ECO 2901
EMPIRICAL INDUSTRIAL ORGANIZATION

Lecture 4:
Static games of incomplete information with non-equilibrium beliefs: Model and Identification

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Lecture 4: Games of incomplete information with non-equilibrium beliefs

[1] **Introduction**

[2] **Quick review of recent empirical evidence on firms' biased beliefs**

[3] **Model**

[4] **Identification & Estimation**

1. Introduction

Firms' Information & Beliefs

- In oligopoly markets, a firm's behavior depends on its **beliefs about the behavior of other firms** in the market.
- Firms form their beliefs under **uncertainty and asymmetric information**.
- Firms are different in their **ability for collecting and processing information**, for similar reasons as they are heterogeneous in their costs of producing goods and services.
- We expect firms to be **heterogeneous in their beliefs**.
- This heterogeneity has implications on their performance and on market outcomes.

Possible sources of firms' biased beliefs

- In reality, firms can face substantial uncertainty about other competitors' strategies.

- There are different sources of bias in players beliefs:

(a) **Limited information / attention**: *Some players do not have information about variables that are know to other players.*

(b) **Bounded rationality**: *Limited capacity to process information / compute;*

(c) **Strategic uncertainty:** *With multiple equilibria, players can have different beliefs about the selected equilibrium. Some players believe that they are playing equilibrium A, other players believe they are playing equilibrium B, ...*

Relaxing Firms' Rational Beliefs

- Despite these arguments, in most fields in economics (and IO in particular), the status quo is **assuming rational expectations**.
- There are reasons to impose assumption of equilibrium beliefs:

  (a) This assumption has identification power.
  (b) Counterfactual analysis: model predicts how beliefs change endogenously.

- But it can be unrealistic in some applications, and can imply serious biases in our views on firms' competition.
- In these lectures, we will review some recent structural empirical papers of oligopoly competition that **relax the assumption of firms' rational beliefs**.

2. Quick Review of Recent Empirical Evidence on Firms' Biased Beliefs

US Telecommunication industry after deregulation

- **Goldfarb and Xiao (AER, 2011)** study entry decisions into local US telecommunication markets following the deregulation Telecommunications Act of 1996, which allowed free competition.

- Holding other market characteristics constant, more experienced and better educated managers have a lower propensity to enter (and a lower propensity to exit after entry) into very competitive markets.

- This suggests that **better-educated managers are better at predicting competitors' behavior**.

- This hypothesis is confirmed from the estimation of a structural game of market entry with Cognitive Hierarchy beliefs.

Learning to bid after market deregulation

- **Doraszelski, Lewis, and Pakes (AER, 2018)** investigate firms' learning about competitors' bidding behavior just after the deregulation of the UK electricity market.
- In the first year after deregulation, **firms' bidding behavior was very heterogeneous and firms made frequent and sizable adjustments** in their bids.
- During the second year, there is a dramatic reduction in the range of bids. After three years, firms' bids become very stable.
- During these three phases, demand and costs were quite stable.
- The authors argue that the changes in firms' bidding strategies can be attributed to strategic uncertainty and learning..

Learning to price after market deregulation

- **Huang, Ellickson, and Lovett (2018)** study firms' price setting behavior in the Washington State liquor market following the privatization of the market in 2012.
- After liberalization, grocery chains newly entered the market. How did these new entrants learn about demand and learn to price optimally over time?
- The authors document **large and heterogeneous price movements in the first two years after the privatization**.
- The authors present evidence consistent with firms' learning about the idiosyncratic and common components of the demand shocks, and about the time persistence of these shocks.

Entry in the early years of UK fast-food restaurant industry

- **Aguirregabiria and Magesan (REStud, 2020)** study competition in store location between McDonalds (MD) and Burger King (BK) during the early years of the fast-food restaurant industry in the UK.
- Reduced form evidence shows that the number of own stores has a strong negative effect on the probability that BK opens a new store but **the effect of the competitor's number of stores is economically negligible**.
- This behavior cannot be rationalized by an equilibrium model of market entry where firms have equilibrium beliefs about the behavior of competitors.

