Mathematics Skill Development - Module 1
Assessment Test

The following questions will evaluate the student's understanding of the material in module 1. Topics include: number types, interval notation of the real number line, simplification of expressions, factoring and long division of polynomials.

1. Find two irrationals whose sum is rational.

Solution:

Since π is an irrational number, so are the numbers $a = \pi + 1$ and $b = -\pi + 1$. Indeed, if a were a rational number, we could find integers m and n to write $a = \frac{m}{n}$. Since $a = \pi + 1$, we could then write $\pi = -1 + \frac{m}{n}$ and contradict the fact that π is an irrational number. The same argument shows that b is also an irrational number (can you explain why?).

However the sum of a and b is $a + b = 2$ which is clearly a rational number.

$\square$

2. The set which contains no elements is referred to as the empty set and is denoted by $\emptyset$. Let $A = (-1, 2), B = (-1, 4], C = \emptyset, D = (2, 5)$ and $E = \mathbb{Z}$.
(a) Write $(A \cap B) \cup C$ in interval notation.
(b) Using the each of the sets A, B, D and E at least once along with the symbols "$\cap$" and "$\cup$", write an expression that equals $\emptyset$.

Solution:

(a) $A \cap B = (-1, 2)$. Since $C = \emptyset$ and contains no elements, $(A \cap B) \cup C = (-1, 2)$.

$\square$

(b) The intersection of the empty set $\emptyset$ with any other set must be empty (how could they have elements in common if $\emptyset$ ahs no elements to begin with?). Therefore any combination of A, B, D and E intersected with C will be empty. For example:

$$(A \cap B \cap D \cap E) \cap C = \emptyset$$

$\square$

3. Use the difference of squares formula to factor $x^{16} - 1$ into a product of five polynomial terms.

Solution: Recall the difference of squares formula: for any numbers a and b, we have that $a^2 - b^2 = (a + b)(a - b)$. Since $x^{16} = \left(x^8\right)^2$ and $1 = 1^2$, we have

$$x^{16} - 1 = (x^8 + 1)(x^8 - 1).$$

In the same way $x^8 = \left(x^4\right)^2$ and

$$x^8 - 1 = (x^4 + 1)(x^4 - 1).$$

Two more steps on the second term give

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1).$$

Putting these 3 equations together, we have

$$x^{16} - 1 = (x^8 + 1)(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$$

□

4. Simplify $\frac{x-3}{\sqrt{x}-1} + \frac{4}{\sqrt{x}+1}$.

Solution: The common denominator of these fractions is $(\sqrt{x}-1)(\sqrt{x}+1)$. Adding the 2 terms together and simplifying:

$$\begin{aligned}\frac{x-3}{\sqrt{x}-1} + \frac{4}{\sqrt{x}+1} &= \frac{(x-3)(\sqrt{x}+1)+4(\sqrt{x}-1)}{x-1} \\ &= \frac{x^{3/2}+x-3x^{1/2}-3+4x^{1/2}-4}{x-1} \\ &= \frac{x^{3/2}+x+x^{1/2}-7}{x-1}\end{aligned}$$

□

5. Solve $x^3+2x^2-5x=6$. *(Hint: try $x=2$ first and then find all the other solutions).*

Solution: The given equation is equivalent to

$$x^3+2x^2-5x-6=0.$$

and we find all the roots of x^3+2x^2-5x-6. It is easy to check that $x=2$ is one of the roots and therefore $(x-2)$ must be one of the factors of this expression. Using long division:

$$\begin{array}{r} x^2+4x+3. \\ x-2 \,\big)\overline{\; x^3+2x^2-5x-6} \\ \underline{-\,x^3+2x^2 \qquad\qquad} \\ 4x^2-5x \quad\; \\ \underline{-\,4x^2+8x \quad\;} \\ 3x-6 \\ \underline{-\,3x+6} \\ 0 \end{array}$$

we see that $x^3+2x^2-5x-6=(x-2)(x^2+4x+3)$. Factoring the quadratic term on the right hand side, we conclude $x^3+2x^2-5x-6=(x-2)(x+3)(x+1)$. The roots of the original expression are precisely the roots of the individual factors, namely $x=2,-3,-1$.

□