# The Normal Table: Read it, Use it

### ECO220Y1Y: 2018/19; Written by Jennifer Murdock

A massive flood in Grand Forks, North Dakota in 1997 cost billions to clean up. The levee could protect the town even if the river rose to 51 feet. But, beyond that, it would flood the town. The Weather Service knew that heavy snowfall would cause high river waters. It forecast levels would rise to 49 feet. Hence, many people felt safe behind the levee, which could withstand up to 51 feet. The town did not take extra precautions. Unfortunately the waters rose to 54 feet. The section "The Importance of Communicating Uncertainty" in the popular 2012 book *The Signal and the Noise: Why So Many Predictions Fail–But Some Don't* by Nate Silver (pp. 177 - 179) features this story.

The Weather Service cannot predict exact water levels: 49 feet is expected but there is a margin of error. On page 178, Silver says there was "about a 35 percent chance of the levees being overtopped" and footnote 8 says that you assume a Normal model to find the number 35. After working through this supplement, you will be able to figure out the exact model assumed and be able to draw a fully-labeled graph, like Figure 1, to illustrate your reasoning. (Make sure to solve Exercise 8 on page 4.) Silver concludes the story on p. 179. "An oft-told joke: a statistician drowned crossing a river that was only three feet deep *on average*. On average, the flood might be forty-nine feet in the Weather Service's forecast model, but just a little bit higher and the town would be inundated."

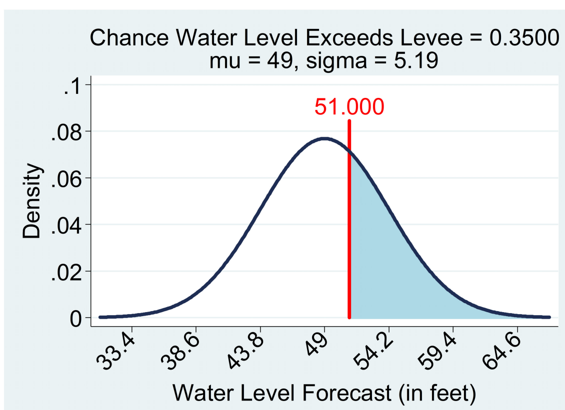



**Figure 1:** A fully-labeled graph (although you would not responsible for ticking the numbers on the *vertical* axis if drawing this graph by hand)

This supplement is required reading for all students in ECO220Y1Y. It shows you how to use a common version of the Standard Normal statistical table. Our textbook uses another common version (pages B-2 and B-3) but it is redundant and takes up two pages whereas only one page is needed. Page 5 provides the Standard Normal table that you will be given for tests and the final examination: it is one page long and saves paper. Use the table on Page 5 when doing your homework.

Remember that to standardize a random variable $X$ with mean $\mu$ and standard deviation $\sigma$ you subtract $\mu$ and divide by $\sigma$: $Z = \frac{X-\mu}{\sigma}$. $Z$ will have mean 0 and standard deviation 1. When $X$ is Normal, $X \sim N(\mu, \sigma^2)$, then $Z$ is Standard Normal, $Z \sim N(0,1)$. Using the Standard Normal table involves both standardizing ($Z = \frac{X-\mu}{\sigma}$) and unstandardizing ($X = \mu + \sigma Z$).

*Examples A - P* illustrate reading the one-page Standard Normal table on page 5. If you understand density functions, a requirement of our course, then you can figure out any statistical table. Hence, working with this table reinforces key course concepts. **Exercises 1 - 8** (on page 4) let you practice using the table in applications: do these and check your understanding.

*Example A.* Looking at the middle of the table find the number 0.3770. What does that mean? Referring to the associated row and column headings, it means that $P(0 < Z < 1.16) = 0.3770$. Notice how that corresponds with the picture at the top right of the table.

*Example B.* Looking at the top left of the table find the number 0.0000. What does that mean? It means that $P(0 < Z < 0.00) = 0.0000$. In other words, there is no area under the Standard Normal curve between zero and zero. Remember that for all continuous distributions the probability of a specific value is zero. The probability is the area under the density function. If there is no width, there can be no area.

*Example C.* Looking at the bottom part of the table find the number 0.4967. What does that mean? It means that $P(0 < Z < 2.72) = 0.4967$.

*Example D.* What is the probability that Z is between 0 and 1? From the table: $P(0 < Z < 1) = 0.3413$.

