Online Mathematics Preparedness Course
Final Examination – Solutions

1. Solve: $3^{x^2} = 9^x$
   We have $3^{x^2} = (3^2)^x$
   $$3^{x^2} = 3^{2x} \rightarrow x^2 = 2x$$
   $$x^2 - 2x = 0$$
   $$x(x-2) = 0$$
   $$x = 0; x = 2$$
   Check: $3^{0^2} = 1$ and $9^0 = 1$
   $3^{2^2} = 9^2 \rightarrow 3^4 = 81; 9^2 = 81$

2. Given that $f(x) = 4x^2 - 3x + 1$, the potential candidates for zero's are:
   Factors of 1: ± 1
   Factors if 4: $\pm 1, \pm 2, \pm 4$
   Based on rational root theorem, the potential candidates for zeros are: $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$

3. If $f(x) = x^2 + 1$
   Domain of $f(x)$ is R
   $$f(a+h) = (a+h)^2 + 1 = a^2 + 2ah + h^2 + 1$$
   $$f(a) = a^2 + 1$$
   $$\frac{f(a+h) - f(a)}{h} = \frac{a^2 + 2ah + h^2 + 1 - a^2 - 1}{h} = \frac{2ah + h^2}{h} = \frac{h(2a+h)}{h} = 2a + h$$

$f(x)$ is not an invertible function as it is a quadratic function and not one-to-one

4. Solve the inequality:
   $$x^2 - 2x > 1$$
   $$x^2 - 2x - 1 > 0$$
   $$\left(x - 1 - \sqrt{2}\right)\left(x - 1 + \sqrt{2}\right) > 0$$
   Intervals of interest are of $\left(-\infty, 1 - \sqrt{2}\right), \left(1 - \sqrt{2}, 1 + \sqrt{2}\right), (1 + \sqrt{2}, \infty)$
   Solution set is $(-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$

5. The graph of functions $f(x) = ax + b;\ g(x) = cx + d$ are lines with slopes a, b respectively. Is the graph of fog a line? If so, what is its slope?
   $$f(g(x)) = f(cx + d) = a(cx + d) + b$$
   $$= acx + ad + d$$

$$= (ac)x + (ad + b)$$

Slope is ac, and y-intercept is $ad + b$

6. Solve the following: $\ln(4 - x) = 2\ln x - \ln 2$

$$\ln(4 - x) = \ln x^2 - \ln 2$$
$$\ln(4 - x) = \ln(\frac{x^2}{2})$$
$$4 - x = \frac{x^2}{2}$$
$$8 - 2x = x^2$$
$$x^2 + 2x - 8 = 0$$
$$(x - 2)(x + 4) = 0$$
$$x = 2; x = -4$$

The only acceptable solution is $x = 2$ since $x = -4$ isn't part of the domain $(0,4)$

7. Solve: $2\sin x - 2\sqrt{3}\cos x - \sqrt{3}\tan x + 3 = 0\ on\ [0,2\pi)$

$$2\sin x - \sqrt{3}\tan x - 2\sqrt{3}\cos x + 3 = 0$$
$$\sin x(2 - \frac{\sqrt{3}}{\cos x}) - \sqrt{3}\cos x(2 - \frac{\sqrt{3}}{\cos x}) = 0$$
$$(\sin x - \sqrt{3}\cos x)(2 - \frac{\sqrt{3}}{\cos x}) = 0, \boldsymbol{x \neq \pi/2, 3\pi/2}$$
$$\sin x - \sqrt{3}\cos x = 0 \rightarrow \sin x = \sqrt{3}\cos x \rightarrow \tan x = \sqrt{3} \rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3}$$
$$2 - \frac{\sqrt{3}}{\cos x} = 0 \rightarrow \cos x = \frac{\sqrt{3}}{2} \rightarrow x = \frac{\pi}{6}, \frac{11\pi}{6}$$

8. The reference angle for $\sin\left(\frac{19\pi}{4}\right)$ $is \frac{\pi}{4}$ and the exact value is $\frac{1}{\sqrt{2}}$

9. Prove the identity $\frac{\sin 3x}{\sin x cos\, x} = 4\cos x - \sec x$

$$\frac{\sin 3x}{\sin x cos\, x} = \frac{\sin(2x + x)}{\sin x \cos x} = \frac{\sin x \cos 2x + \cos x \sin 2x}{\sin x \cos x} =$$

$$\frac{\sin x\,(2\cos^2 x - 1) + \cos x\,(2\sin x \cos x)}{\sin x \cos x} = \frac{\sin x(2\cos^2 x - 1)}{\sin x \cos x} + \frac{\cos x\,(2\sin x \cos x)}{\sin x \cos x}$$
$$\frac{2\cos^2 x - 1}{\cos x} + 2\cos x$$
$$2\cos x - \frac{1}{\cos x} + 2\cos x$$

$$4\cos x - \sec x$$

10. Rewrite the expression as an algebraic expression in x: $\sin(2\sin^{-1} x)$

Let $\boldsymbol{\sin^{-1} x = y}$; $\boldsymbol{\sin y = x}$ for $\boldsymbol{-\pi/2 \leq y \leq \pi/2}$