Bidding behavior in the Texas electricity spot market

- **Hortacsu and Puller (RAND, 2008)** analyze firms' bidding behavior in the Texas electricity spot market.
- Their dataset contains detailed information not only on firms' bids but also on their marginal costs. Using these data, the authors construct the equilibrium bids of the game and compare them to the actual observed bids.
- They find statistically and **economically very significant deviations between equilibrium and actual bids**.
  Small firms don't supply much power even when is profitable to do so.
- This finding is consistent with low strategic ability in the bidding departments of small firms.
- This suboptimal behavior leads to significant efficiency losses.

Key Question

- A key issue in all these applications is how to find convincing evidence that (some) firms have non-equilibrium or biased beliefs, and this is not just **an artifact from the specification (or misspecification) of the model**.
- How can we be (more or less) confident that what we call bias beliefs cannot be explained by observable or unobservable variables affecting firms' demand or costs?
- To answer these questions, we need to study formally the identification of beliefs and structural parameters in profits in our model.
- This is the focus of most of the remaining part of this lecture.

1. MODEL

Static game: Profit function

- N firms competing in a market. The profit function of firm i:

$$\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$$

a_i is the action of firm i,
either continuous or discrete with support $\{0, 1, ..., J\}$
$\mathbf{a}_{-i}$ is the vector with the actions of the other firms
$\mathbf{x}$ represents variables that are common knowledge
ε_i is private information of firm i

- Firms' types $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)$ are drawn from a distribution F.

- Firms choose simultaneously their actions a_i to maximize their respective expected profits.

Static game: Beliefs

- A firm does not know the private information of its competitors and therefore it does not know their actions.
- Firms form probabilistic beliefs about the actions of competitors.
- Let $b_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x})$ be a probability density function that represents the belief of firm i.

Static game: Best response

- Given its beliefs, a **firm's expected profit** is:

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) \quad = \quad \int_{\mathbf{a}_{-i}} \Pi(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \; b_i(\mathbf{a}_{-i} | \varepsilon_i, \mathbf{x}) \; d\mathbf{a}_{-i}$$

- A firm chooses its **strategy function** $\sigma_i(\varepsilon_i, \mathbf{x}; b_i)$, to maximize expected profits:

$$\sigma_i(\varepsilon_i, \mathbf{x}; b_i) = \arg\max_{a_i \in \mathcal{A}} \; \Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i)$$

Characterization of best response strategies

- Let $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$ be the **marginal profit function:**

$$\begin{cases} \text{continuous action:} & \Delta\Pi_i = \partial\Pi_i / \partial a_i \\ \text{discrete action:} & \Delta\Pi_i = \Pi_i(a_i) - \Pi_i(a_i - 1) \end{cases}$$

- **ASSUMPTION 1:** $\Delta\Pi_i$ is strictly monotonic in a_i and additively separable in ε_i:

$$\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) = \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) - \varepsilon_i$$

- **ASSUMPTION 2:** (A) Independent private values ε_i. (B) The marginal CDF F_i is strictly increasing in $\mathbb{R}$. (C) The beliefs function $b_i(a_{-i}|\varepsilon_i, x)$ does not depend on the firm's own type, ε_i.

Characterization of best response strategies [2]

- Let $\Delta\pi_i^e(a_i, \mathbf{x}; b_i)$ be the expected marginal profit up to ε_i.

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) \equiv \int_{\mathbf{a}_{-i}} \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i}$$

- **[A]** A necessary & sufficient condition for best response a_i is:

$$\begin{cases} \text{continuous action:} & \Delta\pi_i^e(a_i, \mathbf{x}; b_i) - \varepsilon_i = 0 \\ \text{discrete action:} & \Delta\pi_i^e(a_i + 1, \mathbf{x}; b_i) < \varepsilon_i \leq \Delta\pi_i^e(a_i, \mathbf{x}; b_i) \end{cases}$$

Characterization of best response strategies [3]