*Example E.* What is the probability that Z is within one standard deviation of its mean? Before looking at the table we must remember that the Standard Normal random variable has a mean of zero and a standard deviation of one and is symmetric about the mean. Hence, the question is asking us to find $P(-1 < Z < 1)$. The table does not tell us that probability directly. However, we know that $P(0 < Z < 1) = 0.3413$ and symmetry implies $P(0 < Z < 1) = P(-1 < Z < 0)$. Hence, $P(-1 < Z < 1) = 2 * 0.3413 = 0.6826$. Hopefully this answer looks familiar to you. Remember the Empirical Rule (pages 292 - 293 of your textbook)? The Empirical Rule tells us that if a sample is taken from a Normal population (i.e. Bell shaped), then about 68% of the observations should lie within one standard deviation of the mean. Now we see that a more precise answer is 68.26%. Because our samples are always subject to sampling error this extra precision is not very helpful: the Empirical Rule only says "about" anyway. However, if you forget the exact numbers associated with Empirical Rule you now know how to find the 68%, the 95% and the 99.7% by using the table.

*Example F.* Looking and the bottom right of the table find the number 0.4990. What does that mean? It means that $P(0 < Z < 3.09) = 0.4990$. What is the $P(0 < Z < \infty)$? That question does not require the use of the table: because the Standard Normal is symmetric around zero we know that $P(0 < Z < \infty) = 0.5$. Notice that 0.4990 is not much smaller than 0.5. In other words, the tails of the Normal distribution become very thin. There is very little chance of getting a random draw from a Standard Normal distribution that is greater than 3.09. So what is $P(Z > 3.09)$ exactly? While the table does not give us that probability directly, we can easily find it. The table does tell us that $P(0 < Z < 3.09) = 0.4990$ and we know $P(0 < Z < \infty) = 0.5$. Hence $P(Z > 3.09) = P(0 < Z < \infty) - P(0 < Z < 3.09) = 0.5 - 0.4990 = 0.001$. In plain English this means that there is a 1 in 1,000 chance of randomly drawing a value of 3.09 or higher from a Standard Normal population.

*Example G.* What is $P(-2.30 < Z < 0)$? Again using symmetry: $P(-2.30 < Z < 0) = P(0 < Z < 2.30)$. From the table: $P(0 < Z < 2.30) = 0.4893$. Hence, $P(-2.30 < Z < 0) = 0.4893$.

*Example H.* What is $P(1.34 < Z < 1.39)$? This is not given directly by our table but we can find it. $P(1.34 < Z < 1.39) = P(0 < Z < 1.39) - P(0 < Z < 1.34) = 0.4177 - 0.4099 = 0.0078$.

*Example I.* What is $P(Z > 2.83)$? This is not given directly by our table but we can find it. $P(Z > 2.83) = P(0 < Z < \infty) - P(0 < Z < 2.83) = 0.5 - 0.4977 = 0.0023$.

*Example J.* What is $P(-2.75 < Z < 0.25)$? This is not given directly by our table but we can find it. $P(-2.75 < Z < 0.25) = P(0 < Z < 0.25) + P(-2.75 < Z < 0) = P(0 < Z < 0.25) + P(0 < Z < 2.75) = 0.0987 + 0.4970 = 0.5957$.

*Example K.* What is $P(Z < -1.96)$? This is not given directly by our table but we can find it. $P(Z < -1.96) = P(Z > 1.96) = P(0 < Z < \infty) - P(0 < Z < 1.96) = 0.5 - 0.4750 = 0.0250$.

*Example L.* What is $P(Z < 2.1)$? $P(Z < 2.1) = 0.5 + P(0 < Z < 2.1) = 0.5 + 0.4821 = 0.9821$.

*Example M.* What is the 78th percentile of the Standard Normal distribution? To help us in answering this let's rewrite the question formally: $P(Z <?) = 0.78$. The ? is what is asked for. First, will the 78th percentile be bigger than 0 or less than 0? No tables needed to answer that question: it should be bigger than zero because we know that the 50th percentile of the Standard Normal is zero. With that in mind rewrite $P(Z <?) = 0.78$ as $P(0 < Z <?) = 0.78 - 0.5 = 0.28$. Now we can use the table to find $P(0 < Z <?) = 0.28$ and see that $? \approx 0.77$: $P(Z < 0.77) \approx 0.78$. If we are very precise then $P(0 < Z < 0.77) = 0.2794$ and not 0.28. But, $0.2794 \approx 0.28$. Some students like to interpolate between values in the table. That is fine. However, simply rounding to the nearest is perfectly acceptable. When you work with Excel you'll be able to get the exact values.

