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COMPARISON THEORY FOR RICCATI EQUATIONS

by

J.-H. Eschenburg and E. Heintze

We give a short new proof for the comparison theory of the matrix valued Riccati equation $B'+B^2+R=0$ with singular initial values. Applications to Riemannian geometry are briefly indicated.

Let E be a finite dimensional real vector space with inner product $\langle , \rangle$ and $S(E)$ the space of self adjoint linear endomorphisms of E . For a given smooth coefficient curve $R : \mathbb{R} \to S(E)$ we consider solutions $B : (0, t_0) \to S(E)$ of the Riccati differential equation ($cf.$[R])

$$(R) \qquad B' + B^2 + R = 0\ .$$

In Riemannian geometry, this is the evolution equation for the shape operators of a familiy of parallel hypersurfaces if R denotes the curvature tensor in normal direction. It has been used to estimate principal curvatures and volumes of spheres and tubes (cf. [Gn], [HE], [Eb], [E1], [HIH], [K1], [GM], [Gr], [Gv], [K2], [E2]). If two coefficient curves $R_1, R_2 : \mathbb{R} \to S(E)$ with $R_1 \geq R_2$ (i.e. $R_1 - R_2$ positive semidefinite) are given, one may compare solutions $B_1, B_2 : (0, t_0) \to S(E)$ of

$$(R_j) \qquad B_j' + B_j^2 + R_j = 0$$

with suitable initial conditions. This has been done in [E2] by first assuming $R_1 > R_2$ and then passing to the limit. But the method was not good enough to discuss equality. In this paper, we give a different and more natural proof of the comparison theorem including the equality discussion.

Theorem. *Let $R_1, R_2 : \mathbb{R} \to S(E)$ be smooth with $R_1 \geq R_2$. For $j = 1,2$, let $B_j : (0,t_j) \to S(E)$ be a solution of (R_j) with maximal $t_j \in (0,\infty]$, such that $U := B_2 - B_1$ has a continuous extension to 0 with $U(0) \geq 0$. Then $t_1 \leq t_2$ and $B_1 \leq B_2$ on $(0,t_1)$. Moreover, $d(t) := \dim\ker(U(t))$ is monotoneously decreasing on $(0,t_1)$. In particular, if $B_1(s) = B_2(s)$ for some $s \in (0,t_1)$, then $B_1 = B_2$ and $R_1 = R_2$ on $[0,s]$.*

Proof. Let $t_0 = \min\{t_1,t_2\}$. By $(R_1),(R_2)$, U satisfies

$$U' = X\cdot U + U\cdot X + S \tag{1}$$

on $(0,t_0)$, where $X = -\frac{1}{2}(B_2+B_1)$ and $S = R_1 - R_2 \geq 0$. Since B_j' is bounded from below ($B_j' \geq -R_j$) , we get that X is bounded from above near 0 , i.e. $X \leq a\cdot I$ for some $a \in \mathbb{R}$.

Let $g : (0,t_0) \to \mathrm{End}(E)$ be a nonsingular solution of the homogeneous equation

$$g' = X\cdot g\ . \tag{2}$$

In fact, any matrix solution g of (2) which is nonsingular at some $s_0 \in (0,t_0)$ is nonsingular everywhere on $(0,t_0)$ since the solution $\bar{g} : (0,t_0) \to \mathrm{End}(E)$ of the initial value problem

$$\bar{g}' = -\bar{g}\cdot X\ ,\ \bar{g}(s_0) = g(s_0)^{-1}$$

satisfies $(g\bar{g})' = 0$. Now any solution U of (1) is obtained as

$$U = g\cdot V\cdot g^t$$

where $V : (0,t_0) \to S(E)$ satisfies

$$V' = g^{-1}\cdot S\cdot (g^{-1})^t\ . \tag{3}$$

From $S \geq 0$ we get $V' \geq 0$ on $(0,t_0)$.

Next we have to show that $\lim\limits_{t\to 0} V(t)$ exists and is positive semidefinite. We have

$$\langle Vx,x\rangle \;=\; \langle g^{-1}\cdot U\cdot (g^{-1})^t x,x\rangle \;=\; \langle U\cdot hx,hx\rangle$$

for any $x \in E$, where $h = (g^{-1})^t$, and therefore,

$$|\langle Vx,x\rangle| \;\leq\; \|U\|\cdot\|hx\|^2\ .$$

This is bounded near 0 : From $(g^{-1})' = -(g^{-1}) \cdot X$ we get $h' = -X \cdot h$ and therefore the function $f = \|hx\|^2$ is bounded near 0 since it satisfies

$$f' = 2\langle h'x, hx\rangle = -2\langle X \cdot hx, hx\rangle \geq -2a \cdot f .$$

Consequently, since $\langle V(t)x, x\rangle$ is monotone in t , the limit $V(0) = \lim_{t\to 0} V(t)$ exists. Moreover, there exists a sequence $s_k \to 0$ such that $y_k := h(s_k)x$ converges to some $y \in E$ as $k \to \infty$. Thus

$$\langle V(0)x, x\rangle = \lim\langle U(x_k)y_k, y_k\rangle = \langle U(0)y, y\rangle \geq 0 .$$

Now from $V(0) \geq 0$ and $V' \geq 0$ we get $V \geq 0$ and hence $U \geq 0$. Thus $B_1 \leq B_2$ on $(0, t_0)$. If $t_2 < \infty$, then (R_2) implies that $\langle B_2(t)x, x\rangle \to -\infty$ for some x as $t \to t_2$, and therefore, $t_1 > t_2$ is impossible. Hence $t_0 = t_1 \leq t_2$.

Since $V(t)$ is monotonously decreasing, so is $\dim\ker V(t)$, but $\dim\ker V(t) = \dim\ker U(t) = d(t)$.

