter θ which is associated with that particular selection mechanism. So, if one considers another consistent selection mechanism, the model would point identify another parameter θ. Hence, the set Θ_I collects the set of (structural) parameters that are consistent with a well defined selection mechanism. Allowing for mixed strategies adds an additional selection function (when $d = 3$) and so the set $\mathcal{C}$ increases in size. Hence, the observational implication of mixed strategies is that any outcome on the support of the mixing distribution is potentially observable. So, mixing, although qualitatively similar to multiplicity, complicates the identification problem (and thus makes the task of identification harder) since we require the data to identify more functions.

Remark *In other models (like in the case where the Δ's have a different sign) it is possible that for some values of the ϵ's the game only admits an equilibrium in mixed strategies (the unique equilibrium is one in mixed strategies). So, it does not seem natural in those games to only allow pure strategies. Hence, one needs to pay special attention to the assumption of not allowing mixed strategy equilibria since it might lead to regions of the exogenous variables where the model does not predict any outcomes (this might not be a big problem in some entry models where unique equilibria in mixed strategies only, are less common).*

Inference on the set Θ_I based on definition (2.21) though theoretically attractive is not practically feasible since one needs to deal with infinite dimensional nuisance parameters (the π's).[15] A practical approach to inference in this class of models, follows the approach in Ciliberto and Tamer (2003) by exploiting the fact that the selection mechanism π is a probability and hence bounded between zero and one. Although this approach does not provide a sharp set, it is practically attractive . So, exploiting the fact that $\Pr(d = i|x, \epsilon_1, \epsilon_2) \in [0, 1]$ for $i = 1, 2, 3$ we can get the following model with inequality restrictions on regressions:

[15]Inference is more practical in settings where one discretizes the model, i.e., uses discrete x's and assumes that the epsilons take finitely many values with given probabilities. This makes it into a finite dimensional problem that is easier to handle.

$$
\begin{pmatrix} \Pr(A^{x,\theta}_{(0,0)}) \\ \Pr(A^{x,\theta}_{(1,1)}) \\ \Pr(A^{x,\theta}_{(1,0)}) \\ \Pr(A^{x,\theta}_{(0,1)}) \end{pmatrix} \le \begin{pmatrix} \Pr(0,0|x) \\ \Pr(1,1|x) \\ \Pr(1,0|x) \\ \Pr(0,1|x) \end{pmatrix} \le \begin{pmatrix} \Pr(A^{x,\theta}_{(0,0)}) + \int_{M^{x,\theta}_{(0,0)}} (1 - \frac{x_1\beta_1-\epsilon_1}{-\Delta_1})(1 - \frac{x_2\beta_2-\epsilon_2}{-\Delta_2})dF \\ \Pr(A^{x,\theta}_{(1,1)}) + \int_{M^{x,\theta}_C} \frac{x_1\beta_1-\epsilon_1}{-\Delta_1} \frac{x_2\beta_2-\epsilon_2}{-\Delta_2} dF \\ \Pr(A_{(1,0)}) + \max\big(\Pr(M^{x,\theta}_C), \int_{M^{x,\theta}_C} \frac{x_1\beta_1-\epsilon_1}{-\Delta_1}(1 - \frac{x_2\beta_2-\epsilon_2}{-\Delta_2})dF\big) \\ \Pr(A_{(0,1)}) + \max\big(\Pr(M^{x,\theta}_C), \int_{M^{x,\theta}_C} (1 - \frac{x_1\beta_1-\epsilon_1}{-\Delta_1}) \frac{x_2\beta_2-\epsilon_2}{-\Delta_2}\big) \end{pmatrix} \tag{2.22}
$$

Notice here that the above inequalities represent a cube that consists of a super set of the feasible probabilities. This superset was generated by taking the extreme points of the selection probabilities equation by equations. For example the $(0,0)$ probability in (2.20) can be as low as $\Pr(A^{x,\theta}_{(0,0)})$ and as high as $A^{x,\theta}_{(0,0)}) + \int_{M^{x,\theta}_{(0,0)}} (1 - \frac{x_1\beta_1-\epsilon_1}{-\Delta_1})(1 - \frac{x_2\beta_2-\epsilon_2}{-\Delta_2})dF$. And, then we repeat this exercise for every outcome. By transforming the predictions of the model into ones with *inequality restrictions*, we are able to bypass the specification of the selection probabilities. Note that, we have ignored in constructing the cube the cross equation restrictions (basically that the selection probabilities sum to one). One can certainly exploit these restrictions but we omit this for simplicity of description. The econometric structure of this model is a method of moments model with inequality restrictions. Inference in models with inequality restrictions is studied in Chernozhukov, Hong, and Tamer (2002) and Andrews, Berry, and Jia (2003) and applied in these settings by Ciliberto and Tamer (2003).

Are Multiple Equilibria and Mixed Strategies a Problem? As we can see from above, dealing with multiple equilibria and mixed strategy equilibria in games with complete information introduces functions that, if left unspecified, complicate the identification problem. In addition, policy analysis in the presence of multiplicity is complicated by the fact that the model is only able to predict uniquely in some regions of multiplicity. Motivated by practical convenience, another approach to inference in games is to make simple equilibrium selection assumption or/and pick an equilibrium on the support of a mixed strategy equilibrium using a predetermined rule. This will yield a complete (nonlinear) econometric model where methods for identifying the model can be studied. So the practical question of interest is whether parameters or policy functions can change significantly with different selection rules. One way to examine this question is to look for set estimates of the parameters of interest without making selection assumptions. The "size" of this set is an indicator of how important selection assumptions are. Finally, the framework above maintains parametric assumptions on the variable profits and the joint distribution of the unobservables. It is thus important to study identification in settings where these distributional assumptions are relaxed. In general, in multivariate discrete models, relaxing these complicates the identification problem (See Bajari, Hong and Ryan (2005) for more on this). This is a largely unexplored area of research.

