# The Identification Power of Equilibrium in Simple Games

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# The Identification Power of Equilibrium in Simple Games

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We examine the identification power that (Nash) equilibrium assumptions play in conducting inference about parameters in some simple games. We focus on three static games in which we drop the Nash equilibrium assumption and instead use rationalizability as the basis for strategic play. The first example examines a bivariate discrete game with complete information of the kind studied in entry models. The second example considers the incomplete-information version of the discrete bivariate game. Finally, the third example considers a first-price auction with independent private values. In each example, we study the inferential question of what can be learned about the parameter of interest using a random sample of observations, under level-$k$ rationality, where $k$ is an integer $\geq 1$. As $k$ increases, our identified set shrinks, limiting to the identified set under full rationality or rationalizability (as $k \to \infty$). This is related to the concepts of iterated dominance and higher-order beliefs, which are incorporated into the econometric analysis in our framework. We are then able to categorize what can be learned about the parameters in a model under various maintained levels of rationality, highlighting the roles of different assumptions. We provide constructive identification results that lead naturally to consistent estimators.

KEY WORDS:   Equilibrium vs rationality; Identification; Partial identification.

## 1.   INTRODUCTION

In this article we examine the identification power of equilibrium in some simple games. In particular, we relax the assumption of Nash equilibrium (NE) behavior and assume that players are rational. Rationality posits that agents play strategies that are consistent with a set of proper beliefs. The object of interest in these games is a parameter vector that parameterizes payoff functions. We study the identified features of the model using a random sample of data under a set of rationality assumptions, culminating with rationalizability, a concept introduced jointly in the literature by Bernheim (1984) and Pearce (1984), and compare those to what we can learn under Nash. We find that in static discrete games with *complete information*, the identified features of the games with more than one level of rationality are similar to those obtained with Nash behavior assumption but allowing for multiple equilibria (including equilibria in mixed strategies). In a bivariate game with *incomplete information*, if the game has a unique (Bayesian) NE, then there is convergence between the identified features with and without equilibrium only when the level of rationality tends to infinity. When there are multiple equilibria, the identified features of the game under rationality and equilibrium are different: smaller identified sets (hence more information about the parameter of interest) when equilibrium is imposed, but computationally easier to construct identification regions under rationality (i.e., no need to solve for fixed points). In the auction game that we study, the situation is different. We follow the work of Battigalli and Sinischalchi (2003) where, under some assumptions given the valuations, rationalizability predicts only upper bounds on the bids. We show how these bounds can be used to learn about learn about the latent distribution of valuation. Another strategic assumptions in auctions resulting in tighter bounds is the concept of $\mathcal{P}$-dominance studied by Dekel and Wolinsky (2003).

Economists have observed that equilibrium play in noncooperative strategic environment is not necessary for rational behavior. Some can easily construct games in which NE strategy profiles are unreasonable, whereas others can find reasonable strategy profiles that are not Nash. Restrictions once Nash behavior is dropped typically are based on a set of "rationality" criteria, as has been enumerated in numerous works under different strategic scenarios. In this article we study the effect of adopting a particular rationality criterion on learning about parameters of interests. We do not advocate one type of strategic assumption over another, but simply explore one alternative to Nash and evaluate its effect on parameter inference. Thus, depending on the application, identification of parameters of interest certainly can be studied under strategic assumptions other than rationalizability. We provide such an example.

Because every Nash profile is rational under our definition, dropping equilibrium play complicates the identification problem, because under rationality only, the set of predictions is enlarged. As Pearce noted, "this indeterminacy is an accurate reflection of the difficult situation faced by players in a game, because logical guidelines and the rules of the game are not sufficient for uniqueness of predicted behavior." Thus it is interesting from the econometric perspective to examine how the identified features of a particular game changes as weaker assumptions on behavior are made.

We maintain that players in the game are rational, where heuristically, we define rationality as behavior consistent with an optimizing agent equipped with a proper set of beliefs or probability distributions about the unknown actions of others. Rationality comes in different levels or orders, where a profile is first-order rational if it is a best response to some profile for the other players. This intersection of layers of rationality constitutes rationalizable strategies. We study the identification question for level-$k$ rationality for $k \geq 1$. When we study the identifying power of a game under a certain set of assumptions on the

strategic environment, we implicitly assume that all players in that game are abiding exactly by these assumptions and playing exactly that game. This is important, because theoretical work has challenged the multiplicity issues that arise under rationalizability. For example, Weinstein and Yildiz (2007) showed that for any rationalizable set of strategies in a given game, there is a local disturbance of that game in which these are the unique rationalizable strategies. This ambiguity about what is the exact game being played is why it is important to study the identified features of a model in the presence of multiplicity.