*Example N.* What is the 5th percentile: $P(Z <?) = 0.05$. Before using the table, we must realize that the 5th percentile will be a negative number: it must be less than the median which is 0. From the table: $P(0 < Z < 1.645) = 0.45$. This implies that $P(Z < -1.645) = 0.05$. (We did interpolate between 1.64 and 1.65, because this is an important case: we'll see 1.645 a lot.)

*Example O.* What is the 95th percentile? Given our work for the last question, this is easy: $P(Z < 1.645) = 0.95$ so the 95th percentile is 1.645.

*Example P.* $P(-? < Z <?) = 0.95$. Because the upper and lower bounds are the same except for the sign the center point must be zero. $P(0 < Z <?) = 0.95/2$ and from the table $P(0 < Z < 1.96) = 0.475$. Hence, $P(-1.96 < Z < 1.96) = 0.95$. Can you see the link to the Empirical Rule?

Using the table, work through these exercises. They give you a chance to *apply* the table in straightforward applications. You may check your answers against those on pages 6 to 7.

**Exercise 1.** A random variable $X$ is Normally distributed with $\mu_X = 0.5$ and $\sigma_X = 0.05$. What is the chance that $X$ is bigger than 0.58? Illustrate your answer with a fully-labeled graph.

**Exercise 2.** A random variable $X$ is Normally distributed with $\mu_X = 20.01$ and $\sigma_X = 4.54$. What is the chance that $X$ is less than 19.19? Illustrate your answer with a fully-labeled graph.

**Exercise 3.** A random variable $X$ is Normally distributed with $\mu_X = 0.4$ and $\sigma_X = 0.13$. What is the chance that $X$ is bigger than 0.25? Illustrate your answer with a fully-labeled graph.

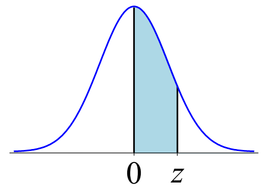
**Exercise 4.** A random variable $X$ is Normally distributed with $\mu_X = 0.2$ and $\sigma_X = 0.0034$. What is the cut-off value such that there is a 5 percent chance of being above this value? (In other words, what is the 95th percentile?) Illustrate your answer with a fully-labeled graph.

**Exercise 5.** A random variable $X$ is Normally distributed with $\mu_X = 69$ and $\sigma_X = 12.23$. What is the cut-off value such that there is a 2.5 percent chance of being below this value? Illustrate your answer with a fully-labeled graph.

**Exercise 6.** A random variable $\hat{P}$ is Normally distributed with $\mu_{\hat{P}} = 0.7$ and $\sigma_{\hat{P}} = 0.0229$. What is the chance that $\hat{P}$ is less than 0.62? Illustrate your answer with a fully-labeled graph. (Note: If the idea that the name of the random variable is $\hat{P}$ confuses you, for now just use $X$. In other words, cross-out all of the $\hat{P}$s and write $X$s instead. However, we will learn about $\hat{P}$ later and you will have to get used to it eventually.)

**Exercise 7.** Continuing with the previous exercise, what are the two cut-off values such that the chance that $\hat{P}$ is less than the lower cut-off value is 0.005 and the chance that $\hat{P}$ is greater than the upper cut-off value is 0.005.

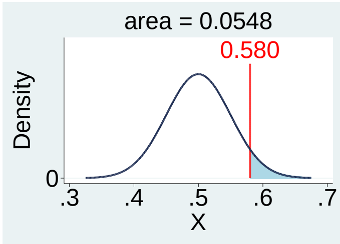
**Exercise 8.** Finally, return to the Grand Forks example that opened this supplement (p. 1). Again suppose the levee could withstand up to 51 feet and that The Weather Service predicted water levels would rise to 49 feet. If there is a 30 percent chance that the flood waters overtake the levee then what is the standard deviation of the predicted water level? Illustrate your answer with a fully-labeled graph.