- **[B]** For any value a^0, the **cumulative choice probability function (CCP)** $P_i(a^0|\mathbf{x}) \equiv \Pr(a_i \leq a^0|\mathbf{x})$ satisfies the condition:

$$P_i(a^0|\mathbf{x}) = F_i\left[\Delta\pi_i^e(a^0, \mathbf{x}; b_i) \mid \mathbf{x}\right] \quad \text{for } a^0 > 0$$

- **[C]** The **quantile function** $Q_i(a^0|x) \equiv F_i^{-1}\left[P_i(a^0|\mathbf{x})\right]$:

$$Q_i(a^0|\mathbf{x}) = \Delta\pi_i^e(a^0, \mathbf{x}; b_i) \quad \text{for } a^0 > 0$$

Example 1: Cournot with independent private costs

- $a_i \in \mathbb{R}_+$ is firm i's amount of output.
- Inverse demand function is $p = p(Q, \mathbf{x})$ where $Q = \sum_{i=1}^{N} a_i$.
- A firm's marginal cost function is $c_i(a_i, \mathbf{x}) + \varepsilon_i$.
- Then, the quantile condition (= expected marginal profit up to ε_i):

$$
\begin{aligned}
Q_i(a^0|\mathbf{x}) &= -c_i(a^0, \mathbf{x}) \\
&\quad + \int_{\mathbf{a}_{-i}} p\left(a^0 + \sum_{j \neq i} a_j, \mathbf{x}\right) + p'\left(a^0 + \sum_{j \neq i} a_j, \mathbf{x}\right) a^0 \; b_i(\mathbf{a}_{-i}|\mathbf{x}
\end{aligned}
$$

Example 2: Market entry with IP entry costs

- $a_i \in \{0, 1\}$ indicator of "firm i is active in the market".
- A firm's profit **if not active** is zero, $\Pi_i(0) = 0$.
- A firm's profit **if active** is: $\Pi_i(1) = v_i(\mathbf{a}_{-i}, \mathbf{x}) - ec_i(\mathbf{x}) - \varepsilon_i$.
- Then, the quantile condition is:

$$Q_i(1|\mathbf{x}) = -ec_i(\mathbf{x}) + \sum_{\mathbf{a}_{-i}} v_i(\mathbf{a}_{-i}, \mathbf{x}) \; b_i(\mathbf{a}_{-i}|\mathbf{x})$$

Example 3: Procurement auction with IP costs

- $a_i \in \mathbb{R}$ represents firm i's bid.
- Profit function: $\Pi_i = (a_i - c_i(\mathbf{x}) - \varepsilon_i)\ 1\{a_j > a_i\ \forall j \neq i\}$
- The expected profit function is:

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) \;=\; (a_i - c_i(\mathbf{x}) - \varepsilon_i)\;\; W(a_i, \mathbf{x}, b_i)$$

where $W(a_i, \mathbf{x}, b_i) \equiv \int_{\mathbf{a}_{-i}} 1\{a_j > a_i\ \forall j \neq i\}\ b_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i}$ is firm i's subjective probability of wining the auction.

Example 3: Procurement auction with IP costs [2]

- The expected marginal profit function is:

$$\Delta\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) \quad = \quad W(a_i, \mathbf{x}, b_i) + (a_i - c_i(\mathbf{x}) - \varepsilon_i) \ \Delta W(a_i, \mathbf{x}, b_i),$$

where $\Delta W(a_i, \mathbf{x}, b_i) = \partial W(a_i, \mathbf{x}, b_i)/\partial a_i$.