Using equilibrium as a restriction to gain identifying power is a well-known strategy in economics. The model of demand and supply uses equilibrium to equate the quantity demanded with quantity supplied, thus obtaining the classic simultaneous equation model. Other literature in econometrics, such as job search models and hedonic equilibrium models, explicitly use equilibrium as a "moment condition." In this article we study the identification question in simple game-theoretic models without the assumption of equilibrium by focusing on the weaker concept of rationality, $k$-level rationality and its limit rationalizability. This approach has two important advantages. First, it leads naturally to a well-defined concept of *levels of rationality*, which is attractive practically. Second, it can be adapted to a very wide class of models without the need to introduce ad hoc assumptions. Ultimately, interim rationalizability allows us to do inference (to varying degrees) both on the structural parameters of a model (e.g., the payoff parameters in a reduced-form game or the distribution of valuations in an auction) and on the properties of *higher-order beliefs* by the agents, which are incorporated into the econometric analysis. The features of this hierarchy of beliefs characterize what we call the *rationality level* of agents. In addition, it is possible to also provide testable restrictions that can be used to find an upper bound on the rationality level in a given data set.

Level-$k$ thinking as an alternative to Nash equilibrium behavior also has been studied by Stahl and Wilson (1995), Nagel (1995), Ho, Camerer, and Weigelt (1998), Costa-Gomes, Crawford, and Broseta (2001), Costa-Gomes and Crawford (2006), and Crawford and Iriberri (2007). These models depart from equilibrium behavior by dropping the assumption that each player has a perfect model of others' decisions and replacing it with the assumption that such subjective models survive $k$ rounds of iterated elimination of dominated decisions. Thus each player's subjective model about others' behavior is consistent with level-$k$ interim rationalizability in the sense of Bernheim (1984). For identification, the aforementioned articles assume the existence of a small number of prespecified types, each of which is associated with a very specific behavior. For example, a particular type of player could perform two mental rounds of deletion of dominated strategies and best response to a uniform distribution over the surviving actions. Using carefully designed experiments, previous researchers sought to explain which type best fits the observed choices. This article differs from the aforementioned works by focusing on bounds for conditional choice probabilities that can be rationalized by beliefs that survive $k$ steps of iterated thinking. We look at the largest possible set of level-$k$ rationalizable beliefs but assume nothing about how players choose their actual (unobserved) beliefs from within this set. In addition, we focus on situations in

which the researcher ignores how "rational" players are and in which other primitives of the game also are the object of interest: payoff parameters in discrete games or the distribution of valuations in an auction. In an experimental data set, the last set of objects are entirely under the control of the researcher, and strong parametric assumptions typically are made about behavioral types.

In Section 2 we review and define rational play in a noncooperative strategic game. Here we mainly adapt the definition provided by Pearce. We then examine the identification power of dropping Nash behavior in some commonly studied games in empirical economics. In Section 3 we consider discrete static games of complete information. This type of game is widely used in the empirical literature on (static) entry games with complete information and under NE (see, e.g., Bjorn and Vuong 1985; Bresnahan and Reiss 1991; Berry 1994; Tamer 2003; Andrews, Berry, and Jia 2003; Ciliberto and Tamer 2003; Bajari, Hong, and Ryan 2005). Here we find that in the $2 \times 2$ game with level-2 rationality, the outcomes of the game coincide with Nash, and thus econometric restrictions are the same. In Section 4 we consider static games with incomplete information. Empirical frameworks for these games have been studied by Aradillas-Lopez (2005), Aguiregabiria and Mira (2007), Seim (2002), Pakes, Porter, Ho, and Ishii (2005), Berry and Tamer (1996), and others. Characterization of rationalizability in the incomplete information game is closely related to the higher-order belief analysis in the global games literature (see Morris and Shin 2003) and to other recently developed concepts, such as those of Dekel, Fudenberg, and Morris (2007) and Dekel, Fudenberg, and Levine (2004). Here we show that level-$k$ rationality implies restrictions on player beliefs in the $2 \times 2$ game that lead to simple restrictions that can be exploited in identification. As $k$ increases, an iterative elimination procedure restricts the size of the allowable beliefs that map into stronger restrictions that can be used for identification. If the game admits a unique equilibrium, then the restrictions of the model converge toward Nash restrictions as the level of rationality $k$ increases. With multiple equilibria, the iterative procedure converges to sets of beliefs that contain both the "large" and "small" equilibria. In particular, studying identification in these settings is simple, because we do not need to solve for fixed points, but simply iterate the beliefs toward the predetermined level of rationality $k$. In Section 5 we examine a first-price independent auction game, where we follow the work of Battigalli and Sinischalchi (2003). Here for any order $k$, we are only able to bound the unobserved valuation from above. Finally, in Section 6 we conclude.

## 2.   NASH EQUILIBRIUM AND RATIONALITY

In noncooperative strategic environments, optimizing agents maximize a utility function that depends on what their opponents do. In simultaneous games, agents attempt to predict what their opponents will play, and then play accordingly. Nash behavior posits that players' expectations of what others are doing are mutually consistent, and so a strategy profile is Nash if no player has an incentive to change strategy given what the other agents are playing. This Nash behavior makes an implicit assumption on players' expectations. But, players "are not compelled by deductive logic" (Bernheim) to play Nash. In this

article we examine the effect of assuming Nash behavior on identification by comparing restrictions under Nash with those obtained under rationality in the sense of Bernheim and Pearce. Here we follow Pearce's framework and first maintain the following assumptions on behavior:

- Players use proper subjective probability distribution, or use the axioms of Savage, when analyzing uncertain events.
- Players are expected utility maximizers.
- Rules and structure of the game are common knowledge.

