Mathematics Skill Development - Module 6
Assessment Test

The following questions will evaluate the student's understanding of circular arc length, the relationship between degree and radian measure, trigonometric identities and trigonometric ratios.

1. Find the radian measure of the following
(a) $-145°$.
(b) $(6 \cdot 600)°$.

Solution:

(a) Using the fact the π radians equals $180°$ and angles which differ by $360°$ are equivalent, we compute

$$-145° = (-145 + 360)° = 215° = \frac{215\pi}{180} \text{ rad} = \frac{43\pi}{36} \text{ rad}.$$

$\square$

(b) Using the fact that angles which differ by $360°$ are equivalent, we compute

$$(6 \cdot 600)° = 3600° = 10 \cdot 360° = 0° = 0 \text{ rad}$$

$\square$

2. Find the degree measure of the following
(a) $\frac{5\pi}{12}$ rad.
(c) 18π rad.

Solution:

(a) Using the fact the π radians equals $180°$, we compute

$$\frac{5\pi}{12} \text{ rad} = \frac{5}{12} \cdot 180° = 75°.$$

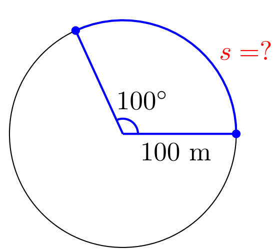
$\square$

(b) Using the fact that angles which differ by 2π radians are equal and -18 is a negative even number, we have -18π rad $= 0$ rad $= 0°$.

$\square$

3. A horse races along a circular track of radius 100m. If the angle subtended by the horse's path from its starting position is $100°$, what distance did it travel along the track?
Solution:

It is helpful to draw a diagram. We have a circular track of radius 100m. Let P be the point where the horse starts running and Q its current location on the track. Then the angle subtended by the horse's path is $100°$ and we want to find the arclength distance it has traveled along the track (denote this distance by s):



Recall that the arclength s is related to the radius of the circle r and the subtended angle θ via

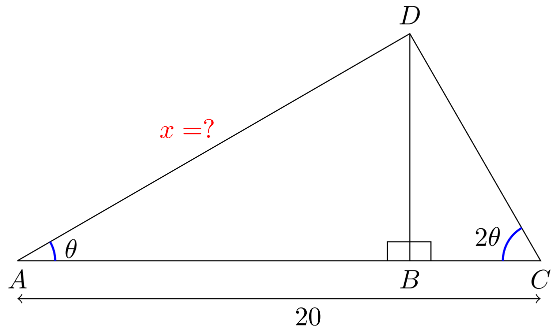
$$s = r\theta$$

where θ is always measured in radians. In this case $r = 100$ and $\theta = 100 \cdot \frac{\pi}{180}$ rad $= \frac{5\pi}{9}$ rad. Therefore the arclength or the distance travelled by the horse is

$$s = r\theta = 100 \cdot \frac{5\pi}{9}\text{m} = \frac{500\pi}{9}\text{m}$$

$\square$

4. Find an expression for $x = |AD|$ as a function of θ in the following triangle:



Solution:

We seek to find $x = |AD|$ as a function of θ to directly relate the two quantities. In order to do this, it was helpful to include the vertical line BD and split the triangle ABC into the two right angled triangles: ABD and CBD. In any right angled triangle we can use the trigonometric ratios to related the length of its sides its angles. In this case, the trigonometric ratios in ABD give

$$\cos\theta = \frac{|AB|}{x}$$
$$\sin\theta = \frac{|BD}{x}$$

while those in the triangle CBD give

$$\cos 2\theta = \frac{|BC|}{|CD|}$$
$$\sin 2\theta = \frac{|BD|}{|CD|}.$$

In addition, since the length of AC is 20, we have the simple relation

$$|AB| + |BC| = |AC| = 20.$$

In a difficult problem, it is always helpful to write down all the information we have at the start of your solution. Then we can try to put it together.

Let us eliminate some of the unknown lengths. Since $\sin\theta = \frac{|BD|}{x}$ and $\sin 2\theta = \frac{|BD|}{|CD}$, we can write the length $|BD|$ in two equivalent ways:

$$|BD| = x\sin\theta$$
$$|BD| = |CD|\sin 2\theta,$$

and therefore,

$$|CD| = x\frac{\sin\theta}{\sin 2\theta}.$$

Using this information in the equation $\cos 2\theta = \frac{|BC|}{|CD|}$, we have

$$\cos 2\theta = \frac{|BC|}{x\sin\theta}\sin 2\theta.$$

It remains to express the length $|BC|$ in terms of x and θ. We use the equation $|AB| + |BC| = 20$ to write $|BC| = 20 - |AB|$ and the trigonometric ratio $\cos\theta = \frac{|AB|}{x}$ gives us $|AB| = x\cos\theta$. Therefore,

$$|BC| = 20 - |AB| = 20 - x\cos\theta$$

and

$$\begin{aligned}\cos 2\theta &= \frac{|BC|}{x\sin\theta}\sin 2\theta \\ &= \left(\frac{20 - x\cos\theta}{x\sin\theta}\right)\sin 2\theta.\end{aligned}$$

This is a relation involving only the two quantities x and θ as we wanted. It only remains to write x as a function θ and simplifying the equation a bit. Using the double angle formulas $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we calculate

$$\begin{aligned}\cos 2\theta &= \left(\frac{20 - x\cos\theta}{x\sin\theta}\right)(2\sin\theta\cos\theta) \\ \cos 2\theta &= \frac{2}{x}(20 - x\cos\theta)\cos\theta \\ x\cos 2\theta &= 40\cos\theta - 2x\cos^2\theta \\ x\cos^2\theta - x\sin^2\theta &= 40\cos\theta - 2x\cos^2\theta \\ x\left(3\cos^2\theta - \sin^2\theta\right) &= 40\cos\theta \\ x &= \frac{40\cos\theta}{3\cos^2 - \sin^2\theta}.\end{aligned}$$

Note: you may find different looking functions but they are all equivalent. For example we can use the identity $\cos^2\theta + \sin^2\theta = 1$ *to rewrite* x *as a function of only* $\cos\theta$ *or of only* $\sin\theta$.

□

5. Show that the expression below is equivalent to $-\sin x$

$$\frac{\cos 2x - \cot x \sin 2x}{\csc x}.$$

Solution:

By rewriting $\cot x = \cos x/\sin x$ and using the formula $\sin 2x = 2\sin x\cos x$, the expression becomes

$$\begin{aligned}\frac{\cos 2x - \cot x\sin 2x}{\csc x} &= \frac{\cos 2x - \frac{\cos x}{\sin x}\cdot 2\sin x\cos x}{\csc x} \\ &= \frac{\cos 2x - 2\cos^2 x}{\csc x}.\end{aligned}$$

Using the double angle formula $\cos 2x = \cos^2 x - \sin^2 x$ and the identity $\sin^2 x + \cos^2 x = 1$, we have

$$\begin{aligned}
\frac{\cos 2x - \cot x \sin 2x}{\csc x} &= \frac{\cos^2 x - \sin^2 x - 2\cos^2 x}{\csc x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\frac{1}{\sin x}} \\
&= -\sin x.
\end{aligned}$$

$\square$