3 Static Games of Incomplete Information

In this section, we examine the question of inference in a discrete game under different informational assumptions. It is maintained that decision makers' profit functions contain a variable that is not observed to them. One may assume that an entrant does not observe another's fixed costs for example. Seim (2002) considers a model of endogenous entry and product positioning where she assumes that some "idiosyncratic sources of profitability are hard to observe by competitors." Another example is the paper by Sweeting (2004) which analyzes an incomplete information model of different radio stations' decision to play a commercial break. Also, a recent paper by Aradillas-Lopez (2005) examines the identified feature in a game similar to the one in Table 1 above but with incomplete information and provides conditions for uniqueness under the assumptions that player belief can depend on variables that do not enter the variable profits function. Most of the work on empirical games with incomplete information[16] deal with the presence of multiple equilibria in two ways. In one way, they provide conditions for uniqueness of equilibria; usually, these are constraints on the shape of the profit function guaranteeing that the equilibrium correspondence admits a unique fixed point. Another way, it is assumed that the same equilibrium is chosen in similar markets or that one uses the same equilibrium selection distribution in observationally similar markets. This equilibrium is the one that can be estimated from the data nonparametrically or using a parametric equilibrium selection mechanism. These assumptions rest on the underlying principle that "similar" equilibria are being played in markets that are observationally equivalent. Here, we provide a different but complementary approach to inference by studying the inferential problem without making equilibrium selection assumptions. We relate our findings to results obtained in the complete information games. We illustrate our insights using a simple bivariate game where we derive the econometric restriction of this game and analyze its identification problem. The objective is to try and nest the different assumptions that are made under a more general framework.

Multiple Equilibria in a Simple Bivariate Game

The nature of equilibria in games with incomplete information is slightly different from the complete information counterpart. Strategies now are mappings from players' types to actions where types are private to the players. We maintain throughout the assumption of common priors , i.e., that the distribution of signals or types is common knowledge to the players and we also restrict ourselves to pure strategies, i,e, we do not allow for mixing. Again, we focus on the bivariate game below where we assume that players have imperfect information about the profit function of their opponent. The

[16]Sweeting (2004) is an exception.

functions u_i and v_i for $i = 1, 2$ are left unspecified as also F the distribution of the unobservables ϵ_1 and ϵ_2 which are assumed to be independent of the observable exogenous variables with F as their joint distribution. Both ϵ_1 and ϵ_2 are not observed by the econometrician but ϵ_1 is private information to player 1 and similarly for player 2.

Table 2: Bivariate Game

	0	1
0	0 , 0	0 , $v_2(x_2) - \epsilon_2$
1	$v_1(x_1) - \epsilon_1$, 0	$v_1(x_1) + u_1(x_1) - \epsilon_1$, $v_2(x_2) + u_2(x_2) - \epsilon_2$

We fix $X = (x_1, x_2)$ and derive the observable implication of the game. It is easy to see that equilibrium mappings are going to be step functions that are decreasing in a threshold. So the Bayesian Nash equilibrium of the game in Table 2 is a pair of mappings $\big(1[\epsilon_1 \le t_1], 1[\epsilon_2 \le t_2]\big)$ where $T = (t_1, t_2)$ are threshold variables that are functions of X, $\theta = (v_1, v_2, u_1, u_2)$ and the common prior distributions. Player 1 believes that ϵ_2 has a distribution G_b^1 and similarly for player 2. These distribution functions can depend on X and θ.[17] So, T solves the following

$$\begin{aligned}
&(v_1(x_1) - t_1)(1 - G_b^1(t_2)) + (v_1(x_1) + u_1(x_1) - t_1)G_b^1(t_2) = 0 \\
&(v_2(x_2) - t_2)(1 - G_b^2(t_1)) + (v_2(x_2) + u_2(x_2) - t_2)G_b^2(t_1) = 0
\end{aligned}$$

where these represent the "zero-profit" conditions. Hence, the equilibrium thresholds T are fixed points of the following mapping

$$H_b(t_1, t_2) = \begin{bmatrix} v_1(x_1) + u_1(x_1)G_b^1(t_2) \\ v_2(x_2) + u_2(x_2)G_b^2(t_1) \end{bmatrix}$$

For example, assuming that $G_b^1 \equiv G_b^2 \equiv G$ then we get that the equilibrium thresholds solve

$$\begin{aligned}
t_1 &= v_1(x_1) + u_1(x_1)G(t_2) \\
t_2 &= v_2(x_2) + u_2(x_2)G(t_1)
\end{aligned} \tag{3.23}$$

The above map can have multiple solutions (equilibria) and so the set of solutions to the above system of equations, $\mathcal{E}$, is defined as

$$\mathcal{E}(X, \theta, G) = \big\{(t_1, t_2) : \ (t_1, t_2) \text{ solves } \ (3.23)\big\}$$

The cardinality of this set is a function of X and (v_1, v_2, u_1, u_2). See the next section for an example where the belief distribution G is normal and the set $\mathcal{E}$ can be shown to have at most three solutions

[17] The belief distribution can also depend on other "excluded" variables Z as in Aradillas-Lopez.

(three equilibria). It is possible to obtain distributions G with a continuum of equilibria (we assume that away below for simplicity). Now, we derive the model predicted probability $\mathcal{P}(X, \theta)$ where $t = (t_1, t_2) \in \mathcal{E}(X, \theta)$:

$$
\begin{aligned}
\mathcal{P}(X,\theta) &= \begin{pmatrix} P_{(0,0)}(X,\theta,G) \\ P_{(1,1)}(X,\theta,G) \\ P_{(1,0)}(X,\theta,G) \\ P_{(0,1)}(X,\theta,G) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{t_i\in\mathcal{E}} \int_{v_1(x_1)+u_1(x_1)G(t_{1i})}^{+\infty} \int_{v_2(x_2)+u_2(x_2)G(t_{2i})}^{+\infty} P(t_i|x,\epsilon_1,\epsilon_2)dF \\ \sum_{t_i\in\mathcal{E}} \int_{-\infty}^{v_1(x_1)+u_1(x_1)G(t_{1i})} \int_{+\infty}^{v_2(x_2)+u_2(x_2)G(t_{2i})} P(t_i|x,\epsilon_1,\epsilon_2)dF \\ \sum_{t_i\in\mathcal{E}} \int_{-\infty}^{v_1(x_1)+u_1(x_1)G(t_{1i})} \int_{v_2(x_2)+u_2(x_2)G(t_{2i})}^{+\infty} P(t_i|x,\epsilon_1,\epsilon_2)dF \\ \sum_{t_i\in\mathcal{E}} \int_{v_1(x_1)+u_1(x_1)G(t_{1i})}^{-\infty} \int_{-\infty}^{v_2(x_2)+u_2(x_2)G(t_{2i})} P(t_i|x,\epsilon_1,\epsilon_2)dF \end{pmatrix}
\end{aligned}
\tag{3.24}
$$

