ECO 3901
EMPIRICAL INDUSTRIAL ORGANIZATION

Lecture 1
Introduction to the course
Dynamic games of oligopoly competition: Models & Solution methods

Victor Aguirregabiria (University of Toronto)

January 13, 2022

Organization of the Course

- **Class Meetings:** Thursdays, 3-5pm
  - Online Lectures via Zoom, Weeks 1 to 3
  - In Person Lectures, Weeks 4 to 13: O.I.S.E. Room OI8214
- **Office hours**: Tuesdays, 3-5pm
- **Evaluation:** Problem Set 1 (50%); Final Exam (50%)
- **What do I expect from you?**
  (1) attend every class meeting
  (2) read papers/material before each lecture
  (3) participate in class
  (4) go through class notes and understand them
  (5) do the problem set on time
  (6) prepare for the final exam

A General Description of this Course

- This courses deals with **models, methods, and empirical applications** in Industrial Organization (IO).
- We will study the determinants of firms' behaviour and market outcomes in the context of problems of market entry/exit, investment, innovation, product design, networks, matching, and natural resources.
- The course focuses on research papers using **empirical dynamic games** to investigate firms' strategies and competition.
- This course emphasizes the combination of **data, models, and econometric techniques** to understand how markets operate.

Topics

1. Dynamic games of oligopoly competition: Models & solution methods
2. Identification of structural dynamic games
3. Structural estimation dynamic games
4. Market entry and exit
5. Dynamic spatial competition in retail markets
6. Uncertainty and firms' investment decisions
7. Dynamic games of innovation
8. Dynamic price competition
9. Airline networks
10. Mergers and dynamics
11. Dynamic search and matching
12. Dynamic games with firms' non-equilibrium beliefs and learning
13. Dynamic games of natural resources extraction

Today's Lecture: Dynamic games of oligopoly competition: Models

1. Introduction & Examples
2. Structure of empirical dynamic games
3. Markov Perfect Equilibrium
4. Dynamic Games with Incomplete Information
5. Solution Methods
6. Some extensions of the basic framework

1. Introduction

Dynamic Games: Introduction

- In oligopoly industries, firms compete in **decisions** that:
  - have returns in the future (forward-looking)
  - involve substantial uncertainty
  - have important effects on competitors' profits

- Some examples are:
  - Investment in R&D, innovation
  - Investment in capacity, physical capital
  - Product design / quality
  - Market entry / exit
  - Pricing ...

Dynamic Games: Introduction [2]

- Measuring and understanding the **dynamic strategic interactions** between firms decisions is important to understand the forces behind the evolution of an industry or to evaluate policies.

- Investment costs, uncertainty, and competition effects play an important role in these decisions.

- Estimation of these parameters is necessary to answer some empirical questions.

- Empirical dynamic games provide a framework to estimate these parameters and perform policy analysis.

Examples of Empirical Applications

- **Competition in R&D and product innovation**
  - Intel & AMD: Goettler and Gordon (JPE, 2011)
  - Incumbents & new entrants (hard drives): Igami (JPE, 2017).

- **Regulation and industry dynamics**
  - Environmental regulations, entry-exit and capacity in cement industry: Ryan (ECMA, 2012)
  - Land use regulation in the hotel industry: Suzuki (IER; 2013)
  - Subsidies to entry in small medical markets: Dunne et al. (RAND, 2013).

Examples of Empirical Applications [2]

- **Product Design, Preemption, and Cannibalization**
  - Choice of format of radio stations: Sweeting (ECMA, 2013)
  - Hub-and-spoke networks and entry deterrrence in the airline industry: Aguirregabiria and Ho (JoE, 2012)
  - Cannibalization and preemption strategies in fast-food industry: Igami and Yang (QE, 2016).
- **Demand uncertainty, Time to build, and Investment**
  - Concrete industry: Collard-Wexler (ECMA, 2013)
  - Shipping industry: Kalouptsidi (AER, 2014)
- **Dynamic price competition**
  - Price adjustment costs: Kano (IJIO, 2013)
  - Frictions (adjustment costs) both in demand and supply: Mysliwski, Sanches, Silva & Srisuma (WP, 2020)

Examples of Empirical Applications [3]

- **Dynamic effects of mergers**
  - Dynamic response after airline mergers: Benkard, Bodoh-Creed, and Lazarev (WP, 2010)
  - Endogenous mergers: Jeziorski (RAND, 2014).