We next describe heuristically what is meant by rationalizable strategies; precise definitions have been given by Pearce (1984), for example:

- We say that a strategy profile for player $i$ (which can be a mixed strategy) is *dominated* if there exists another strategy for that player that does better no matter what other agents are playing.
- Given a profile of strategies for all players, a strategy for player $i$ is a *best response* if that strategy does better for that player than any other strategy given that profile.

To define rationality, we use the following notation. Let $\mathcal{R}^i(0)$ be the set of all (possibly mixed) strategies that player $i$ can play and $\mathcal{R}^{-i}(0)$ be the set of all strategies for players other than $i$. Then, heuristically, we have the following:

- *Level-1 rational strategies* for player $i$ are strategy profiles $s^i \in \mathcal{R}^i(0)$ such that there exists a strategy profile for other players in $\mathcal{R}^{-i}(0)$ for which $s^i$ is a best response. The set of level-1 strategies for player $i$ is $\mathcal{R}^i(1)$.
- *Level-2 rational strategies* for player $i$ are strategy profiles $s^i \in \mathcal{R}^i(0)$ such that there exists a strategy profile for other players in $\mathcal{R}^{-i}(1)$ for which $s^i$ is a best response.
- *Level-t rational strategies* are defined recursively from level 1.

Note that by construction, $\mathcal{R}^i(t) \subseteq \mathcal{R}^i(t-1)$. Finally, *rationalizable strategies* are ones that lie in the intersection of the $\mathcal{R}$'s as $t$ increases to infinity. In the complete information game of Section 3, we show that there exists a finite $k$ such that for $\mathcal{R}^i(t) = \mathcal{R}^i(k)$ for all $t \geq k$. In the incomplete information models of Sections 4 and 5, we show that we can have $\mathcal{R}^i(t) \subset \mathcal{R}^i(k)$ for all $t > k$. In all of these settings, a strategy is level-$k$ rational for a player if it is a best response to some strategy profile in $\mathcal{R}^i(k-1)$ by his opponents. Iterating this further, we arrive at the set of rationalizable strategies. Pearce provided properties of the rationalizable set; for example, NE profiles are always included in this set, and the set contains at least one profile in pure strategies.

## 3. BIVARIATE DISCRETE GAME WITH COMPLETE INFORMATION

Consider the following bivariate discrete 0/1 game where $t_p$ is the payoff that player $p$ obtains by playing 1 when player $-p$ is playing 0. Parameters $\alpha_1$ and $\alpha_2$ are of interest. The econometrician does not observe $t_1$ or $t_2$ and is interested in learning about the $\alpha$'s and the joint distribution of $(t_1, t_2)$. (See Table 1.) Assume also, as in entry games, that the $\alpha$'s are negative. In this

Table 1. Bivariate discrete game

|  | $a_2 = 0$ | $a_2 = 1$ |
|---|---|---|
| $a_1 = 0$ | $0, 0$ | $0, t_2$ |
| $a_1 = 1$ | $t_1, 0$ | $t_1 + \alpha_1, t_2 + \alpha_2$ |

example and the next, we assume that we have access to a random sample of observations $(y_{1i}, y_{2i})_{i=1}^N$, which represent, for example, market structures in a set of $N$ independent markets. To learn about the parameters, we map the observed distribution of the data (the choice probabilities) to the distribution predicted by the model. Because this is a game of complete information, players observe all of the payoff-relevant information. In particular, in the first round of rationality, player 1 will play 1 if $t_1 + \alpha_1 \geq 0$, because this will be a dominant strategy. In addition, if $t_1$ is negative, then player 1 will play 0. But when $t_1 + \alpha_1 \leq 0 \leq t_1$, both actions 1 and 0 are level-1 rational; action 1 is rational because it can be a best response to player 2 playing 0, whereas action 0 is a best response to player 2 playing 1. The set $\mathcal{R}(1)$ is summarized in Figure 1. Consider, for example the upper right corner. For values of $t_1$ and $t_2$ lying there, playing 0 is not a best response for either player. Thus $(1, 1)$ is the unique level-1 rationalizable strategy (which is also the unique NE). Consider now the middle region on the right side, that is, $(t_1, t_2) \in [-\alpha_1, \infty) \times [0, -\alpha_2]$. In level-1 rationality, 0 is not a best reply for player 1, but player 2 can play either 1 or 0; 1 is a best reply when player 1 plays 0, and 0 is a best reply for player 2 when player 1 plays 1. However, in the next round of rational play, given that player 2 now believes that player 1 will play 1 with probability 1, then player 2's response is to play 0. Thus $\mathcal{R}(1) = \{\{1\}, \{0, 1\}\}$ while the rationalizable set reduces to the outcome $(1, 0)$. Here $\mathcal{R}(k) = \mathcal{R}(2) = \{\{1\}, \{0\}\}$ for all $k \geq 2$. In the middle square, we see that the game provides no observable restrictions; any outcome can be *potentially observable*, because both strategies are rational at any level of rationality. Note also that in this game, the set of rationalizable strategies is the set of profiles that are undominated. This is a property of bivariate binary games.