where $\sum_{t_i\in\mathcal{E}} P(t_i|x,\epsilon_1,\epsilon_2) = 1$ for all x, ϵ a.e. Then, one can define identification constructively as we have done above. See (2.21). It is clear that without prior assumptions on the shape of G and the functions u and v that the identified set is not a singleton (and can be large). Note here, that the joint distribution of ϵ's played no role in the cardinality or construction of the equilibria. What matters is the assumption of common priors which we maintain[18], and its distribution. In addition, there is the added complication that one has to "solve" for the set of equilibria for every "parameter" iteration. Notice, that the above formulation in (3.24) is general and hence does not allow one to characterize the identified features of the model. The identification problem becomes less complicated if one imposes assumptions on $P(t|x, \epsilon_1, \epsilon_2)$. For example, one assumption is requiring that $P(t|x, \epsilon_1, \epsilon_2) \equiv P(t|x) = P_\lambda(t|x)$ where P_λ is a known probability distribution up to the parameter λ. This assumption requires that in markets with similar observables, the same equilibrium selection distribution is used. The presence of multiple equilibria introduces the unknown parameter λ and so the inferential problem becomes one of trying to identify the parameters of interest in the presence of the "selection" parameter λ. A more stringent assumption would require that $P(t|x, \epsilon_1, \epsilon_2)$ be degenerate at one of the equilibria. This would be the one that can be estimated from the data. If we also require that the the distribution $G \equiv F$ (i.e. the common prior distribution is the same as the distribution of the ϵ's) then the choice probabilities solve a fixed point theorem.

The presence of multiple equilibria here is of a different nature than in games with complete information in that one is not able to just focus on games where the outcome is uniquely predicted.

[18] In reality, it is not clear why a common prior assumption is usually made in these settings. In addition, even with this assumption, it is not clear why one would want to assume that the common distribution is known to the econometrician.

So, to that extent it might appear that inference in this class of games is simpler. As we can see, this is so only if one makes equilibrium selection assumptions.

The nature of the data is important in models with incomplete information. For example, if one has a data set where the number of players goes to infinity from one market, then one can assume that the equilibrium that is being played is the "observed" equilibrium. This is usually done in "social interactions" models as in Brock and Durlauf (2001). However, in industrial organization, it is more common to have a cross section of markets or games where each data "point" represents an observable outcome of a game. In this case, there is a subtle but crucial difference between the distribution of the data and the underlying economic model. For a given set of observable covariates, the distribution of the data provides an "aggregator" that tells us a summary of the observed outcomes. This data distribution in turns needs to be related to an economic model. In each market, a different game (different equilibrium) might be played since in the general case, equilibrium selection depends on the unobservables. Hence, one has a single observation per game (market). So, the link between the observed data distribution and the underlying distribution of interest is complicated in the presence of multiplicity[19]. The statistical structural is similar to inference in a mixture model with unknown mixing distributions. It is possible to characterize the identified set in more parametric examples. See section 3.2 below. But basically, without making selection assumption, the characterizing identification can be complicated.

3.1 Complete vs Incomplete Information

The sharp econometric restrictions from a model with incomplete information (equations (3.24) above) look strikingly similar to the restrictions from a model with complete information (2.20). Basically, restrictions from both models share the same statistical structure due to the presence of unknown equilibrium selection probabilities. Both models are a version of a discrete mixture with an unknown mixing distribution. So, methods for inference in games with incomplete information without selection assumptions can be used to analyze models with complete information. This mixture framework is similar to one studied in Honoré and Tamer (2005) in a different setup. There, the incompleteness in the model is a result of the ambiguity regarding the initial condition in a single agent dynamic discrete choice model. In cases with discrete x's and discrete ϵ's, the identified regions for these models can be characterized as solutions to linear programming problems. So, this strategy of discretizing the data and the distribution of equilibrium selection will transform

[19] For example, in general games with multiple equilibria, standard descriptive statistics from the observed data are related in a nontrivial way to the underlying statistics from the economic model. For example, the average number of entrants in observationally similar markets is an average with respect to a mixture where the mixing distributions correspond to different equilibria. For more on this point, see Echenique and Komunjer (2005).

the structure of the problem of inference in both complete and incomplete information games into one of solving a linear (and sometimes nonlinear) programming problem. We leave this connection to future work. Finally, it is also worth mentioning that the approach to inference above requires that one is able to solve for the equilibria of the game repeatedly which increases the computational burden.

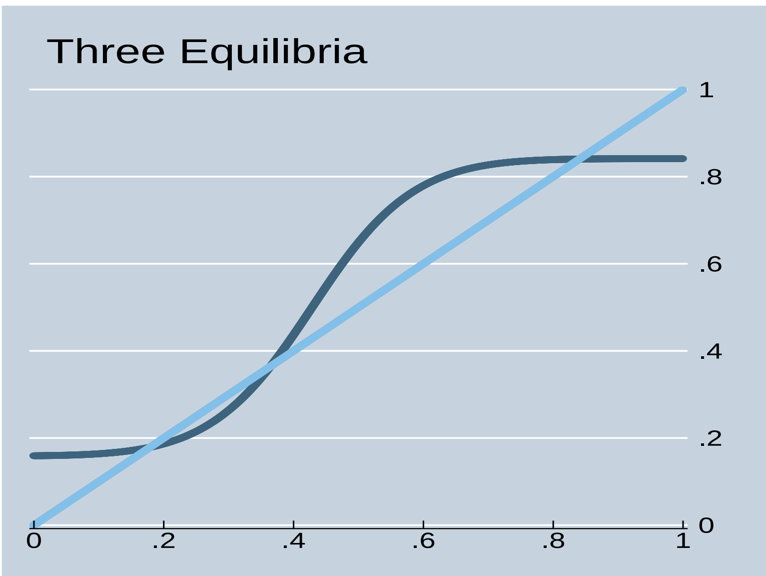
3.2 A Parametric Example

Consider again the example of Table 2 above where now we assume that player i does not observe player $1-i$'s fixed costs ϵ_{1-i} for $i = 0, 1$. Assume that ϵ_1, ϵ_2 are independent of x with a joint distribution F and also that the common belief distribution G is the normal CDF Φ_θ with mean μ and variance σ^2 (normality is assumed here without loss of generality). Then to examine multiplicity, one needs to look at the map

$$\begin{aligned} t_1 &= x_1\beta_1 + \Delta_1\Phi_\theta(t_2) \\ t_2 &= x_2\beta_2 + \Delta_2\Phi_\theta(t_1) \end{aligned} \tag{3.25}$$