Remark 1. Of course, the way we solved (1) is the well known variation of constant method. In general, this can be stated as follows. Let V be a vector space and $\varrho : G \to \text{Aut } (V)$ a representation of a (matrix) Lie group G on V with Lie algebra $\underline{G}$. Let $X : I \to \underline{G}$ and $s : I \to V$ be smooth curves, where I is some real interval. The solutions $u : I \to V$ of the linear ODE

$$(1)' \qquad u' = \varrho_*(X)u + s$$

can be written as $u = \varrho(g)v$ where $v : I \to V$ is smooth and $g : I \to G$ be a solution of the homogeneous equation

$$(2)' \qquad g' = X \cdot g .$$

(This is matrix notation. For an abstract Lie group, $X \cdot g$ has to be replaced by $(R_g)_*X$ where R_g denotes the right translation.) Since $\varrho(g)' = \varrho_*(X)\varrho(g)$, we get $u' = g_*(X)u + \varrho(g)v'$. Hence u is a solution if and only if $\varrho(g)v' = s$, i.e. iff

$$(3)' \qquad v' = \varrho(g^{-1})s .$$

We have applied this to the vector space $V = S(E)$ and the Lie group $G = \text{ Aut } (E)$ with the representation $\varrho(g)u = g \cdot u \cdot g^t$ where $u \in V$, $g \in G$. Note that $\varrho_*(X)u = X \cdot u + u \cdot X^t$ for $X \in \underline{G}$, $u \in V$. Thus (1) has the form (1)'.

Remark 2. Acutally, an arbitrary solution B of (R) in $(0, t_0)$ has the following behavior at 0 :

$$B(t) = P/t + C(t)$$

where P is an orthogonal projection (i.e. $P \in S(E)$, $P^2 = P$) and $C(t)$ extends continuously at $t = 0$ with im $P \subset \ker C(0)$ (see the following proposition). Thus, the initial condition of the theorem may be stated as follows:

$$P_1 = P_2 \qquad \text{and} \qquad C_2(0) \geq C_1(0)$$

where $B_j(t) = P_j/t + C_j(t)$. This fits nicely to the geometric situation alluded to in the introduction. In fact, if we interpret $B(t)$ as the shape operator of the tube of distance t around some submanifold L along a geodesic γ_v perpendicular to L in some Riemannian manifold, then $N =$ im P is the normal space perpendicular to v and $T = \ker P$ the tangent space of L , and $C(0)|T$ is the shape operator of L in the direction v .

Remark 3. A slight modification of the proof of the theorem shows that the continuity of U at 0 is not necessary to assume: Let $u(t)$ be the smallest eigenvalue of $U(t)$. Then we may replace the assumption $U(0) \geq 0$ by

$$\liminf_{t \to 0} u(t) \geq 0 \ .$$

Thus, in the geometric application where B_j corresponds to a submanifold L_j , one can also treat cases where $\dim L_1 \geq \dim L_2$.

Proposition. *Let $B : (0, t_0) \to S(V)$ be a solution of (R) . Then there is a projection $P \in S(V)$, $P^2 = P$, such that*

$$C(t) := B(t) - P/t$$

has a continuous extension to $t = 0$, and im $P \subset \ker C(0)$.

Proof. Let $s_0 \in (0, t_0)$ and $Y : (0, t_0) \to$ End (V) the solution of

$$Y' = B \cdot Y \ , \ Y(s_0) = I \ .$$

Then Y has inverse $Y^{-1} = Z$ which is the solution of

$$Z' = -Z \cdot B \ , \ Z(s_0) = I \ .$$

Thus $B = Y'Y^{-1}$. Differentiating (R) we see that Y also satisfies

$$Y'' + R \cdot Y = 0 \ , \ Y(s_0) = I \ , \ Y'(s_0) = B(s_0) \ ,$$

and so Y extends smoothly to all of $\mathbb{R}$, and

$$Y(t) = Y(0) + t \cdot V(t)$$

where V is smooth with $V(0) = Y'(0)$. Since $Y^t \cdot Y' = Y^t \cdot B \cdot Y$ is symmetric, it vanishes on $W := \ker(Y)$. Thus $Y(W^\perp) \perp Y'(W)$ and consequently $Y(W^\perp) + Y'(W) = E$. (Since Y solves a second order equation, we have $\ker(Y) \cap \ker(Y') = 0$.) Now for an orthonormal basis $e_1, \ldots, e_n$ of E , where $e_1, \ldots, e_k$ is a basis of $W(0) = \ker Y(0)$, we have that

$$t^{-k} \cdot \det Y(t) = \det(V(t)e_1, \ldots, V(t)e_k, Y(t)e_{k+1}, \ldots, Y(t)e_n)$$

has smooth extension to 0 with nonzero limit. Thus $t^k \cdot Y^{-1}(t)$ and hence $t^k \cdot B(t)$ have smooth extensions to 0 . So there are $B_1, \ldots, B_k \in S(E)$ and a smooth $C : [0, t_0) \to S(E)$ such that

$$B(t) = t^{-k} \cdot B_k + \ldots + t^{-1} \cdot B_1 + C(t) .$$

Using (R) to compare coefficients we see that only B_1 is nonzero, and moreover,

$$B_1^2 = B_1 \ , \ B_1 \cdot C(0) + C(0) \cdot B_1 = 0 .$$

Thus, $P := B_1$ is a projection, and $C_0 := C(0)$ vanishes on the image of P since $y = C_0 P x$ is in the $(-1)-$ eigenspace of P which is zero (note that $Py = PC_0 x = -C_0 P x = -y$) .

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Institut für Mathematik
Universitätsstr. 8
D - 8900 Augsburg

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