### 3.1 Implications of Level-*k* Rationality

A random sample of observations allows us to obtain a consistent estimator of the choice probabilities (or the data). The object of interest here is $\theta = (\alpha_1, \alpha_2, F(\cdot, \cdot))$, where $F(\cdot, \cdot)$ is the joint distribution of $(t_1, t_2)$. One interesting approach to conduct inference on the identified set, $\Theta_I$, is to assume that both $t_1$ and $t_2$ are discrete random variables with identical support on $s_1, \ldots, s_K$ such that $P(t_1 = s_i; t_2 = s_j) = p_{ij} \geq 0$ for $i, j \in \{1, \ldots, k\}$ with $\sum_{i,j} p_{ij} = 1$. Thus we make inference on the set of probabilities $(p_{ij}, i, j \leq k)$ and $(\alpha_1, \alpha_2)$. We highlight this for level-2 rationality. In particular, we say that

$$\theta = ((p_{ij}), \alpha_1, \alpha_2) \in \Theta_I$$

if and only if

$$P_{11} = \sum_{i,j} p_{ij} \big( 1[s_i \geq -\alpha_1; s_j \geq -\alpha_2]$$

$$+ l_{ij}^{(1,1)} 1[0 \leq s_i \leq -\alpha_1; 0 \leq s_j \leq -\alpha_2] \big),$$
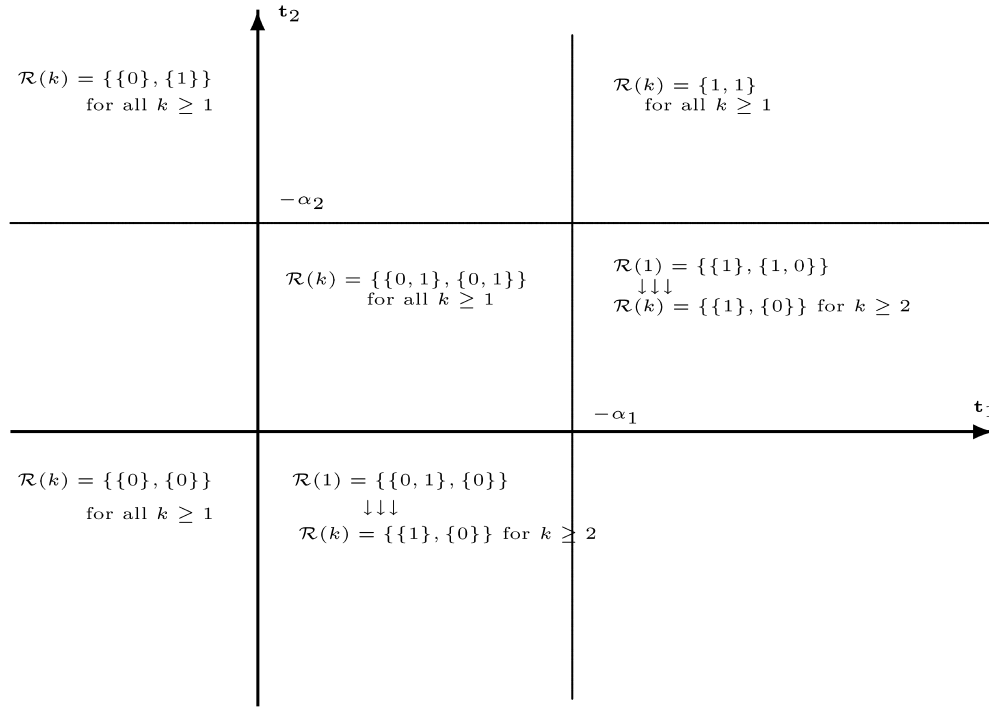
Figure 1. Rationalizable profiles in a bivariate game with complete information.

$$P_{00} = \sum_{i,j} p_{ij}\big(1[s_i \le 0; s_j \le 0]$$
$$+ l_{ij}^{(0,0)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big),$$
$$P_{10} = \sum_{i,j} p_{ij}\big(1[s_i \ge 0; s_j \le 0] + 1[s_i \ge -\alpha_1; 0 \le s_j \le -\alpha_2]$$
$$+ l_{ij}^{(1,0)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big),$$

and

$$P_{01} = \sum_{i,j} p_{ij}\big(1[s_i \le 0; s_j \ge 0] + 1[0 \le s_i \le -\alpha_1; s_j \ge -\alpha_2]$$
$$+ l_{ij}^{(0,1)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big)$$

for some $(l_{ij}^{(1,1)}, l_{ij}^{(0,0)}, l_{ij}^{(0,1)}, l_{ij}^{(1,0)}) \ge 0$ and $l_{ij}^{(1,1)} + l_{ij}^{(0,0)} + l_{ij}^{(0,1)} + l_{ij}^{(1,0)} = 1$ for all $i,j \le k$. The $l$'s can be thought of as the "selection mechanisms" that choose an outcome in the region where the model predicts multiple outcomes. We treat the support points as known, but this is without loss of generality, because those also can be made part of $\theta$. The foregoing equalities (and inequalities) for a given $\theta$ are similar to first-order conditions from a linear programming problem and thus can be solved rapidly using linear programming algorithms. In particular, consider the objective function in (1). Note first that $Q(\theta) \le 0$ for all $\theta$'s in the parameter space. Moreover,

$$\theta \in \Theta_I$$

if and only if $Q(\theta) = 0$.