For some values of x, θ and $(\beta, \Delta) = (\beta_1, \beta_2, \Delta_1, \Delta_2)$, this map has multiple solutions. An example is illustrated in Figure (2). It can be shown that in the case of normal beliefs, we can have at most 3 equilibria in this model.

Figure 2: Equilibria for some value of X and θ when $F = \Phi$



$$\mathcal{P}(X,\theta,\beta,\Delta)=\begin{pmatrix}\sum_{i=1}^{2}\int_{x_1\beta_1+\Delta_1\Phi_\theta(t_{1i})}^{+\infty}\int_{x_2\beta_2+\Delta_2\Phi_\theta(t_{2i})}^{+\infty}P(t_i|x,\epsilon_1,\epsilon_2)dF\\ \sum_{i=1}^{2}\int_{-\infty}^{x_1\beta_1+\Delta_1\Phi_\theta(t_{1i})}\int_{+\infty}^{x_2\beta_2+\Delta_2\Phi_\theta(t_{2i})}P(t_i|x,\epsilon_1,\epsilon_2)dF\\ \sum_{i=1}^{2}\int_{-\infty}^{x_1\beta_1+\Delta_1\Phi_\theta(t_{1i})}\int_{x_2\beta_2+\Delta_2\Phi_\theta(t_{2i})}^{+\infty}P(t_i|x,\epsilon_1,\epsilon_2)dF\\ \sum_{i=1}^{2}\int_{x_1\beta_1+\Delta_1\Phi_\theta(t_{1i})}^{-\infty}\int_{-\infty}^{x_2\beta_2+\Delta_2\Phi_\theta(t_{2i})}P(t_i|x,\epsilon_1,\epsilon_2)dF\end{pmatrix}\tag{3.26}$$

The restrictions in (3.24) can be adapted here (we restrict ourselves to the two stable equilibria for simplicity) where for some values of the parameters, the equilibrium is unique, and the set $\mathcal{E}$ is a singleton. The presence of the selection mechanisms $P(t_i|\epsilon_1,\epsilon_2)$ for $i=1,2$ complicate the identification. But, for example, assume that x_{11} has large support conditional on the other regressors with a nonzero positive coefficient β_{11}. Then, for example by looking at the second equation in (3.25) above, we see that as x_{11} becomes large for a given value of the other regressors, t_1 becomes large, and hence t_2 becomes close to $x_2\beta_2+\Delta_2$ and the equilibrium correspondence is unique. For these values of x_{11}, the model becomes a standard bivariate discrete choice model where one can use usual methods for identification. Another way to make inference in this model without making equilibrium selection assumptions, is to use the requirement that the function P is a probability and is hence between zero and one. This implies that

$$\begin{aligned}\begin{pmatrix}\min\left(P(0,0|x,\beta,\Delta,t_1),P(0,0|x,\beta,\Delta,t_2)\right)\\ \min\left(P(1,1|x,\beta,\Delta,t_1),P(1,1|x,\beta,\Delta,t_2)\right)\\ \min\left(P(1,0|x,\beta,\Delta,t_1),P(1,0|x,\beta,\Delta,t_2)\right)\\ \min\left(P(0,1|x,\beta,\Delta,t_1),P(0,1|x,\beta,\Delta,t_2)\right)\end{pmatrix}&\le\begin{pmatrix}\Pr(0,0|x)\\ \Pr(1,1|x)\\ \Pr(1,0|x)\\ \Pr(0,1|x)\end{pmatrix}\\ &\le\begin{pmatrix}\max\left(P(0,0|x,\beta,\Delta,t_1),P(0,0|x,\beta,\Delta,t_2)\right)\\ \max\left(P(1,1|x,\beta,\Delta,t_1),P(1,1|x,\beta,\Delta,t_2)\right)\\ \max\left(P(1,0|x,\beta,\Delta,t_1),P(1,0|x,\beta,\Delta,t_2)\right)\\ \max\left(P(0,1|x,\beta,\Delta,t_1),P(0,1|x,\beta,\Delta,t_2)\right)\end{pmatrix}\end{aligned}\tag{3.27}$$

where

$$\begin{pmatrix}P(0,0|x,\beta,\Delta,t_i)\\ P(1,1|x,\beta,\Delta,t_i)\\ P(1,0|x,\beta,\Delta,t_i)\\ P(0,1|x,\beta,\Delta,t_i)\end{pmatrix}=\begin{pmatrix}\int_{x_1\beta_1+\Delta_1\Phi_\theta(t_i)}^{+\infty}\int_{x_2\beta_2+\Delta_2\Phi_\theta(t_i)}^{+\infty}dF\\ \int_{-\infty}^{x_1\beta_1+\Delta_1\Phi_\theta(t_i)}\int_{+\infty}^{x_2\beta_2+\Delta_2\Phi_\theta(t_i)}dF\\ \int_{-\infty}^{x_1\beta_1+\Delta_1\Phi_\theta(t_i)}\int_{x_2\beta_2+\Delta_2\Phi_\theta(t_i)}^{+\infty}dF\\ \int_{x_1\beta_1+\Delta_1\Phi_\theta(t_i)}^{-\infty}\int_{-\infty}^{x_2\beta_2+\Delta_2\Phi_\theta(t_i)}dF\end{pmatrix}$$

and t is the solution to the set of equations in (3.25). If this equilibrium fixed point relation has a unique solution, then the inequalities in (3.27) become equalities. This system of *inequality restrictions* is not sharp, but is more practically feasible to use. Similarly to games with complete information where one needs to solve the equilibria of the game at each parameter value and for each market, here one needs to solve the fixed point problem at each iteration.[20]

[20]We can take a transformation of both sides in (3.25) to get

$$\Phi_\theta(t_1) = \Phi_\theta\big(x_1\beta_1 + \Delta_1\Phi_\theta(t_2)\big)$$
$$\Phi_\theta(t_2) = \Phi_\theta\big(x_2\beta_2 + \Delta_2\Phi_\theta(t_1)\big)$$

One can solve for $\Phi_\theta(t_i)$ as a fixed point of the above mapping (as opposed to t_i). This is easier since solving for Φ can be done using a grid search method on $[0, 1]$.