- We have that $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = a^0$ if and only if

$$W(a^0, \mathbf{x}, b_i) + (a^0 - c_i(\mathbf{x}) - \varepsilon_i)\Delta W(a^0, \mathbf{x}, b_i) = 0$$

- Then, $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) \leq a^0$ iff $\varepsilon_i \leq a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$ such that:

$$Q_i(a^0|\mathbf{x}) = a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$$

General model of firms' beliefs

$$P_i\left(a^0|\mathbf{x}\right) = F_i\left[\int \Delta\pi_i(a^0, \mathbf{a}_{-i}, \mathbf{x})\ b_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i}\right]$$

- These best response conditions contain all the restrictions of the model on beliefs function b_i and profit function $\Delta\pi_i$.
- Many models of competition in IO – under different types of equilibrium concepts are particular versions of this model.
- **Auctions, Bertrand competition, Cournot competition, Entry models** under different types of restrictions on firms' beliefs:
    - o **Bayesian Nash equilibrium**.
    - o **Level-K and Cognitive Hierarchy Beliefs**.
    - o **Rationalizability.**

Bayesian Nash Equilibrium

- Under Bayesian Nash Equilibrium (with independent private values):

$$b_i(\mathbf{a}_{-i} \mid \mathbf{x}) = \Pr(\mathbf{a}_{-i} \mid \mathbf{x})$$

- This is the most commonly used solution concept in games of incomplete information in IO.
- It has received particular attention in **auction games** and in discrete choice models of **market entry**, but it has been also applied to games of quantity or price competition.

Cognitive Hierarchy and Level-K models

- Equilibrium concepts where firms have biased beliefs, that is, $b_i(\mathbf{a}_{-i} \mid \mathbf{x}) \neq \Pr(\mathbf{a}_{-i} \mid \mathbf{x})$.

- There is a finite number K of belief types that correspond to different levels of strategic sophistication.

- Believes for Level-0 can be arbitrary, $b^{(0)}$.

- Level-1 firms believe that all the other firms are level 0:

$$b^{(1)}\left(\mathbf{a}_{-i} \mid \mathbf{x}\right) = \prod_{j \neq i} F\left[\int \Delta\pi(a_j, \mathbf{a}_{-j}, \mathbf{x})\ b^{(0)}(\mathbf{a}_{-j}|\mathbf{x})\ d\mathbf{a}_{-j}\right]$$

Cognitive Hierarchy and Level-K models [2]

- In **Level-k model**, a level-k firm believes that all the other firms are $k-1$.

$$b^{(k)}\left(\mathbf{a}_{-i} \mid \mathbf{x}\right) = \prod_{j \neq i} F\left[\int \Delta\pi(a_j, \mathbf{a}_{-j}, \mathbf{x})\ b^{(k-1)}(\mathbf{a}_{-j}|\mathbf{x})\ d\mathbf{a}_{-j}\right]$$

- In **Cognitive Hierarchy model**, a level-k firm believes that all other firms come from a probability distribution over levels 0 to $k-1$.

- These models impose restrictions on beliefs.

o There is a finite number K of belief types (typically 2 or 3).
o These belief functions satisfy a hierarchical equilibrium.

Rationalizability

- The concept of *Rationalizability* (Bernheim, 1984; Pearce, 1984) imposes two restrictions on firms' beliefs and behavior.
  [A.1] Every firm is rational in the sense that it maximizes its own expected profit given beliefs.
  [A.2] This rationality is common knowledge, i.e., every firms knows that all the firms know that it knows ... that all the firms are rational.

- Aradillas-Lopez & Tamer (2008) study identification under Rationalizability.

- In a game **with multiple equilibria**, the solution concept of **Rationalizability allows for biased beliefs**.

- Each firm has beliefs that are consistent with a BNE, but these beliefs may not correspond to the same BNE.

2. IDENTIFICATION

Data

- The researcher has a sample of M local markets, indexed by m, where she observes firms' actions and state variables (**firms' choice data**):

$$\{a_{imt},\ \mathbf{x}_{mt} : i = 1, 2, ..., N;\ t = 1, 2, ..., T^{data}\}$$

- In addition to these data, the researcher may have data on some components of the profit function.

- I distinguish three cases, from the best to the worst case scenario:

  (a) Only Choice Data
  (b) Choice data + Revenue function
  (c) Choice data + Revenue function + Cost function

Revenue and Costs

- It is convenient now to distinguish between revenue and costs in the profit function:

$$\pi_i = r_i - c_i$$

- Such that:

$$\Delta\pi_i = \Delta r_i - \Delta c_i$$

- Both Δr_i and Δc_i may depend on the actions of other firms, $\mathbf{a}_{-i}$. It depends on the model, on the type of decision variable.