- **Exploitation of a common natural resource**
  - Fishing: Huang and Smith (AER, 2014).

- **Dynamic Search & Matching**
  - NYC Taxi industry: Buchholz (AER, 2022)
  - World trade and transoceanic shipping industry: Brancaccio, Kalouptsidi, and Papageorgiou (ECMA, 2020).

2. Structure of Dynamic Games of Oligopoly Competition

Model: Basic Structure (Ericson-Pakes, 1995)

- Time is discrete and indexed by t.
- The game is played by N firms that we index by i.
- The action is taken to maximize the expected and discounted flow of profits in the market,

$$E_t \left({}^{\infty}_{s=0} \delta_i^s \ \pi_{it+s} \right)$$

$\delta_i \in (0, 1)$ is the discount factor, and π_{it} is firm i's profit at period t.

- Every period t, firms make two decisions: a static, and a dynamic.
- For instance (Pakes & McGuire, 1994): firms compete in prices (static competition), and make investments to improve the quality of their products (dynamic decision).

Model: Decisions, States, Profits

- We represent firm $i's$ investment / dynamic decision by a_{it}. It can be continuous, discrete, or mixed.

- Current profit π_{it} depends on the firms's own action a_{it}, other firms' actions, $\boldsymbol{a}_{-it} = \{a_{jt} : j \neq i\}$, and a vector of state variables $\boldsymbol{x}_t$.

$$\pi_{it} = \pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t \right)$$

- We should interpret $\pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t \right)$ as an **"indirect" profit function** that comes from the static equilbrium of the model: e.g., Bertrand equilibrium in prices, Cournot equilibrium in quantities.

- $\boldsymbol{x}_t$ includes:
 - Endogenous state variables that depend on the firms' investment decisions at previous periods, e.g., capital stocks.
 - Exogenous state variables affecting costs and consumer.

Example: Dynamic Quality Competition

- Each firm has a differentiated product. Consumer demand depends on products' qualities (k_{it}) and prices (p_{it}).

- State $\boldsymbol{x}_t$ consists of product qualities $\boldsymbol{k}_t = (k_{1t}, k_{2t}, ..., k_{Nt})$, and exogenous variables affecting firms' marginal costs ($\boldsymbol{z}_t$).

- Given $\boldsymbol{x}_t$, firms' compete in prices a la Bertrand, and this determines Bertrand equilibrium variable profits for each firm: $r_i(\boldsymbol{x}_t)$.

- The total profit, π_{it}, consists on $r_i(\boldsymbol{x}_t)$ minus the cost of investing in quality improvement: $IC_i(a_{it}, k_{it})$:

$$\pi_{it} = r_i(\boldsymbol{x}_t) - IC_i(a_{it}, k_{it})$$

- Quality stock evolves endogeneously according to the transition rule:

$$k_{i,t+1} = k_{it} + a_{it}$$

Example: Dynamic Quality Competition (2)

- We can be more specific in the specifica of variable profit r_{it}:

$$r_{it} = (p_{it} - mc_i(k_{it}, \mathbf{z}_t))\ q_{it}$$

p_{it} and q_{it} are the price and the quantity sold by firm i.

- The demand system could have a simple Logit structure:

$$q_{it} = \frac{\exp\{\mathbf{z}_{it}\beta_z + \beta_k\ k_{it} - \alpha\ p_{it}\}}{1 + \sum_{j=1}^{N} \exp\{\mathbf{z}_{jt}\beta_z + \beta_k\ k_{jt} - \alpha\ p_{jt}\}}$$

- Bertrand equilibrium implies the **"indirect" variable profit function**:

$$r_i(\mathbf{k}_t,\mathbf{z}_t) = (p_i^*[\mathbf{k}_t,\ \mathbf{z}_t] - mc_i[k_{it},\mathbf{z}_t])\ q_i^*[\mathbf{k}_t,\ \mathbf{z}_t]$$

Timing of the model: Time-to-Build or Not

- The previous example incorporates an assumption of **time-to-build**.
- **Time-to-build**: The investment decision at period t, a_{it}, takes one period to affect thet quality stock. Therefore, variable profit depends $\boldsymbol{k}_t$ but noy on a_{it} or $\boldsymbol{a}_{-it}$:

$$\pi_{it} = r_i(\boldsymbol{k}_t, \boldsymbol{z}_t) - IC_i\left(a_{it}, k_{it}\right) = \pi_i\left(a_{it}, \boldsymbol{x}_t\right)$$