4 Some thoughts on Identification in Dynamic Entry Games

There has been a set of mostly methodological papers exploring the inferential properties of dynamic models with multiple decision makers. A summary of these papers is contained in Ackerberg, Benkard, Berry, and Pakes (2006). Most of the recent papers[21] build on existing results in the single agent dynamic optimization models as explored in Rust (1994) and further studied in Hotz and Miller (1993). In this paper, we extend the identification framework we explored earlier to a dynamic setting. Most of the discussion will be a review of existing work with some emphasis on some of the subtle details. In the first section, we examine a simple monopoly entry model and show that how the model can be identified under a set of assumptions, some of which are strong and suspect. For example, we show that if the distribution of fixed costs is known (up to a finite dimensional parameter), then one can identify variable profits under the assumptions that the fixed costs are iid over time and markets and that the same equilibrium is played in observationally similar markets. Then, we examine the reverse question, i.e., what can we learn about the distribution of fixed costs if one observes variable profits. Here, we see that under some support conditions, one can identify the distribution of fixed costs. But that generally, this distribution is partially identified. In the last section we provide some heuristics about dealing with multiplicity in dynamic models.

4.1 Identification in a Simple Dynamic Exit Game

Consider the problem of a monopolist operating in a market and facing the choice of staying the next period or exiting. Once out, the monopolist receives zero profits and stays out forever. At the beginning of every period, the monopolist observes ϵ, a random variable representing fixed cost of staying an extra period. This random variable is not observed by the economist and is the only unobservable in the model. The crucial assumption is that ϵ is independent and identically distributed over time and markets which restricts the presence say of unobserved market heterogeneity that can affect profits. The economist observes the sequence of exit decisions by monopolists in an independent cross section of markets. As usual, the objective of the problem is to see what we can learn about variable profits and the distribution of fixed costs under a set of assumptions. The data identifies the choice probability of exit $P_0(x)$ (or entry $P_1(x) = 1 - P_1(x)$) as a function of the state variable x. The question becomes one of relating these choice probabilities to the underlying parameters of interest. To start with, this simple set up, which can be generalized to oligopoly

[21] The papers are Pakes, Ostrovsky, and Berry (2005), Bajari, Benkard, and Levin (2005), Aguiregabiria and Mira (2004) and Pesendorfer and Schmidt-Dengler (2004). For a summary of these papers see Ackerberg, Benkard, Berry, and Pakes (2006).

cases, imposes strong assumptions that are unlikely to hold in practice. These assumptions though, provide a benchmark for analysis (and are used by most of the recent papers) and are a natural starting point for studying identification.

The Bellman equation for the model above is

$$v(x,\epsilon) = \max\left\{u(x) - \epsilon + \beta \int v(x',\epsilon')\gamma(x',\epsilon'|x,\epsilon)d\epsilon' dx', 0\right\}$$

where $u(.)$ is variable period profit and is an object of interest. Moreover, $\gamma(x',\epsilon'|x,\epsilon) = \gamma(x'|x)\gamma(\epsilon')$ is the equilibrium transition probability that we assume here is the same across markets and hence $\gamma(x'|x)$ can be consistently estimated. The expected Bellman equation is

$$\begin{aligned} v(x) = \int v(x,\epsilon)dF &= P_1(x)\left\{u(x) + E[\epsilon|\epsilon \le \delta(x)] + \beta \int v(x')\gamma(x'|x)dx'\right\} \\ &= P_1(x)\left\{\delta(x) + E[\epsilon|\epsilon \le \delta(x)]\right\} \end{aligned} \tag{4.28}$$

where

$$\delta(x) = u(x) + \beta \int v(x')\gamma(x'|x)dx' \tag{4.29}$$

is the deterministic part of present and future returns. In addition, the choice probabilities are related to $\delta(.)$ in the obvious manner,

$$P_1(x) = F(\delta(x)) \tag{4.30}$$

where $F(.)$ is the distribution function of ϵ. Without strong assumptions on $F(.)$ or $\delta(.)$, one can use results from the binary choice literature by Matzkin to identify both these functions up to some homogeneity and normalizations. Assuming that F is known (up to a finite dimensional θ) and is sufficiently smooth, we have $\delta(x) = F^{-1}(P_1(x))$ and so we can "solve" for v by substituting δ from (4.30) into (4.28). This will give us a functional space map $v = \Psi(F)$ (this is the map that Aguiregabiria and Mira use). Notice here, that other than support conditions and smoothness on F, all we require to get the conditional expected value function is knowledge of F. To solve for $u(x)$ we use the equation for δ to back out u as a function of the discount parameter β. We summarize identification in the following lemma.

Result 5 *Let the model above hold. Assume that ϵ has a continuous density on the real line with distribution F with finite mean.*

1. *If F is known, then $v(.)$ is identified on the support of x. In addition, if β is known, then $u(.)$ is also identified on the support of x. If β is not known, then if there exists an $\bar{x}$ such that $u(\bar{x}) = a$ where a is known, then β and $u(.)$ can be identified.*

2. *If F is known up to a finite dimensional parameter θ and β is an unknown parameter, then we can get v_θ and $u_{\theta,\beta}$ on the support of x. The identified set Θ_I is defined as*

$$\Theta_I = \left\{(\theta,\beta) : \Pr\left[x : P_1(x) = F_\theta(u_{\theta,\beta}(x) + \beta \int F_\theta^{-1}(P_1(x'))\gamma(x'|x)dx'\right] = 1\right\}$$

Notice here, that we do not require discrete support on the x's. In addition, the results above can be extended to multinomial models. For more on this see Magnac and Thesmar (2002), Hotz and Miller (1993) and Aguireghaberia and Mira. Of course identification is easier (the set Θ_I is "smaller") if one makes more parametric assumptions on the variable profit function $u(.)$. The question of point identification thus focuses on conditions under which the set Θ_I shrinks to a singleton. Those are not simple to obtain especially in nonlinear models like the one above.