$$\sin(2\sin^{-1}x) = \sin(2y) = 2\sin y\cos y = 2(x)(\sqrt{1-x^2})$$

11. This expression $\frac{3}{2}\ln(x+2) + \frac{1}{2}[\ln x - \ln(x^2+3x+2)]$ in a simple logarithmic is

$$\frac{3}{2}\ln(x+2) + \frac{1}{2}\ln x - \frac{1}{2}\ln(x^2+3x+2)]$$

$$\ln(x+2)^{\frac{3}{2}} + \ln x^{\frac{1}{2}} - \ln((x+1)(x+2))^{\frac{1}{2}}$$

$$\ln\left(\frac{(x+2)^{\frac{3}{2}}x^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}}(x+2)^{\frac{1}{2}}}\right) = \ln\left(\frac{(x+2)x^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}}}\right)$$

12. Evaluate the expression: $\log_2 80 - \log_2 5$

$$\log_2\frac{80}{5} = \log_2 16 = \log_2 2^4 = 4\log_2 2 = 4$$

13. The domain of $\log(x^2+1) - \frac{1}{x} + \sqrt{x^2-2x-3}$

Domain of $\log(x^2+1)$ is R

Domain of $\frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$

Domain of $\sqrt{x^2-2x-3}$ is $(-\infty, -1] \cup [3, \infty)$

Domain is $(-\infty, -1] \cup [3, \infty)$

14. The vertices of the hyperbola $9x^2 - 16y^2 = 144$ are $(\pm 4, 0)$

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$$
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

15. Do these points lie on the same line: $(1,4), (4,10), (9,20)$

Checking for slope with first two points:

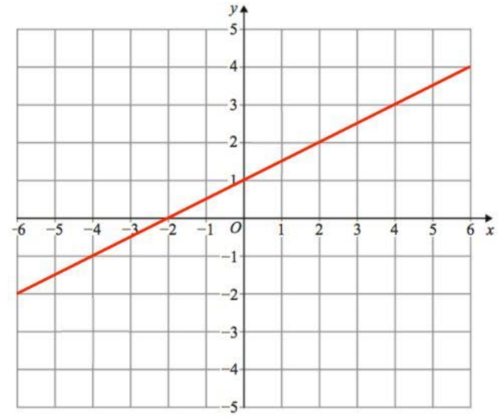
$$m = \frac{10-4}{4-1} = \frac{6}{3} = 2$$

Checking for slope with the second and third point:

$$m = \frac{20-10}{9-4} = \frac{10}{5} = 2$$

They are indeed on the same line since the slope is the same.

16. The equation of this line is:



The two points we can see from the graph are $(-2,0)$ & $(2,2) \to m = \frac{1}{2}$

Equation of line $y = \frac{1}{2}x + b$

$$0 = \frac{1}{2}(-2) + b \to b = 1$$

$$y = \frac{1}{2}x + 1$$

$$2y - x = 2$$

17. If $f(x) = 2x + \ln x$, $f^{-1}(2) = 1$

18. Solve for x: $e^{ax} = Ce^{bx}$

$$\ln e^{ax} = \ln Ce^{bx}$$

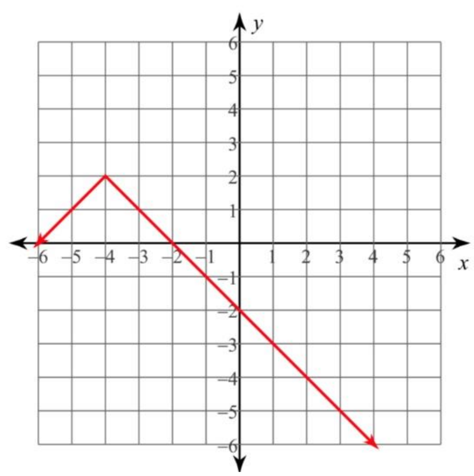
$$ax \ln e = \ln C + bx \ln e$$

$$ax = \ln C + bx$$

$$(a-b)x = \ln C$$

$$\frac{\ln C}{a-b} = x, a \neq b$$

19. This is a graph of an absolute value function, that has been reflected about x-axis, moved 4 units to the left, and 2 units up



$$y = -(x+4) + 2$$

20. $\tan(arc \sin 1)$ is undefined

Let $\theta = arc \sin 1 \to \sin\theta = 1 \to \theta = \frac{\pi}{2}$

$\tan \pi/2$ is undefined

21. Solve for P: $\log P = \log P_0 - clog\ (t+1)$

$$\log P = \log P_0 - log\ (t+1)^c$$

$$\log P = \log \frac{P_0}{(t+1)^c}$$

$$P = \frac{P_0}{(t+1)^c}$$

22. $\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\cos 2x = \cos^2 x - (1 - cos^2 x) = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1$

$\cos 2x = \dfrac{1}{\sec 2x}$