- We still have a (dynamic) game, as future profits depend on $\boldsymbol{a}_{-it}$.
- **Without Time-to-build:** We can consider a version of the model where investment has an instantaneous effect on quality: demand, marginal costs, and variable profits depend on $\boldsymbol{k}_t + \boldsymbol{a}_t$ instead of $\boldsymbol{k}_t$:

$$\pi_{it} = r_i(\boldsymbol{k}_t + \boldsymbol{a}_t, \boldsymbol{z}_t) - IC_i\left(a_{it}, k_{it}\right) = \pi_i\left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t\right)$$

Model: Evolution of the state variables

- **Exogenous common knowledge state variables:** follow an exogenous Markov process with transition probability function $F_z(\mathbf{z}_{t+1}|\mathbf{z}_t)$.

- **Endogenous state variables:** The form of the transition rule depends on the application:
  - Market entry: $k_{it} = a_{it-1}$, such that $k_{i,t+1} = a_{it}$
  - Investment without depreciation: $k_{i,t+1} = k_{it} + a_{it}$.
  - Investment - deterministic depreciation: $k_{i,t+1} = \lambda(k_{it} + a_{it})$
  - Investment - stochastic depreciation: $k_{i,t+1} = k_{it} + a_{it} - \xi_{i,t+1}$

- In a compact way, we use $F_x(\mathbf{x}_{t+1}|\mathbf{a}_t, \mathbf{z}_t)$ to represent the transition probability function of all the state variables.

3. Markov Perfect Equilibrium

Markov Perfect Equilibrium: Key Assumption

- Most dynamic IO models assume Markov Perfect Equilibrium (MPE), (Maskin and Tirole, ECMA 1988).

- A key condition in this solution concept is that **players' strategies are functions of only payoff-relevant state variables**, $\boldsymbol{x}_t$.

- **Why this restriction?**:

  • **Rationality:** if other players have this type of strategies, a player cannot make better by conditioning its behavior on non-payoff relevant information (e.g., lagged values of the state variables)

  • **Dimensionality:** It is convenient because it reduces the dimensionality of the state space.

- It is straightforward to extend results below to an equilibrium concept where strategy functions can depend on $(\boldsymbol{x}_t, \boldsymbol{x}_{t-1})$, or on $(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}, \boldsymbol{x}_{t-2})$, and so on.

Markov Perfect Equilibrium: Definition

- Let $\boldsymbol{\alpha} = \{\alpha_i(\mathbf{x}_t) : i = 1, 2, ..., N\}$ be a set of strategy functions.
- A MPE is an N-tuple of strategy functions **ff** such that every firm is maximizing its value given the strategies of the other players.
- For given strategies of the other firms, the decision problem of a firm is a single-agent dynamic programming (DP) problem.

Markov Perfect Equilibrium: Best Response DP

- Let $V_i^{\boldsymbol{\alpha}}(\mathbf{x}_t)$ be the value function of the DP problem that describes the best response of firm i to the strategies of the other firms in $\boldsymbol{\alpha}$.
- This value function is the unique solution to the Bellman equation:

$$V_i^{\boldsymbol{\alpha}}(\mathbf{x}_t) = \max_{a_{it}} \left\{ \pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \delta_i \int V_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1}) \; dF_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \right\}$$

- with:

$$\pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) \quad = \quad \pi_i \left(a_{it}, \boldsymbol{\alpha}_{-i}(\mathbf{x}_t), \mathbf{x}_t \right)$$

- and:

$$F_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \quad = \quad F_x(\mathbf{x}_{t+1} | a_{it}, \boldsymbol{\alpha}_{-i}(\mathbf{x}_t), \mathbf{x}_t)$$

Markov Perfect Equilibrium: Definition

- A Markov perfect equilibrium (MPE) is an N-tuple of strategy functions $\boldsymbol{\alpha}$ such that for any player i and for any $\mathbf{x}_t$, we have that:

$$\alpha_i(\mathbf{x}_t) = \arg\max_{a_{it}} \quad v_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t)$$

with $v_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t)$ being the **Conditional-Choice Value Function:**

$$v_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) \equiv \pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1}) \; dF_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$$

MPE: Existence

- Doraszelski & Satterhwaite (2010) show that existence of a MPE in pure strategies is not guaranteed in this model.
- When firms make discrete choices, the existence of a MPE cannot be ensured without allowing firms to randomize over discrete actions.
- A possible approach to guarantee existence is to allow for mixed strategies. However, computing a MPE in mixed strategies poses important computational challenges.
- To establish equilibrium existence, D&S propose incorporating private information state variables.
- This incomplete information version of Ericson-Pakes model has been the one adopted in most empirical applications.