* In an entry model or in a Cournot model, Δc_i typically does not depend on $\mathbf{a}_{-i}$.

*In a model of price competition with differentiated product, Δc_i typically depends on $\mathbf{a}_{-i}$: Δc_i depends on the quantity produced & sold by i, this quantity depends (through demand) on the own price and the price of competitors.

Binary choice – Two-player game

- Game of price competition where firms choose between a low price ($a_i = 0$) and a high price ($a_i = 1$).

- Notation:

$$\begin{array}{l} \Delta r_i(a_{-i}, \mathbf{x}) \equiv r_i(1, a_{-i}, \mathbf{x}) - r_i(0, a_{-i}, \mathbf{x}) \\ \Delta c_i(a_{-i}, \mathbf{x}) \equiv c_i(1, a_{-i}, \mathbf{x}) - c_i(0, a_{-i}, \mathbf{x}) \\ p_i(\mathbf{x}) \equiv p_i(1|\mathbf{x}) = \text{prob. choosing high price} \\ b_i(\mathbf{x}) \equiv b_i(1|\mathbf{x}) = \text{belief prob. competitor chooses high price.} \end{array}$$

- Expected marginal profit (up to ε_i):

$$\begin{array}{rcl} \Delta\pi_i^e(\mathbf{x}) & = & [1 - b_i(\mathbf{x})] \; [\Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x})] \\ & & \\ & + & b_i(\mathbf{x}) \; [\Delta r_i(1, \mathbf{x}) - \Delta c_i(1, \mathbf{x})] \end{array}$$

Binary choice – Two-player game [2]

- Best response probability:

$$p_i(\mathbf{x}) = F_i \left(\begin{array}{c} \Delta r_i(0,\mathbf{x}) - \Delta c_i(0,\mathbf{x}) \\ +b_i(\mathbf{x}) \; [\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x}) - \Delta c_i(1,\mathbf{x}) + \Delta c_i(0,\mathbf{x})] \end{array} \right)$$

- Using quantile $Q_i(\mathbf{x}) \equiv F_i^{-1}(p_i(\mathbf{x}))$:

$$\begin{aligned} Q_i(\mathbf{x}) &= \Delta r_i(0,\mathbf{x}) - \Delta c_i(0,\mathbf{x}) \\ &+ b_i(\mathbf{x}) \; [\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x}) - \Delta c_i(1,\mathbf{x}) + \Delta c_i(0,\mathbf{x})] \end{aligned}$$

- For the moment, I assume that F_i is known to the researcher.

Identification with revenue and cost data

- In the static case with two-players, beliefs are identified:

$$b_i(\mathbf{x}) = \frac{Q_i(\mathbf{x}) - \Delta r_i(0, \mathbf{x}) + \Delta c_i(0, \mathbf{x})}{\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x}) - \Delta c_i(1, \mathbf{x}) + \Delta c_i(0, \mathbf{x})}$$

- This belief function can be compared to the actual choice probability of the competitor to test unbiased / rational beliefs:

$$b_i(\mathbf{x}) - p_{-i}(\mathbf{x}) = 0 \quad ?$$

- We can also test other restrictions on beliefs such as level-K or Cognitive Hierarchy models.

- If panel data, we can study how beliefs evolve over time (learning).

Identification with revenue but not cost data

- MR functions $r_i(0, \mathbf{x})$ and $r_i(1, \mathbf{x})$ are known to the researcher but the MC is not known.
- Without further restrictions, the system of equations

$$
\begin{aligned}
Q_i(\mathbf{x}) &= \Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x}) \\
&+ b_i(\mathbf{x}) \; [\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x}) - \Delta c_i(1, \mathbf{x}) + \Delta c_i(0, \mathbf{x})]
\end{aligned}
$$

cannot identify the unknown functions $b_i(\mathbf{x})$ and Δc_i.