$$Q(\theta) = \max_{v_i,\ldots,v_8,(l_{ij}^{(1,1)},l_{ij}^{(0,0)},l_{ij}^{(0,1)},l_{ij}^{(1,0)})} -(v_1 + \cdots + v_8) \quad \text{s.t.}$$

$$P_{11} - \sum_{i,j} p_{ij}\big(1[s_i \ge -\alpha_1; s_j \ge -\alpha_2]$$
$$+ l_{ij}^{(1,1)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big)$$
$$= v_1 - v_2,$$
$$P_{00} - \sum_{i,j} p_{ij}\big(1[s_i \le 0; s_j \le 0]$$
$$+ l_{ij}^{(0,0)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big)$$
$$= v_3 - v_4,$$
$$P_{10} - \sum_{i,j} p_{ij}\big(1[s_i \ge 0; s_j \le 0]$$
$$+ 1[s_i \ge -\alpha_1; 0 \le s_j \le -\alpha_2]$$
$$+ l_{ij}^{(1,0)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big)$$
$$= v_5 - v_6,$$
$$P_{01} - \sum_{i,j} p_{ij}\big(1[s_i \le 0; s_j \ge 0] + 1[0 \le s_i \le -\alpha_1; s_j \ge -\alpha_2]$$
$$+ l_{ij}^{(0,1)} 1[0 \le s_i \le -\alpha_1; 0 \le s_j \le -\alpha_2]\big)$$
$$= v_7 - v_8,$$
$$v_i \ge 0; \qquad (l_{ij}^{(1,1)}, l_{ij}^{(0,0)}, l_{ij}^{(0,1)}, l_{ij}^{(1,0)}) \ge 0;$$
$$l_{ij}^{(1,1)} + l_{ij}^{(0,0)} + l_{ij}^{(0,1)} + l_{ij}^{(1,0)} = 1 \quad \text{for all } 1 \le i,j \le k.$$

(1)

First, note that for any $\theta$, the program is feasible; for example, set $(l_{ij}^{(1,1)}, l_{ij}^{(0,0)}, l_{ij}^{(0,1)}, l_{ij}^{(1,0)}) = 0$ and then set $v_1 = P_{11} - \sum_{i,j} p_{ij} 1[s_i \ge -\alpha_1; s_j \ge -\alpha_2]$ and $v_2 = 0$ if $P_{11} - \sum_{i,j} p_{ij} 1[s_i \ge -\alpha_1; s_j \ge -\alpha_2] \ge 0$, or set $v_2 = -(P_{11} - \sum_{i,j} p_{ij} 1[s_i \ge -\alpha_1; s_j \ge -\alpha_2])$ and $v_1 = 0$ and similarly for the rest. Moreover,

$\theta \in \Theta_I$ if and only if $Q(\theta) = 0$. We can collect all of the parameter values for which the foregoing objective function is equal to 0 (or approximately equal to 0). A similar linear programming procedure was used by Honoré and Tamer (2006). The sampling variation comes from having to replace the choice probabilities $(P_{11}, P_{12}, P_{21}, P_{22})$ with their sample analogs, which results in a sample objective function $Q_n(\cdot)$ that can be used to conduct inference.

More generally, and without making support assumptions, a practical way to conduct inference with one level of rationality, say, is to use an implication of the model. In particular, under $k = 1$ rationality, the statistical structure of the model is one of moment inequalities,

$$\Pr(t_1 \geq -\alpha_1; t_2 \geq -\alpha_2) \leq P(1,1) \leq \Pr(t_1 \geq 0; t_2 \geq 0),$$

$$\Pr(t_1 \leq 0; t_2 \leq 0) \leq P(0,0) \leq \Pr(t_1 \leq -\alpha_1; t_2 \leq \alpha_2),$$

$$\Pr(t_1 \geq -\alpha_1; t_2 \leq 0) \leq P(1,0) \leq \Pr(t_1 \geq 0; t_2 \leq -\alpha_2),$$

$$\Pr(t_1 \leq 0; t_2 \geq -\alpha_2) \leq P(0,1) \leq \Pr(t_1 \leq -\alpha_1; t_2 \geq 0).$$

The foregoing inequalities do not exploit all of the information, and thus the identified set based on these inequalities is not sharp. But these inequality-based moment conditions are simple to use and can be generalized to large games. Heuristically, then, by definition the model identifies the set of parameters $\Theta_I$ such that the above inequalities are satisfied. Moreover, we say that the model *point identifies* a unique $\theta$ if the set $\Theta_I$ is a singleton. Figure 2 shows the mapping between the predictions of the game and the observed data under Nash and level-$k$ rationality. The observable implication of Nash is different depending on whether or not we allow for mixed strategies. In particular, without allowing for mixed strategies, in the middle square of Figure 2(a), the only observable implication is $(1,0)$ and $(0,1)$; however, it reverts to all outcomes once the mixed strategy equilibrium is considered. To get an idea of the identification gains when we assume rationality versus equilibrium, we simulated a stylized version of the foregoing game in the case where $t_p$ is standard normal for $p = 1, 2$ and the only object of interest is the vector $(\alpha_1, \alpha_2)$. We compare the identified set of the foregoing game under $k = 1$ rationality and NE when we consider only pure strategies. Figure 3 shows that there is identifying power in assuming Nash equilibrium. In particular, under Nash, the identified set is a somehow tight "circle" around the simulated truth, whereas under rationality, the model provides only upper bounds on the alphas. But if we add exogenous variations in the profits ($X$'s), then the identified region under rationality will shrink. In the next section we examine the identifying power of the same game under incomplete information.