4. Dynamic Games with Incomplete Information

Incomplete Information Shocks

- Suppose that the profit function depends also on a vector of state variables $\varepsilon_{it} = (\varepsilon_{it}(a_{it}) : a_{it} \in A\}$.

- ε_{it} is **private information of firm** i, independently distributed over time and across individuals with CDF G_i that has full support on $\mathbb{R}^{|A|}$.

- Strategy functions are now $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$.

- It is very convenient to represent a firm's strategy using **Conditional Choice Probability (CCP) function**:

$$P_i(a|\mathbf{x}) \equiv \Pr\left(\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a \mid \mathbf{x}_t = \mathbf{x}\right)$$

- Since choice probabilities are integrated over the continuous variables in ε_{it}, they are lower dimensional objects than the strategies $\boldsymbol{\alpha}$.

Conditional Choice Probabilities

- When $\varepsilon_{it}(a_{it})$ enter additively in the profit function, there is a **one-to-one relationship** between best-response strategy functions $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ and its CCP function $P_i(.|\mathbf{x}_t)$.
- It is obvious that given $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ there is a unique $P_i(.|\mathbf{x}_t)$.
- The inverse relationship – given $P_i(.|\mathbf{x}_t)$ there is a unique best response function $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ – is a corollary of **Hotz-Miller inversion Theorem**.

- **Conditional Choice Probabilities (2)**

- **Hotz-Miller inversion Theorem** (Hotz & Miller, REStud, 1993)

  "Let $\alpha_i(x_t, \varepsilon_{it})$ be a best response strategy and let $P_i^{\alpha}(a|\mathbf{x})$ be its corresponding CCP such that:

$$P_i^{\alpha}(a|\mathbf{x}) = \int 1\{\arg\max_{a_{it}}[v_i^{\alpha}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it})] = a\} \ dG_i(\varepsilon_{it})$$

  This mapping from the vector of conditional-choice values $\{v_i^{\alpha}(a, x_t) : a \in A\}$ into the vector of CCPs $\{P_i(a|x_t) : a \in A\}$ is invertible."

- Therefore, given $P_i^{\alpha}(.|\mathbf{x}_t)$ we have a unique $v_i^{\alpha}(., \mathbf{x}_t)$, and then a unique best response strategy function:

$$\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it}}[v_i^{\alpha}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it})]$$

MPE as Fixed Point Mapping in CCPs

- Given strategy functions described by CCP functions $\mathbf{P}$, we can define $\pi_i^{\mathbf{P}}$ and $F_i^{\mathbf{P}}$ as:

$$\begin{aligned} \pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) &= \sum_{a_{-it}} \left[\prod_{j \neq i} P_j \left(a_{jt} \mid \mathbf{x}_t \right) \right] \pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t \right) \\ F_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) &= \sum_{a_{-it}} \left[\prod_{j \neq i} P_j \left(a_{jt} \mid \mathbf{x}_t \right) \right] F_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) \end{aligned}$$

- We also define:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \equiv \pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\mathbf{P}}(\mathbf{x}_{t+1}) \, dF_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t)$$

- with:

$$V_i^{\mathbf{P}}(\mathbf{x}_t) = \int \max_{a_{it}} \left\{ v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it}) \right\} dG_i(\varepsilon_{it})$$

MPE as Fixed Point Mapping in CCPs [2]

- A MPE is a vector of CCPs, $\mathbf{P} \equiv \{P_i(a|\mathbf{x}) : \text{for any } (i, a, \mathbf{x})\}$, such that, for any $(i, a, \mathbf{x}$:

$$P_i(a|\mathbf{x}) = \Pr\left(a = \arg\max_{a_i} \left\{ v_i^{\mathbf{P}}(a_i, \mathbf{x}) + \varepsilon_i(a_i) \right\} \mid \mathbf{x}\right)$$

- This system of equations defines a Fixed Point mapping from the space of CCPs $\mathbf{P}$ into itself:

$$\mathbf{P} = \Psi(\mathbf{P})$$

- Under the conditions of the model, the mapping $\Psi(.)$ is continuous. Therefore, by Brower's Fixed Point Theorem an equilibrium exists.
- In general, this model has multiple equilibria.