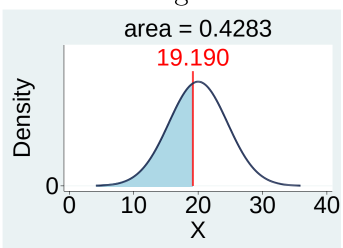

**The Standard Normal Distribution:**

| | *Second decimal place in z* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0199 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| 0.1 | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0753 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1844 | 0.1879 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| 0.6 | 0.2257 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 | 0.2454 | 0.2486 | 0.2517 | 0.2549 |
| 0.7 | 0.2580 | 0.2611 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2967 | 0.2995 | 0.3023 | 0.3051 | 0.3078 | 0.3106 | 0.3133 |
| 0.9 | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3531 | 0.3554 | 0.3577 | 0.3599 | 0.3621 |
| 1.1 | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3830 |
| 1.2 | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3980 | 0.3997 | 0.4015 |
| 1.3 | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4131 | 0.4147 | 0.4162 | 0.4177 |
| 1.4 | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| 1.5 | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.4418 | 0.4429 | 0.4441 |
| 1.6 | 0.4452 | 0.4463 | 0.4474 | 0.4484 | 0.4495 | 0.4505 | 0.4515 | 0.4525 | 0.4535 | 0.4545 |
| 1.7 | 0.4554 | 0.4564 | 0.4573 | 0.4582 | 0.4591 | 0.4599 | 0.4608 | 0.4616 | 0.4625 | 0.4633 |
| 1.8 | 0.4641 | 0.4649 | 0.4656 | 0.4664 | 0.4671 | 0.4678 | 0.4686 | 0.4693 | 0.4699 | 0.4706 |
| 1.9 | 0.4713 | 0.4719 | 0.4726 | 0.4732 | 0.4738 | 0.4744 | 0.4750 | 0.4756 | 0.4761 | 0.4767 |
| 2.0 | 0.4772 | 0.4778 | 0.4783 | 0.4788 | 0.4793 | 0.4798 | 0.4803 | 0.4808 | 0.4812 | 0.4817 |
| 2.1 | 0.4821 | 0.4826 | 0.4830 | 0.4834 | 0.4838 | 0.4842 | 0.4846 | 0.4850 | 0.4854 | 0.4857 |
| 2.2 | 0.4861 | 0.4864 | 0.4868 | 0.4871 | 0.4875 | 0.4878 | 0.4881 | 0.4884 | 0.4887 | 0.4890 |
| 2.3 | 0.4893 | 0.4896 | 0.4898 | 0.4901 | 0.4904 | 0.4906 | 0.4909 | 0.4911 | 0.4913 | 0.4916 |
| 2.4 | 0.4918 | 0.4920 | 0.4922 | 0.4925 | 0.4927 | 0.4929 | 0.4931 | 0.4932 | 0.4934 | 0.4936 |
| 2.5 | 0.4938 | 0.4940 | 0.4941 | 0.4943 | 0.4945 | 0.4946 | 0.4948 | 0.4949 | 0.4951 | 0.4952 |
| 2.6 | 0.4953 | 0.4955 | 0.4956 | 0.4957 | 0.4959 | 0.4960 | 0.4961 | 0.4962 | 0.4963 | 0.4964 |
| 2.7 | 0.4965 | 0.4966 | 0.4967 | 0.4968 | 0.4969 | 0.4970 | 0.4971 | 0.4972 | 0.4973 | 0.4974 |
| 2.8 | 0.4974 | 0.4975 | 0.4976 | 0.4977 | 0.4977 | 0.4978 | 0.4979 | 0.4979 | 0.4980 | 0.4981 |
| 2.9 | 0.4981 | 0.4982 | 0.4982 | 0.4983 | 0.4984 | 0.4984 | 0.4985 | 0.4985 | 0.4986 | 0.4986 |
| 3.0 | 0.4987 | 0.4987 | 0.4987 | 0.4988 | 0.4988 | 0.4989 | 0.4989 | 0.4989 | 0.4990 | 0.4990 |
| 3.1 | 0.4990 | 0.4991 | 0.4991 | 0.4991 | 0.4992 | 0.4992 | 0.4992 | 0.4992 | 0.4993 | 0.4993 |
| 3.2 | 0.4993 | 0.4993 | 0.4994 | 0.4994 | 0.4994 | 0.4994 | 0.4994 | 0.4995 | 0.4995 | 0.4995 |
| 3.3 | 0.4995 | 0.4995 | 0.4995 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4997 |
| 3.4 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4998 |
| 3.5 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 |
| 3.6 | 0.4998 | 0.4998 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |

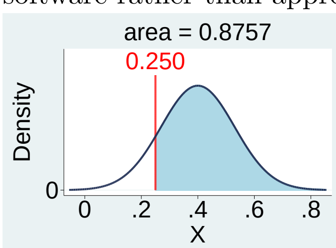
**A1.** First, write question using formal notation: $P(X > 0.58 \mid \mu_X = 0.5, \sigma_X = 0.05) = ?$. Next, standardize and then use the table. $P(X > 0.58 \mid \mu_X = 0.5, \sigma_X = 0.05) = P(Z > \frac{0.58-0.5}{0.05}) = P(Z > 1.6) = 0.5 - P(0 < Z < 1.6) = 0.5 - 0.4452 = 0.0548$.



