provided an appropriate branch of the logarithm is chosen. A careful examination of the mapping $w = \sin z$ (see the exercises) shows that $\sin z$ maps the strip

$$\left\{x + iy: -\frac{\pi}{2} < x < \frac{\pi}{2}, -\infty < y < \infty\right\}$$

both one-to-one and onto the region D obtained from the plane by deleting the two intervals $(-\infty, -1]$ and $[1, \infty)$ (see Fig. 1.26). Thus, on D we can solve uniquely for z in terms of w, and the formula is valid. Interchanging the roles of z and w, write

$$\text{Arcsin } z = -i \text{ Log}(iz + \sqrt{1 - z^2}), \qquad z \in D.$$

Similarly,

$$\text{Arccos } z = -i \text{ Log}(z + \sqrt{z^2 - 1}),$$

and

$$\text{Arctan } z = \frac{i}{2} \text{Log}\left(\frac{1 - iz}{1 + iz}\right), \qquad z \neq \pm i,$$

with appropriate interpretations of the resulting logarithms and roots.

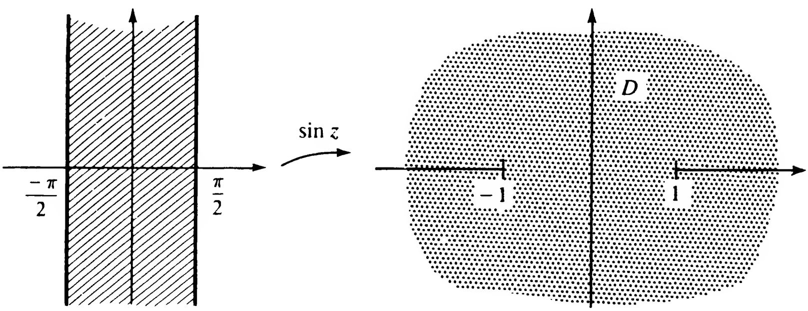

Figure 1.26

EXERCISES FOR SECTION 1.5

Find the values(s) of the given expression in Exercises 1 to 14.

1. $e^{i\pi/4}$
2. $e^{5\pi i/4}$
3. $\log(1 + i\sqrt{3})$
4. $\log(-i)$
5. $(1 + i)^i$
6. 2^{-1-i}
7. $e^{-7\pi i/3}$
8. $\exp(\text{Log}(3 + 2i))$
9. $\text{Log}(4 - 4i)$
10. $\text{Log}(-1)$
11. $i^{\sqrt{3}}$
12. $\log(\sqrt{3} - i)$
13. $\log((1 - i)^4)$
14. $\exp\left[\pi\left(\frac{i + 1}{\sqrt{2}}\right)^4\right]$

15. Establish the following relations:
    (a) $\exp(\bar{z}) = \overline{\exp(z)}$
    (b) $\sin(\bar{z}) = \overline{\sin(z)}$
    (c) $\cos(\bar{z}) = \overline{\cos(z)}$
16. Establish the formulas

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

where

$$\cosh u = \frac{1}{2}(e^u + e^{-u}), \qquad u \text{ real}$$

$$\sinh u = \frac{1}{2}(e^u - e^{-u}), \qquad u \text{ real.}$$

17. Show that $\cos z = 0$ if and only if $z = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$; show that $\sin z = 0$ if and only if $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. That is, extending $\sin z$ and $\cos z$ from the real axis to the whole plane does not introduce any new zeros.
18. Verify that

$$\cos(z + w) = \cos z \cos w - \sin z \sin w$$

and

$$\sin(z + w) = \sin z \cos w + \cos z \sin w,$$

for all complex numbers z and w.
19. Show that both $\cos z$ and $\sin z$ are unbounded if $z = iy$ and $y \to \infty$. Also show that

$$|\cos(x + iy)| \leqslant e^y \qquad \text{if} \qquad y \geqslant 0, \ -\infty < x < \infty;$$

and

$$|\sin(x + iy)| \leqslant e^y \qquad \text{if} \qquad y \geqslant 0, \ -\infty < x < \infty.$$

20. Prove that $\cos^2 z + \sin^2 z = 1$ for all z.
21. Define $\cosh z$ and $\sinh z$ by

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z}).$$

Show that the following identities hold:
(i) $\cosh^2(z) - \sinh^2(z) = 1$
(ii) $\cosh z = \cos(iz)$
(iii) $\sinh z = -i\sin(iz)$
(iv) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$
(v) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

22. Show that $\sin(-z) = -\sin z$ and $\cos z = \cos(-z)$ for all z.

Mappings with the Exponential, Logarithm, and Trigonometric Functions

23. Show that $F(z) = e^z$ maps the strip $S = \{x + iy : -\infty < x < \infty, -\pi/2 \leqslant y \leqslant \pi/2\}$ onto the region $\Omega = \{w = s + it : s \geqslant 0, w \neq 0\}$ and that F is one-to-one on S (see Fig. 1.27). Furthermore, show that F maps the boundary of S onto all the boundary of Ω except $w = 0$. Explain what happens to each of the horizontal lines $\{\operatorname{Im} z = \pi/2\}$ and $\{\operatorname{Im} z = -\pi/2\}$.

