Mathematics Skill Development - Module 7
Assessment Test

The following questions will evaluate the student's understanding of inverse trigonometric functions and ability to solve trigonometric equations and sketch trigonometric functions.

1. Evaluate the following expression as an exact number

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

Solution:

We know that $\sin(\frac{\pi}{3}) = \sqrt{3}/2$ and $\sin(x)$ is an odd function,

$$\sin(-x) = -\sin(x), \quad \forall x \in \mathbb{R}.$$

Combining these facts, we have that

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}.$$

□

2. Write $\sin(2\cos^{-1}(x))$ as an algebraic expression in x for $-1 \leq x \leq 1$.

Solution:

Let $\theta = \cos^{-1}(x)$ so that $\cos(\theta) = x$. Using the double angle formula

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

we have that $\sin(2\cos^{-1}(x)) = 2x\sin(\cos^{-1}(x))$ (since $\cos(\cos^{-1}(x)) = x$ by the definition of an inverse function). To solve for $\sin(\theta) = \sin(\cos^{-1}(x))$, we could either use the trigonometric identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

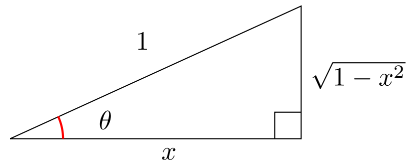
to conclude that

$$\begin{aligned}
\sin^2(\theta) &= 1 - \cos^2(\theta) \\
\sin(\theta) &= \sqrt{1 - \cos^2(\theta)} \\
&= \sqrt{1 - \cos^2\cos^{-1}x} \\
&= \sqrt{1 - x^2}
\end{aligned}$$

where the last line follows from $\cos\cos^{-1}x = x$. Therefore

$$\sin\left(2\cos^{-1}(x)\right) = 2x\sqrt{1 - x^2}.$$

An alternative, geometrical, approach is to construct a right-angled triangle corresponding to the trigonometric ratio $\cos(\theta) = x = \frac{x}{1}$:



Using the Pythagoras theorem, we can solve for $\sin\theta = \sqrt{1-x^2}$ and obtain the same result.

□

3. Solve the following equation

$$\cos x + 1 = \sin x.$$

in the interval $[0, 2\pi)$.

Solution:

We square both sides and then use the trigonometric identity $\sin^2 x + \cos^2 x = 1$ to compute

$$\begin{aligned}
\cos x + 1 &= \sin x \\
\cos^2 x + 2\cos x + 1 &= \sin^2 x \\
\cos^2 x + 2\cos x + 1 &= 1 - \cos^2 x \\
2\cos^2 x + 2\cos x &= 0 \\
2\cos x(\cos x + 1) &= 0.
\end{aligned}$$

Implying either $\cos x = 0$ or $\cos x + 1 = 0$. In the interval $[0, 2\pi)$, $\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $\cos x = -1$ when $x = \pi$. Thus the only possible solutions to our equation are $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Since we squared both sides of the original equation, it is very important that we check that we have not introduced any spurious solutions. We have that

$$\begin{aligned}
\cos\frac{\pi}{2} + 1 &= 1 \\
\cos\pi + 1 &= 0 \\
\cos\frac{3\pi}{2} + 1 &= 1
\end{aligned}$$

while

$$\begin{aligned}
\sin\frac{\pi}{2} &= 1 \\
\sin\pi &= 0 \\
\sin\frac{3\pi}{2} &= -1.
\end{aligned}$$

Therefore $x = \frac{\pi}{2}$ and $x = \pi$ are the only solutions to the original equation.
Note: $x = \frac{3\pi}{2}$ *is just spurious solution originating from the fact that* $1^2 = (-1)^2$ *and does not solve the original equation.*

□

4. Explain how to obtain the graph of $-2\cos(x-\pi)+3$ from the graph of $\sin(x)$.

Solution:

Recall that $\sin(x) = \cos(x-\pi/2)$. and therefore in order to obtain the graph of $-2\cos(x-\pi)+3$, we must shift the graph of $\sin(x)$ by $\pi/2$ units to the right, stretch it vertically by a factor of -2 and then translate up by 3 units.

$\square$