## 4. DISCRETE GAME WITH INCOMPLETE INFORMATION

Consider now the discrete game presented in Table 1 but under the assumptions that player 1 (2) does not observe $t_2$ ($t_1$) or that the signals are private information. We denote player $p \in \{1, 2\}$'s opponent by $-p$. We let $\mathcal{I}_p$ denote the signals used by player $p$ to obtain information about $t_{-p}$, where $t_p \in \mathcal{I}_p$ could be a special case. Player $p$ holds beliefs about his opponent's type conditional on $\mathcal{I}_p$, and those beliefs can be summarized by a subjective distribution function. Let $\pi_2(\mathcal{I}_1)$ denote player 1's *subjective* probability of entry for player 2, and define $\pi_1(\mathcal{I}_2)$ analogously for player 2. Given his beliefs, the expected utility function of player 1 is

$$U(a_1, t_1, \mathcal{I}_1) = \begin{cases} t_1 + \alpha_1 \pi_2(\mathcal{I}_1) & \text{if } a_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for player 2, we have

$$U(a_2, t_2, \mathcal{I}_2) = \begin{cases} t_2 + \alpha_1 \pi_1(\mathcal{I}_2) & \text{if } a_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Both players are assumed to be expected-utility maximizers who make choices simultaneously and independently. (This includes NE behavior as a special case.) This yields threshold-crossing decision rules

$$Y_1 = \mathbb{1}\{U(1, t_1, \mathcal{I}_1) \geq 0\}, \qquad Y_2 = \mathbb{1}\{U(1, t_2, \mathcal{I}_2) \geq 0\}. \quad (2)$$

Incomplete information makes it impossible for player $p$ to randomize in a way that makes his opponent exactly indifferent between his two actions. In addition, because we focus on the case where $t_p$ is continuously distributed, the event $U(1, t_p, \mathcal{I}_p) = 0$ occurs with probability 0. Our assumptions differ from NE be-
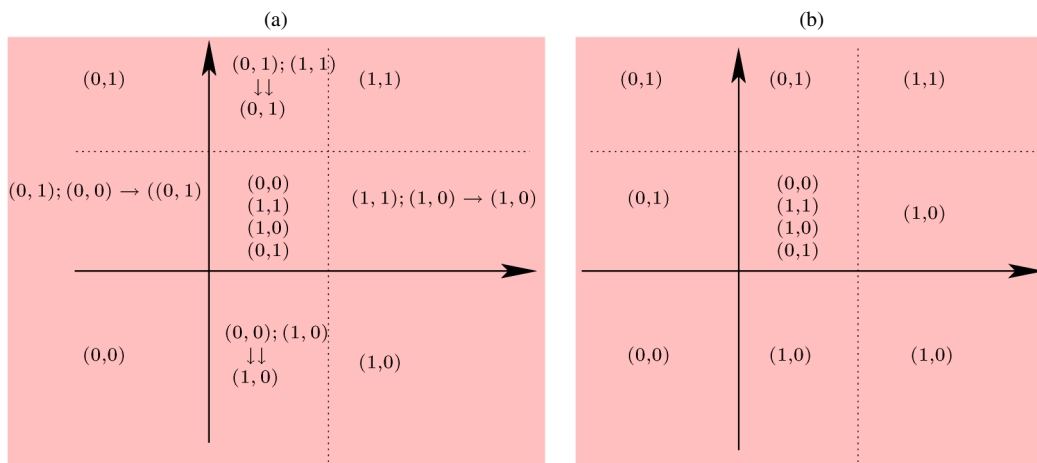