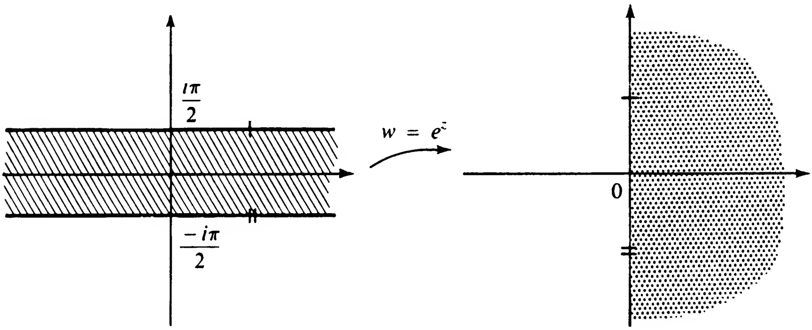

Figure 1.27

24. Let D be the domain obtained by deleting the ray $\{x : x \leqslant 0\}$ from the plane, and let $G(z)$ be a branch of $\log z$ on D. Show that G maps D onto a horizontal strip of width of 2π,

$$\{x + iy: -\infty < x < \infty, c_0 < y < c_0 + 2\pi\},$$

and that the mapping is one-to-one on D.

25. Show that $w = \sin z$ maps the strip $-\pi/2 < x < \pi/2$ both one-to-one and onto the region obtained by deleting from the plane the two rays $(-\infty, -1]$ and $[1, \infty)$ (see Fig. 1.26). (**Hint:** Use Exercise 22 and the fact that $\sin(\bar{z}) = \overline{\sin(z)}$.)

26. Show that the function $w = \cos z$ maps the strip $\{0 < x < \pi\}$ one-to-one and onto the region D shown in Figure 1.26. Use this to define the inverse function to $\cos z$. Derive the formula for $\arccos z$ given in the text.

27. Let $0 < \alpha < 2$. Show that an appropriate choice of $\log z$ for $f(z) = z^\alpha = \exp[\alpha \log z]$ maps the domain $\{x + iy: y > 0\}$ both one-to-one and onto the domain $\{w: 0 < \arg w < \alpha\pi\}$. Show that f also carries the boundary to the boundary (see Fig. 1.28).

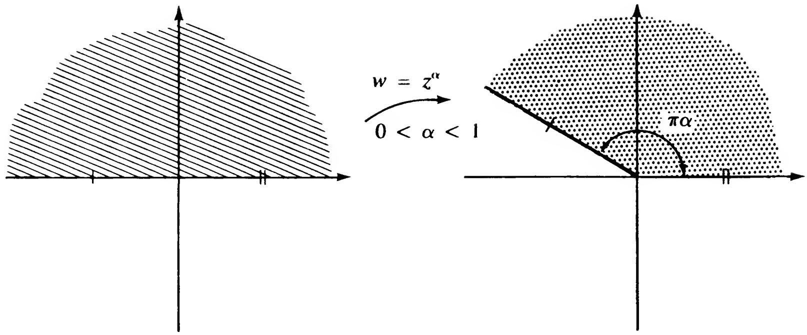

Figure 1.28

28. Show that the function $w = g(z) = e^{z^2}$ maps the lines $x = y$ and $x = -y$ onto the circle $|w| = 1$. Show further that g maps each of the two pieces of the region $\{x + iy: x^2 > y^2\}$ onto the set $\{w: |w| > 1\}$ and each of the two pieces of the region $\{x + iy: x^2 < y^2\}$ onto the set $\{w: |w| < 1\}$.

Inverse Trigonometric Functions

29. Show directly that if ζ is any value of

$$-i \log(iz + \sqrt{1 - z^2}),$$

then $\sin \zeta = z$. Likewise, show that if ξ is any value of

$$\frac{i}{2}\log\left(\frac{1 - iw}{1 + iw}\right),$$

then $\tan \xi = w$.

30. Use the result in Exercise 29 and your knowledge of the branches of the logarithmic function to explain the branches of $\arcsin z$.

1.6 Line Integrals and Green's Theorem

The fundamental theorems of complex variables depend on line integrals, so this section is devoted to that topic and to formulating Green's Theorem, the basic theorem about line integrals. Several consequences of Green's Theorem are also covered.

Curves

A **curve** γ is a continuous complex-valued function $\gamma(t)$ defined for t in some interval $[a, b]$ in the real axis. The curve γ is **simple** if $\gamma(t_1) \neq \gamma(t_2)$ whenever $a \leqslant t_1 < t_2 < b$, and it is **closed** if $\gamma(a) = \gamma(b)$. The famous **Jordan* Curve Theorem** asserts that the complement of the range of a curve that is both simple and closed consists of two

* Camille Jordan, 1838–1922.