MPE in terms of CCPs: Example

- Suppose that vector ε_{it}'s are iid Extreme Value Type I.
- Then, a MPE is a vector $\mathbf{P} \equiv \{P_i(a|\mathbf{x}) : \text{for any } (i, a, \mathbf{x})\}$, such that:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{v_i^{\mathbf{P}}(a, \mathbf{x})\right\}}{\sum_{a'} \exp\left\{v_i^{\mathbf{P}}(a', \mathbf{x})\right\}}$$

- where

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \equiv \pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\mathbf{P}}(\mathbf{x}_{t+1}) \; dF_i^{\mathbf{P}}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$$

- and $V_i^{\mathbf{P}}$ is the unique solution to the Bellman equation:

$$V_i^{\mathbf{P}}(\mathbf{x}_t) = \ln\left(\sum_{a_i} \exp\left\{\pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta V_i^{\mathbf{P}}(\mathbf{x}_{t+1}) \; F_i^{\mathbf{P}}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)\right\}\right)$$

5. Solution Methods

Equilibrium Mapping in Vector Form

- Suppose that $\mathbf{x}_t$ is discrete: $\mathbf{x}_t \in \{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^{|X|}\}$.
- The primitives of the model are:

1. **Vectors of payoffs**: $\mathbf{\Pi}_i(a_i, a_{-i})$ with dimension $|X| \times 1$, for every value of (a_i, a_{-i}).
2. **Matrices of transition probabilities**: $\mathbf{F}_x(a_i, a_{-i})$ with dimension $|X| \times |X|$, for every value of (a_i, a_{-i}).
3. **Discount factor**: δ.
4. **Distribution of private information shocks**: $G(\varepsilon_i(a_i) : a_i \in \mathcal{A})$.

Equilibrium Mapping in Vector Form [2]

- Let $\mathbf{P}_i(a_i)$ be a vector of CCPs with dimension $|X| \times 1$ and with the probs. that firm i chooses a_i for every state $\mathbf{x}$.

- Let $\mathbf{P}_i \equiv \{\mathbf{P}_i(a_i) : \text{for every } a_i \in A\}$.

- We can define the $|X| \times 1$ **vectors of expected payoffs**:

$$\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a_i) \equiv \sum_{a_{-i}} \left[\prod_{j \neq i} \mathbf{P}_j(a_j) \right] \ * \ \mathbf{\Pi}_i(a_i, a_{-i})$$

- And the $|X| \times |X|$ **matrices of expected transition probabilities**:

$$\mathbf{F}_i^{\mathbf{P}_{-i}}(a_i) \equiv \sum_{a_{-i}} \left[\prod_{j \neq i} \mathbf{P}_j(a_j) \right] \ * \ \mathbf{F}_x(a_i, a_{-i})$$

- where $*$ represents the "element-by-element" or Hadamard product.

Equilibrium Mapping in Vector Form [3]

- A MPE is a vector $\mathbf{P} \equiv \{\mathbf{P}_i : i \in I\}$ such that:

$$\mathbf{P}_i = \Psi_i(\mathbf{P}_{-i}) \ \textit{for every } i \in I$$

where $\Psi_i(.)$ is i's **best response mapping** that is the composition of:

$$\Psi_i = \Lambda_i \circ \Gamma_i$$

- $\mathbf{V}_i^{\mathbf{P}_{-i}} = \Gamma_i(\mathbf{P}_{-i})$ gives the vector of values that solves Bellman's equation for firm i given $\mathbf{P}_{-i}$: (for Logit case):

$$\mathbf{V}_i^{\mathbf{P}_{-i}} = \ln\left(\sum_{a_i} \exp\left\{\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a_i) + \delta\ \mathbf{F}_i^{\mathbf{P}_{-i}}(a_i)\ \mathbf{V}_i^{\mathbf{P}_{-i}}\right\}\right)$$

- $\mathbf{P}_i = \Lambda_i(\mathbf{V}_i^{\mathbf{P}_{-i}})$ gives optimal CCPs given $\mathbf{V}_i^{\mathbf{P}_{-i}}$: (for Logit case):

$$\mathbf{P}_i(a_i) = \Lambda_i(a_i, \mathbf{V}_i^{\mathbf{P}_{-i}}) = \frac{\exp\left\{\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a_i) + \delta\ \mathbf{F}_i^{\mathbf{P}_{-i}}(a_i)\ \mathbf{V}_i^{\mathbf{P}_{-i}}\right\}}{\sum_{a'} \exp\left\{\mathbf{\Pi}_i^{\mathbf{P}_{-i}}(a') + \delta\ \mathbf{F}_i^{\mathbf{P}_{-i}}(a')\ \mathbf{V}_i^{\mathbf{P}_{-i}}\right\}}$$

Methods / Algorithms to Compute a MPE

- We study three algorithms that have been used to compute MPE in this class of models.