4.2 Identification of the Fixed Costs Distribution

Pakes, Ostrovsky, and Berry (2005) note that we often have information about variable profits from the analysis of data on prices, demand, costs and so forth. Entry data is called on to learn about the distribution of fixed costs. This is the reverse of the question we asked above, i.e., given that we know $u(.)$, what can we learn about $F(.)$. We know from above that given F, we can get v through the map $v = \Psi(F)$. Given that we know v (and u), we can get $\delta(x)$ through (4.29). Then, this allows us to get F on the support of $\delta(x)$ through $P_1(x) = F(\delta(x))$. Hence, one can define the map $F = \Phi(v)$. Sufficient condition for point identification of F (on the support of δ) can be obtaining by analyzing the fixed points of the following map

$$\begin{bmatrix} F \\ v \end{bmatrix} = \begin{bmatrix} \Phi(v) \\ \Psi(F) \end{bmatrix}$$

Sufficient conditions for uniqueness of fixed points in Banach exist, but in general those are hard to satisfy. Below, we characterize solutions to the fixed point map in two cases. The first case provides necessary conditions for local solutions to the value function $v(.)$ which then allows one to construct the fixed cost distribution (modulo measurability). The second case constructs a consistent fixed costs distribution in the case where x takes discretely many values.

Case 1: Continuous x

Interestingly, one can provide a necessary condition for a value function $v(x)$ to be a solution to the above problem in the case F is unknown. In this analysis, we deliberately omit exact conditions for differentiability, measurability for the sake of simplicity. Start with the fact that

$$P_1(x)E[\epsilon|\epsilon \leq \delta(x)] = \int^{\delta(x)} \epsilon f(\epsilon)d\epsilon$$

This means that from (4.28),

$$v(x) = \delta(x)P_1(x) + \int^{\delta(x)} \epsilon f(\epsilon)d\epsilon$$

Taking a derivative with respect to x (examining $v(x)$ locally), we get

$$\begin{aligned} v'(x) &= \big(\delta(x)P_1(x)\big)' + \delta(x)f(\delta(x))\delta'(x) \\ &= \big(\delta(x)P_1(x)\big)' + \delta(x)P_1'(x) \\ &= \delta'(x)P_1(x) + 2\delta(x)P_1'(x) \end{aligned} \tag{4.31}$$

The second equality follows from $P_1(x) = F(\delta(x))$ which implies that $P_1'(x) = f(\delta(x))\delta'(x)$. Hence, replacing $\delta(x) = u(x) + \beta \int v(x')\gamma(x'|x)$ we get a nonlinear differential equation in v and v' where methods can be used to examine existence and uniqueness of solutions. Given a solution v, that will give us the function $\delta(x)$ which in turns gives us $F(.)$ on the support of $\delta(x)$. Another avenue for identification is making parametric assumptions on $v(.)$. Conditions for identifications can then be obtained more easily (of course, making assumptions on $v(.)$ is a little tenuous since v is not a model primitive).

Case 2: Discrete x

We present insights about an interesting case where x has discrete support. In particular, let $x \in \{x_1, \ldots, x_K\}$. Moreover, let the order be such that $\delta(x_1) < \ldots < \delta(x_K)$. We know this order since $P_1(x) = F(\delta(x))$ (here, the strict inequality is without loss of generality since one can only learn $F(.)$ on the support of δ) and $P_1(x)$ is known. We construct a distribution function which is consistent with the model and the data. Consider a distribution function F_L that takes K values and has support on $\delta(x_1), \ldots, \delta(x_K)$ with probability $P_1(x_1), P_1(x_2) - P_1(x_1), P_1(x_3) - P_1(x_2) \ldots, P_1(x_K) - P_1(x_{K-1})$. By construction, this distribution is consistent with the empirical evidence (in fact, this provides a lower bound on the true underlying $F(.)$). What remains is to derive the implied function $v(.)$. The implied conditional expectation is

$$\begin{aligned} E[\epsilon|\epsilon \leq \delta(x_i)] &= \frac{\sum_{j<i} \delta(x_j)(P_1(x_j) - P_1(x_{j-1}))}{P_1(x_i)} \\ &= \frac{\sum_{j<i} \big(u(x_j) + \beta \sum_{i=1}^{K} v(x_i)\gamma(x_i|x_j)\big)(P_1(x_j) - P_1(x_{j-1}))}{P_1(x_i)} \end{aligned} \tag{4.32}$$

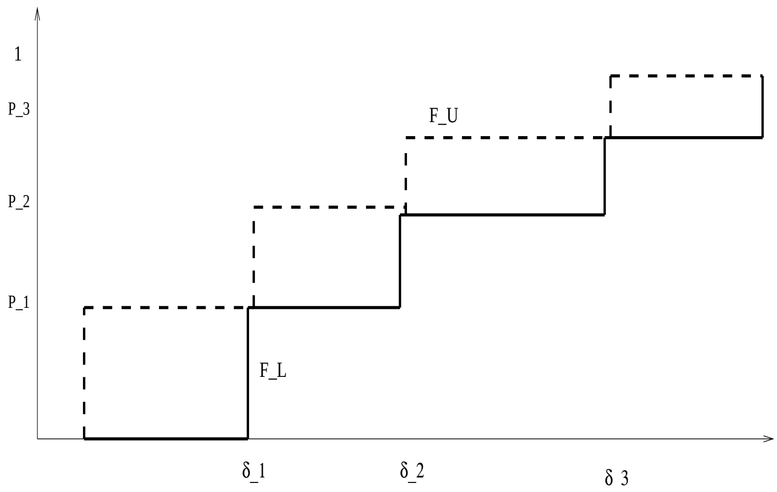
Using a discrete version of the optimized Bellman equation we have,

$$v(x) = P_1(x)\big\{u(x) + E[\epsilon|\epsilon \leq \delta(x)] + \beta \sum_{i=1}^{K} v(x_i)\gamma(x_i|x)\big\} \tag{4.33}$$

substituting for $E[\epsilon|\epsilon \leq \delta(x)]$ from (4.32) above, one can set up a system of equation to obtain information about $v(.)$ given that we know $u(.)$.

