Mathematics Skill Development - Module 5
Assessment Test

The following questions will evaluate the student's understanding of compositions of functions, even and odd functions, one-to-one functions and their inverses, logarithmic functions and exponential functions.

1. Let $h(x) = \sqrt[3]{\sqrt{x}+1}$ and $g(x) = x - 1$. Find two functions $f(x)$ and $a(x)$ such that we may write h as the composition

$$h(x) = f \circ g \circ a(x).$$

Solution:

Since $g \circ a(x) = g(a(x)) = a(x) - 1$, the simplest choice is to set $f(x) = \sqrt[3]{x}$. Then

$$f \circ g \circ a(x) = \sqrt[3]{a(x) - 1}$$

and we need only choose $a(x)$ so that the expressions inside the cubic root match up. Solving $a(x) - 1 = \sqrt{x}+1$ gives $a(x) = \sqrt{x} + 2$ and

$$f \circ g \circ a(x) = \sqrt[3]{\left(\sqrt{x}+2\right) - 1} = \sqrt[3]{\sqrt{x}+1} = h(x)$$

□

2. Determine if the following functions are even, odd or neither:
(a) $f(x) = 2 - x^2 - x^4$.
(b) $f(x) = \frac{x^4}{x^2+1}$.
(c) $f(x) = 1 - \sqrt{x}$.

Solution:

(a) Since f is a sum of 3 even functions, we expect it to be even also. Computing

$$\begin{aligned} f(-x) &= 2 - (-x)^2 - (-x)^4 \\ &= 2 - (-1)^2x^2 - (-1)^4x^4 \\ &= 2 - x^2 - x^4 \\ &= f(x), \end{aligned}$$

and f is even as expected.

□

(b) In this case, f is a ratio of 2 even functions and we expect that it is also even. Computing

$$\begin{aligned} f(-x) &= \frac{(-x)^4}{(-x)^2+1} \\ &= \frac{(-1)^4x^4}{(-1)^2x^2+1} \\ &= \frac{x^4}{x^2+1} \\ &= f(x) \end{aligned}$$

explicitly shows that f is an even function.
(c) Since $\sqrt{x}$ is defined only when $x \geq 0$, negative numbers are not in the domain of f. Therefore, we cannot make sense of the expression for $f(-x)$ when $x > 0$ and f is neither even nor odd (since it cannot satisfy the definition of either).

$\square$

3. Suppose $e^{g(x)} = x + 2$. Use this fact to solve the equation

$$g(x) = \ln\left(\frac{3}{x}\right).$$

Solution:

Taking the natural logarithm of both sides of the equation $e^{g(x)} = x + 2$, we get $g(x) = \ln(x + 2)$. Therefore, we solve the equation $\ln(x+2) = \ln\left(\frac{3}{x}\right)$ and use the properties of logarithms to get

$$\begin{aligned} \ln(x+2) &= \ln 3 - \ln x \\ \ln(x+2) + \ln x &= \ln 3 \\ \ln(x(x+2)) &= \ln 3 \\ \ln(x^2 + 2x) &= \ln 3. \end{aligned}$$

From here, we exponentiate both sides to obtain

$$\begin{aligned} x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0. \end{aligned}$$

Factoring this quadratic equation gives 2 possible solutions: $x = -3$ and $x = 1$. However, when $x = -3$, $\ln(x+2)$ and $\ln\left(\frac{3}{x}\right)$ are both undefined (we can only take logarithms of positive numbers). Therefore $x = 1$ is the only solution of $\ln(2 + x) = \ln\left(\frac{3}{x}\right)$.

$\square$

4. Assume that the function $f(x) = \frac{e^x}{2e^x+1}$ is one-to-one. Find $f^{-1}(x)$.

Solution:

Let $y = f(x)$ and solve for x in terms of y:

$$\begin{aligned} y &= \frac{e^x}{2e^x + 1} \\ 2ye^x + y &= e^x \\ e^x(2y - 1) &= -y \\ e^x &= \frac{-y}{2y - 1}. \end{aligned}$$

Taking the natural logarithm of both sides of this equation gives

$$x = \ln\left(\frac{-y}{2y - 1}\right).$$

Therefore,

$$f^{-1}(x) = \ln\left(\frac{-x}{2x - 1}\right).$$

$\square$