Figure 2. Observable implications of equilibrium (b) versus rationality (a).
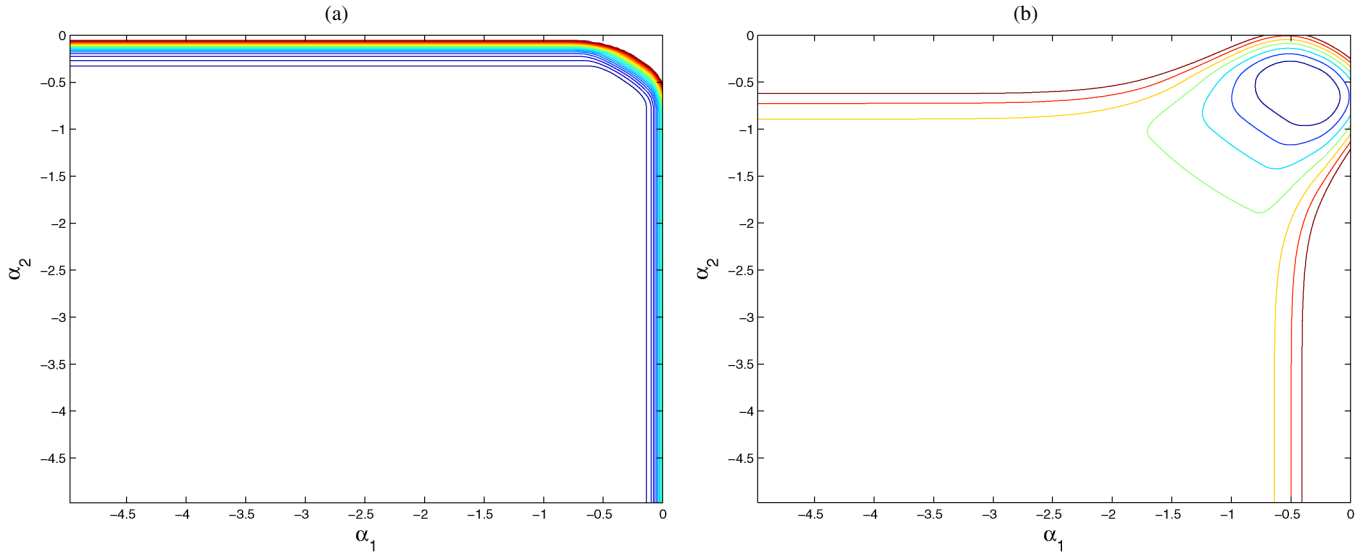
Figure 3. Identification set under Nash and 1-level rationality. Shown the identified regions for $(\alpha_1, \alpha_2)$ under $k = 1$ rationality (a) and Nash (b). We set in the underlying model $(\alpha_1, \alpha_2) = (-.5, -.5)$. The model was simulated assuming Nash with $(0, 1)$ selected with probability one in regions of multiplicity. Note that in (a), the model only places upper bounds on the alphas, whereas in (b) $(\alpha_1, \alpha_2)$ are constrained to lie a much smaller set (the inner "circle").

cause we do not impose the restriction that subjective beliefs are consistent with players' actual behavior. Again, here we assume that both $\alpha_1$ and $\alpha_2$ are negative.

### 4.1    Implications of Level-1 Rationality

We maintain the expected utility maximization assumption and the resulting decision rules (2). In the first round of rationality, we know that for any belief function, or without making any common prior assumptions, the following hold:

$$t_1 + \alpha_1 \geq 0 \implies U(1, t_1, \mathcal{I}_1) = t_1 + \alpha_1 \pi_2(\mathcal{I}_1) \geq 0$$
$$\forall \pi_2(\mathcal{I}_1) \in [0, 1],$$
$$t_1 \leq 0 \implies U(1, t_1, \mathcal{I}_1) = t_1 + \alpha_1 \pi_2(\mathcal{I}_1) \leq 0 \quad (3)$$
$$\forall \pi_2(\mathcal{I}_1) \in [0, 1].$$

Well-defined beliefs satisfy $\pi_2(\cdot) \in [0, 1]$. This implies that if player 1 is an expected-utility maximizer and holds well-defined beliefs, then he must satisfy

$$t_1 + \alpha_1 \geq 0 \implies a_1 = 1$$

and

$$t_1 \leq 0 \implies a_1 = 0.$$

Now, let $0 \leq t_1 \leq -\alpha_1$. For a player that is rational of order 1, there exists well defined beliefs that rationalizes either 1 or 0. Thus when $0 \leq t_1 \leq -\alpha_1$, both $a_1 = 1$ and $a_1 = 0$ are rationalizable. So, the implication of the game are summarized in Figure 4. Note here that the $(t_1, t_2)$ space is divided into nine regions: four regions where the outcome is unique, four regions with two potentially observable outcomes, and the middle square where any outcome is potentially observed. To make inference based on this model, we need to map these regions into predicted choice probabilities. To obtain the sharp set of parameters that is identified by the model, we can supplement this

model with consistent "selection rules" that specify, in regions of multiplicity, the probability of observing the various outcomes, which would be a function of both $(t_1, \mathcal{I}_1)$ and $(t_2, \mathcal{I}_2)$. The probabilities can be given a "structural" interpretation in which they would be interpreted as proper selection mechanisms. Given the level-1 behavioral assumptions, the only valid selection mechanisms are those that can be produced (rationalized) by the choice rules (2) for some well-defined beliefs. Expected utility maximization explains [through (2)] how players' choices are produced in an incomplete information environment given beliefs. Finally, let the joint distribution of $(t_1, t_2)$ be noted by $F(\cdot)$.

*Result 1.* For the game with incomplete information, let the players be rational with order 1 (level-1 rational) and write $W_p \equiv t_p \cup \mathcal{I}_p$. Then the choice probabilities predicted by the model are

$$P(1, 1) = \int_{III} dF + \int_{II} S_{(1,1)}^{II}(W_1, W_2) \, dF$$
$$+ \int_{VI} S_{(1,1)}^{VI}(W_1, W_2) \, dF + \int_V S_{(1,1)}^V(W_1, W_2) \, dF,$$

$$P(0, 0) = \int_{VII} dF + \int_{VIII} S_{(0,0)}^{VIII}(W_1, W_2) \, dF$$
$$+ \int_{IV} S_{(0,0)}^{IV}(W_1, W_2) \, dF + \int_V S_{(0,0)}^V(W_1, W_2) \, dF, \quad (4)$$