- Without further restrictions, any belief function (including the BNE belief) is consistent with observed behavior, $Q_i(\mathbf{x})$, given the appropriate Δc_i function.

Identification: Firm-specific cost shifter

- **Exclusion Restriction (Firm specific cost shifter)**:
  The vector $\mathbf{x}$ has a firm-specific components that affect the marginal profit of a firm but not the marginal profit of other firms.

- That is, $\mathbf{x} = (\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i})$ such that:

$$\Delta\pi_i(a_i, a_{-i}, \mathbf{x}) = \Delta\pi_i(a_i, a_{-i}, \widetilde{\mathbf{x}}, \mathbf{z}_i)$$

Identification of beliefs

- Let $\mathbf{z}_{-i}^1$, $\mathbf{z}_{-i}^2$, and $\mathbf{z}_{-i}^3$ be three values for $\mathbf{z}_{-i}$.

$$\begin{cases} Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^2) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1) \\ = \left[b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^2) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1) \right] \; \left[\Delta\pi_i(1, \widetilde{\mathbf{x}}, \mathbf{z}_i) - \Delta\pi_i(0, \widetilde{\mathbf{x}}, \mathbf{z}_i) \right] \\ \\ Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^3) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1) \\ = \left[b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^3) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1) \right] \; \left[\Delta\pi_i(1, \widetilde{\mathbf{x}}, \mathbf{z}_i) - \Delta\pi_i(0, \widetilde{\mathbf{x}}, \mathbf{z}_i) \right] \end{cases}$$

- And taking the ratio between these two differences, we have that:

$$\frac{b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^2) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1)}{b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^3) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1)} = \frac{Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^2) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1)}{Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^3) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i}^1)}$$

Testing different models of beliefs

- Given this identified beliefs object, we can test different models or restrictions on beliefs such that:
    - Unbiased beliefs of firm i
    - Bayesian Nash equilibrium
    - Rationalizability
    - Cognitive Hierarchy model; Level-k
- If we have panel data over several periods of time, we can also test different models of learning:
    - Bayesian learning
    - Fictitious play and other types of adaptive learning.

Extensions

- The paper extends this identification result in different ways:
  - More than two players.
  - Continuous choice games.
  - Ordered multinomial choice games
  - Dynamic games (discrete and continuous choice)
  - **Nonparametric distribution of private information** F_i

Identification with nonparametric distribution private info

- When the decision variable is continuous, there is identification of beliefs even if F_i is nonparametrically specified.

- Suppose that:
  (i) the distribution F_i is independent of z_i and z_{-i} but it may depend on $\widetilde{\mathbf{x}}$;
  (ii) z_i and z_{-i} are continuous random variables; (
  (iii) $p_i(\widetilde{\mathbf{x}}, z_i, z_{-i})$ is strictly monotonic in z_i and z_{-i} and asymmetric in these two arguments;
  (iv) the researcher knows the revenue function;
  (v) firm i's marginal cost does not depend on a_{-i}.

Identification with nonparametric distribution private info

- Let (z_i^A, z_{-i}^A) and (z_i^B, z_{-i}^B) be two arbitrary values of (z_i, z_{-i}). Under (i) to (v), the following results hold.

- (A) There exist values z_{-i}^{AB*} and z_{-i}^{BA*} which are uniquely identified and satisfy the following three properties:
  (1) $z_{-i}^{AB*} \neq z_{-i}^{BA*}$;
  (2) $p_i(z_i^A, z_{-i}^A) = p_i(z_i^B, z_{-i}^{AB*})$ & $p_i(z_i^B, z_{-i}^B) = p_i(z_i^A, z_{-i}^{BA*})$.

- Using (A), we can show that the following condition holds:

$$\frac{b_i(z_i^A, z_{-i}^A) - b_i(z_i^A, z_{-i}^{BA*})}{b_i(z_i^B, z_{-i}^B) - b_i(z_i^B, z_{-i}^{AB*})} = -\frac{\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A)}{\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B)}$$

  such that an object that depends only on beliefs is identified using the firm's observed behavior and revenue function.