Another distribution function that is consistent with the empirical evidence is F_U in the figure below. This distribution obeys (4.30). One can then derive the function v which in turns gives us the "support" functions (the δ's).

Figure 3: Two Discrete Distributions F



4.3 Multiple Equilibria in a Simple Dynamic Model

In this section, we provide insights about the effect of multiple equilibria on inference in a simple dynamic game. In the above, we have always assumed that one knows (or can consistently estimate) the transition probability function $\gamma(.|.)$. Most existing papers, this function is replaced by its empirical counterpart. What rationalizes this is a common equilibrium assumption, i.e., that conditional on observables, the same equilibrium is played across markets. If one allows for multiple equilibria in different markets then the empirical transition probability (the empirical analog of $\gamma(.)$) will have no particular structural meaning but rather it aggregates observations across markets that come from potentially different equilibria. In this section we provide simple insights about cases when one drops this equilibrium selection assumption.

Consider the entry decision of two firms 1 and 2. Firm 1 makes profit $u(1, y_2) - \epsilon_1$ from entry where ϵ_1 are the fixed costs of staying next period. If firm 1 decides to stay out next period, it gets zero profits forever. The same holds for firm 2. In this example, all markets are identical to the econometrician (no x's). The belief for firm 1 about firm two's actions next period are $P_1^b(y_2'|y_1, y_2)$ where y_2' is player 2's decision to enter next period. The objective function for player 1 (similarly for player 2) is to enter next period if

$$
\begin{aligned}
\epsilon_1 &\leq u(1, y_2) + \beta \sum_{y_2'} W(1, y_2') P_1^b(y_2'|y_1, y_2) \\
&= u(1, y_2) + \beta\big((W(1,1) - W(0,1)) P_1^b(1|y_1, y_2) + W(0,1)\big)
\end{aligned}
$$

where W is the value function that depends on (y_1, y_2). Again as in the static game with incomplete information the equilibrium here is a threshold $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ for the ϵ's at which the firms are indifferent between dropping out and staying in. These thresholds are solutions to the following fixed point map:

$$
\begin{aligned}
\bar{\epsilon}_1 &= u(1, y_2) + \beta\big((W(1,1) - W(1,0)) F^b(\bar{\epsilon}_2|y_1, y_2) + W(1,0)\big) \\
\bar{\epsilon}_2 &= u(y_1, 1) + \beta\big((W(1,1) - W(0,1)) F^b(\bar{\epsilon}_1|y_1, y_2) + W(0,1)\big)
\end{aligned}
$$

where F^b is the common prior belief distribution. This map can have multiple solutions. This depends on F^b and the other parameters u, W and β. One can immediately see that without further assumptions, the model adds an equilibrium selection function that can depend on the unobservables. For example, instead of the moment condition in (4.30), we have the following

$$
P_1(x) = \sum_i \int^{\delta_i(x)} P(t_i|\epsilon, x) dF_\epsilon
$$

where $P(.)$ is the equilibrium selection function and the index i runs over the set of equilibria (assumed discrete here). Then, identification in these settings involves the set of variable profits and fixed cost distributions that obey the Bellman equation and the above modified choice probability map. This is further complicated by the fact that the above map involves solving for the equilibria of the game at each iteration. As we can see that without further assumptions, identification in dynamic Markov games while allowing for players to play different equilibria in different markets is hard. There are other avenues that one might want to consider. One avenue might be one where one knows ex-ante that there are two types of markets and that in one type one equilibrium is played and another is played in the other market. We leave this and other identification results for dynamic games as a topic for further research.

5 Conclusion

In this chapter, we review and study the identification question in parametric and nonparametric models of entry. We find for example that combinations of economic insights into the nature of competition coupled with results from the binary choice literature help clarify what can and cannot be learned from the data. We also pay particular attention to the role multiple equilibria and mixed strategies play. The common econometric specification to both models of complete and incomplete information is the mixture model with unknown mixing distributions. We also provide some insights for identification in dynamic models. But, in general, there has not been much work on identification in dynamic discrete games that clarifies the role different assumptions (especially ones related to selection functions) play.

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A Entry game with 2 types: An illustration

This example shows that in games with heterogenous effects, multiple equilibria occur that are not unique in the number. Applications to these games are studied in Ciliberto and Tamer (2003). Here, consider an entry game similar to the BR baseline model in (2.5) above, but where we allow for two types, type 1 and 2 with the following profits functions:

$$\pi_1(y_{1m}, y_{2m}, x_m, f_{1m}) = v_1(y_{1m} + y_{2m}, x_m) - f_{1m}$$
$$\pi_2(y_{1m}, y_{2m}, x_m, f_{2m}) = v_2(y_{1m}, y_{2m}, x_m) - f_{2m}$$

We assume that the variable profits for type 1 firms does not depend on the types of firms in the market but only on the total number of firms in the market while the profits for type 2 firms depends on the number of type 1 and type 2 firms. In addition, assume that $v_2(1, 1, x_m) < v(0, 2, x_m)$ for all x a.e. (this is not essential). We will also make a set of assumptions that simplify the model. First, assume that

$$v_1(y_1 + y_2, x) = \alpha_0 + \alpha_1(y_1 + y_2) + \alpha_2 x - f_1$$

$$v_1(y_1 + y_2, x) = \beta_0 + \beta_1 y_1 + \beta_2 y_2 + \beta_3 x - f_2$$

where β_2, β_1 and α_1 are strictly negative and $\beta_2 > \beta_1$ (profits for type 2 firms will decrease by a larger amount if a type 1 firm enters the market vs a type 2 firms). Second, assume that only 4 firms, two of each type, can be in any market (just for simplicity). One can then easily write down the probability of all the outcomes. For example,

$$\Pr(0, 0|x) = \Pr(f_1 > \alpha_0 + \alpha_1 + \alpha_2 x; f_2 > \beta_0 + \beta_2 + \beta_3 x)$$
$$\Pr(1, 0|x) = \Pr(\alpha_0 + \alpha_1 + \alpha_2 x \geq f_1 \geq \alpha_0 + 2\alpha_1 + \alpha_2 x;\ \beta_0 + \beta_1 + \beta_2 + \beta_3 x \leq f_2)$$
$$\Pr(0, 2|x) = \Pr(f_1 \geq \alpha_0 + 3\alpha_1 + \alpha_2 x;\ f_2 \leq \beta_0 + 2\beta_2 + \beta_3 x|x)$$