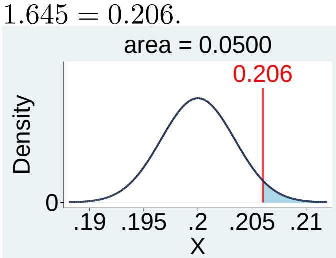

**A2.** First, write question using formal notation: $P(X < 19.19 \mid \mu_X = 20.01, \sigma_X = 4.54) = ?$. Next, standardize and then use the table. $P(X < 19.19 \mid \mu_X = 20.01, \sigma_X = 4.54) = P(Z < \frac{19.19-20.01}{4.54}) = P(Z < -0.181) = 0.5 - P(-0.181 < Z < 0) = 0.5 - P(0 < Z < 0.181) \approx 0.5 - P(0 < Z < 0.18) = 0.5 - 0.0714 = 0.4286$. (Note: The graph below has the exact area obtained using software rather than approximating using the table.)



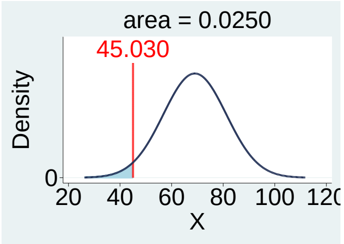

**A3.** First, write question using formal notation: $P(X > 0.25 \mid \mu_X = 0.4, \sigma_X = 0.13) = ?$. Next, standardize and then use the table. $P(X > 0.25 \mid \mu_X = 0.4, \sigma_X = 0.13) = P(Z > \frac{0.25-0.4}{0.13}) = P(Z > -1.154) = 0.5 + P(-1.154 < Z < 0) = 0.5 + P(0 < Z < 1.154) \approx 0.5 + P(0 < Z < 1.15) = 0.5 + 0.3749 = 0.8749$. (Note: The graph below has the exact area obtained using software rather than approximating using the table.)



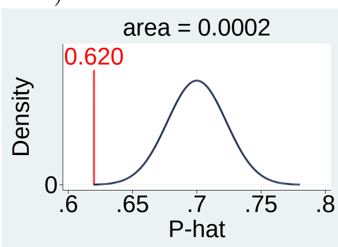

**A4.** First, write question using formal notation: $P(X > ? \mid \mu_X = 0.2, \sigma_X = 0.0034) = 0.05$. Next, use the table then un-standardize. $P(Z > 1.654) = 0.05$ and $1.645 = \frac{x-0.2}{0.0034}$ so $x = 0.2 + 0.0034 * 1.645 = 0.206$.



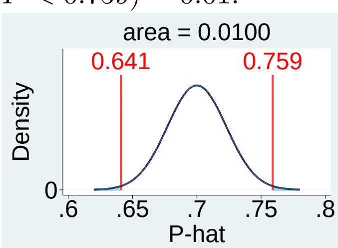

**A5.** First, write question using formal notation: $P(X < ? \mid \mu_X = 69, \sigma_X = 12.23) = 0.025$. Next, use the table then un-standardize. $P(Z < -1.96) = 0.025$ and $-1.96 = \frac{x-69}{12.23}$ so $x = 69 - 12.23 * 1.96 = 45.03$.




**A6.** First, write question using formal notation: $P(\hat{P} < 0.62 \mid \mu_{\hat{P}} = 0.7, \sigma_{\hat{P}} = 0.0229) =?$. Next, standardize and then use the table. $P(\hat{P} < 0.62 \mid \mu_{\hat{P}} = 0.7, \sigma_{\hat{P}} = 0.0229) = P(Z < \frac{0.62-0.7}{0.0229}) = P(Z < -3.493) = 0.5 - P(-3.493 < Z < 0) = 0.5 - P(0 < Z < 3.493) \approx 0.5 - P(0 < Z < 3.49) = 0.0002$.




**A7.** First, write question using formal notation: $P(\hat{P} <?) = 0.005$ and $P(\hat{P} >??) = 0.005$. Next, use the table then unstandardize. $P(Z < -2.575) = 0.005$ and $P(Z > 2.575) = 0.005$. $? = 0.7 - 0.0229 * 2.575 = 0.641$ and $?? = 0.7 + 0.0229 * 2.575 = 0.759$. In other words, $P(0.641 < \hat{P} < 0.759) = 0.01$.




**A8.** Using formal notation, $P(X > 51) = 0.30$ where $\mu = 49$ and $\sigma =?$. From the Normal table: $P(Z > 0.525) \approx 0.30$. Given $\frac{51-49}{\sigma} = 0.525$ solve for $\sigma \approx 3.8$ feet. Using software, $\sigma = 3.814$.