1. Fixed point iterations in the best response mapping Ψ.
2. Newton's method.
3. Spectral residual method(s)

- Method [1] does not guarantee convergence. [2] does, but it is impractical in most applications. [3] has advantages relative to [1] and [2].

Fixed Point Iterations

- Let $\mathbf{P}^0 \equiv \{\mathbf{P}_i^0 : \text{for any } i\}$ be arbitrary vector of CCPs.

- At iteration n, for any player i:

$$\mathbf{P}_i^n = \Psi_i\left(\mathbf{P}_{-i}^{n-1}\right)$$

- We check for convergence:

$$\begin{cases} \text{if } \left\|\mathbf{P}^n - \mathbf{P}^{n-1}\right\| \leq \kappa & \text{then} \quad \mathbf{P}^n \text{ is a MPE} \\ \text{if } \left\|\mathbf{P}^n - \mathbf{P}^{n-1}\right\| > \kappa & \text{then} \quad \text{Proceed to iteration } n+1 \end{cases}$$

where κ is a small positive constant, e.g., $\kappa = 10^{-6}$.

- **Convergence is NOT guaranteed**. This is a serious limitation.

Newton's Method

- Define the function $f(\mathbf{P}) \equiv \mathbf{P} - \Psi(\mathbf{P})$.
- Finding a fixed point of Ψ is equivalent to finding a zero (root) of f.
- We can use Newton's method to find a root of f.
- At iteration n: $\quad$ ($\nabla f(\mathbf{P})$ is the Jacobian matrix)

$$\mathbf{P}^{n} = \mathbf{P}^{n-1} + \left[\nabla f(\mathbf{P}^{n-1})\right]^{-1} f(\mathbf{P}^{n-1})$$

- We check for convergence: $\left\|\mathbf{P}^{n} - \mathbf{P}^{n-1}\right\| \leq \kappa$
- **Convergence is guaranteed** (to one of the multiple equilibria).

Newton's Method [2]

- The main computational cost of a Newton's iteration comes from the computation of Jacobian matrix $\nabla f(\mathbf{P})$.

- There is not a closed-form expression for the derivatives in this matrix. And in this class of models, this matrix is not sparse.

- This matrix is of dimension $N|\mathcal{A}||\mathcal{X}| \times N|\mathcal{A}||\mathcal{X}|$, and the computation of one single element in this matrix involves solving many single-agent dynamic programming problems, each of them with a complexity $O(|\mathcal{X}|^3)$.

- In summary, Newton's method is not practical in most empirical applications, in which $|\mathcal{X}|$ is greater than 10^5.

Spectral Residual Method

- It is a general method for solving high-dimension systems of nonlinear equations, $f(\mathbf{P}) = 0$.
- It has two very attractive features:

1. It is derivative free, and the cost of one iteration is equivalent to evaluation $f(\mathbf{P})$ – the same cost as one fixed point iteration.
2. It converges to a solution under mild regularity conditions – similar good convergence properties to Newton's.

Spectral Residual Method [2]

- Spectral methods propose the following updating rule/iteration:

$$\mathbf{P}_{n+1} = \mathbf{P}_n - \alpha_n \, f\left(\mathbf{P}_n\right)$$

where α_n is the spectral steplength, which is a scalar.

- Different updating rules have been proposed in the literature. Barzilai and Borwein (1988) is commonly used:

$$\alpha_n = \frac{[\mathbf{P}_n - \mathbf{P}_{n-1}]'[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]}{[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]'[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]}$$

- The intuition for the convergence of the Spectral Residual method is that the updating of α_n can guarantee the right direction to convergence.

6. Some Extensions of the Basic Framework

Some Extensions of the Basic Framework

1. **Continuous time**
2. **Oblivious equilibrium**
3. **Dealing with large state spaces**
4. **Persistent asymmetric information**
5. **Firms' biased beliefs**