It is easy to see that if (f_1, f_2) has wide support, then there is region for which $(1, 0)$ and $(0, 2)$ are multiple pure strategy equilibria of the game if $\beta_2 > \beta_1$. This heterogeneity in the effect on type 2 variable profits of the two types of firms entering causes the model to have multiple equilibria where each equilibrium involves a *different number* of firms. Looking at figure 4, we see that $(1, 2)$ and $(2, 0)$ can also appear as equilibria of the game. Finally, we assume that we are going to restrict ourselves to games with pure strategies only. So, given a random sample of markets, where in each market m we observe its configuration in terms of the number of type 1 and type 2 firms and a vector of regressors x_m (that can be type specific), we can relate the (conditional) distribution of the total number of firms to the predictions of the model. For example, Borzekowski and Cohen (2004) study a technology adoption game with network effects where the decision of a credit union to adopt a particular technology depends on the the number of other credit unions that adopt the technology. Strategies for identification in these models is similar to ones we highlighted in section 3 above. To relate the observed choice probabilities to ones predicted by the model, one can use figure 4 below. For example, for values of (f_1, f_2) when $(0,0)$ is the unique equilibrium, $\Pr((0,0)|x; A_0) = 1$ and so

$$\Pr((0,0)|x) = \int_{A_0} dF_{f_1;f_2} \equiv \Pr(A_0)$$

where $A_0 = \{f_1, f_2 : v(1,x) \leq f_1; v_2(0,1,x) \leq f_2\}$ since $(0,0)$ is a potentially observable outcome in cases when it is a unique equilibrium of a game (assuming no mixing). On the other hand, to write down the the probabilities of observing $(1,2)$ predicted by the model, one needs a selection mechanism as in section 2 above. In general though, entry models with types reduces the dimension of the problem and makes inference more practically feasible since instead of dealing with whether a particular firm enters, now we deal with the number of firms of a given type enter.

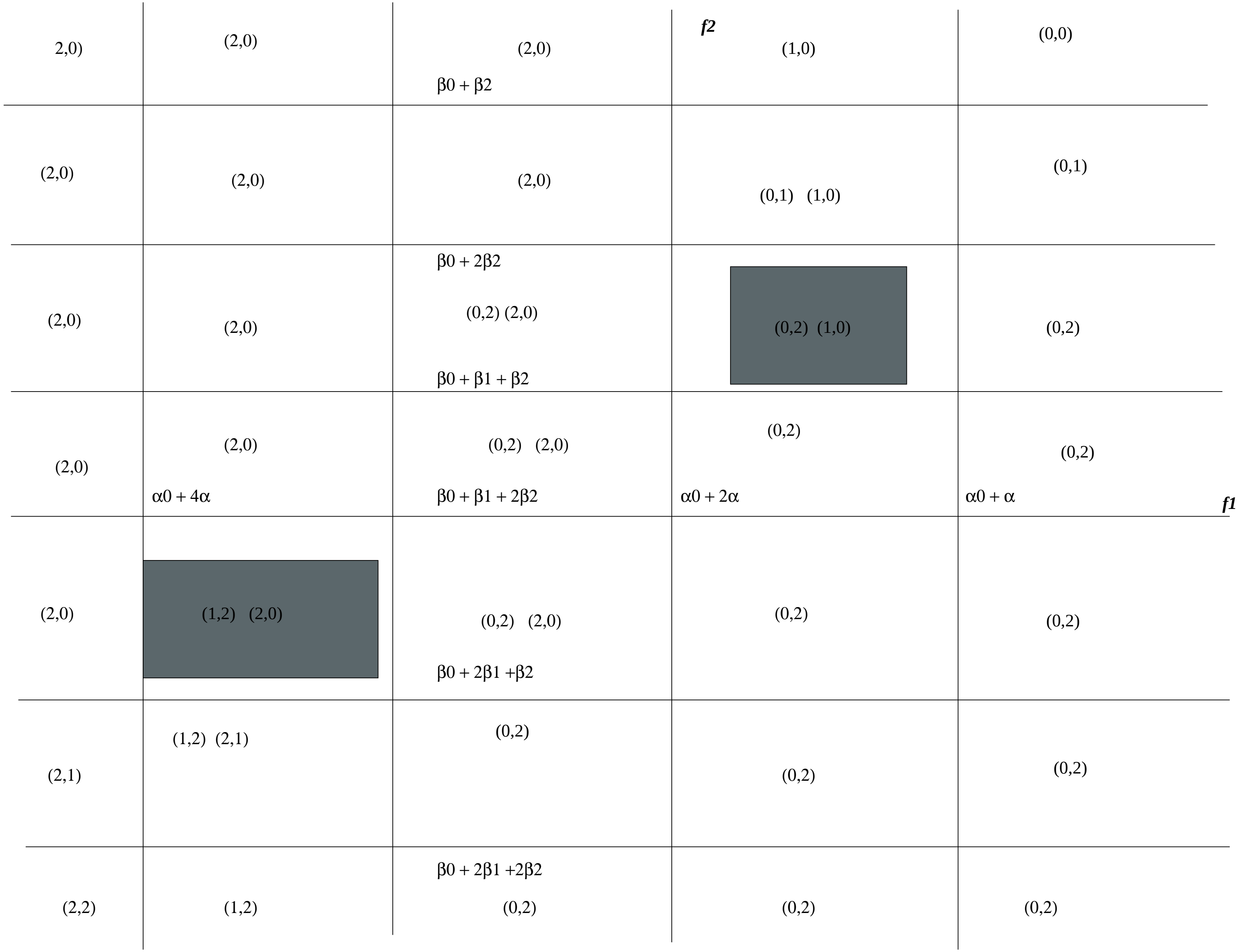

Figure 4: Equilibrium Maps for a Two-Type Game