$$P(0, 1) = \int_I dF + \int_{II} S_{(0,1)}^{II}(W_1, W_2) \, dF$$
$$+ \int_{IV} S_{(0,1)}^{VI}(W_1, W_2) \, dF + \int_V S_{(0,1)}^V(W_1, W_2) \, dF,$$

where $S_j^i \geq 0$ are such that for example, $S_{(1,0)}^{II} + S_{(1,1)}^{II} = 1$, and so on, and I, II, III, IV, V, VI, and VIII are regions for $(t_1, t_2)$ shown in Figure 4.
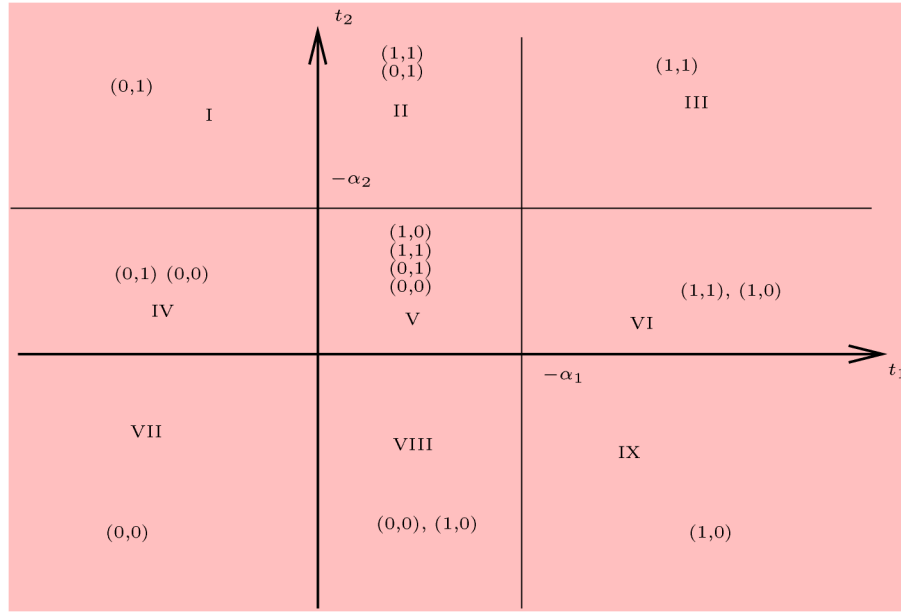
Figure 4. Observable implications of level-1 rationality.

The functions $S$ are unknown and represent "selection" functions that represent the probabilities of selecting a particular outcome in a region of multiplicity. Suppose, for simplicity, that $\mathcal{I}_p = t_p$, so that players condition their beliefs only on the realization of their own type. The (sharp) identified set $\Theta_I$, is the set of parameters where there exists proper selection functions $S$ such that the predicted choice probabilities in (4) are equal to the ones obtained from data. The restrictions in (4) can be exploited by, for example, discretizing the joint distribution of $(t_1, t_2)$, such as discussed for the complete-information case, to construct the identified set. The latter, $\Theta_I$, is the set of parameters for which the equalities in (4) are satisfied for well-defined selection functions. Implications of the foregoing set of equalities is a set of moment *inequalities* constructed by exploiting the fact that the $S$ functions are probabilities and thus are positive. So, for example, an implication of Result 1 is that $\int_{III} dF \leq P(1,1) \leq \int_{III} dF + \int_{II \cup III \cup V \cup VI} dF$, where the bounds of this inequalities do not involve the unknown functions $S$.

Next we analyze the behavior of players who assume that their opponents are (at least) level-$k$ rational for $k \geq 1$. Level-2 rational players are those whose second-order beliefs for their opponent are compatible with the bounds implied by (3). As we show, by eliminating beliefs that violate (3), we are able to reduce the set of level-2 rational beliefs from the entire $[0, 1]$ interval to a segment of it. Further rounds of iterated thinking will refine these bounds even more. Unlike in the Bayesian Nash equilibrium (BNE) case, we do not impose the requirement that beliefs are correct; we will rule only out those that are not compatible with the assumption that opponents are level-$k$ rational.

### 4.2 Implications of Level-$k$ Rationality

Level-1 rationality is characterized simply by expected utility maximization and any arbitrary system of well-defined beliefs. We now generalize the results of the previous section by characterizing bounds for beliefs that are compatible with assuming that opponents are level-$k$ rational. This means that, for example, level-2 rational players are all of those whose beliefs are consistent with the bounds implied by (3). As we show later, by eliminating beliefs that violate (3), we will be able to reduce the set of level-2 rational beliefs from the entire $[0, 1]$ interval to a segment of it. Level-3 rational players are those whose beliefs are compatible with the bounds for level-2 rational beliefs. This iterative construction can then be used to characterize bounds for level-$k$ rational beliefs. Each "round of rationality" refines these bounds by deleting all beliefs that assign positive probability to opponents' dominated strategies. As a reminder, the realization of $t_p$ is privately observed by player $p$, who conditions his beliefs about the expected action of his opponent on the realization of signals $\mathcal{I}_p$, with $t_p \in \mathcal{I}_p$ being a special case. The true distribution of $(t_1 \cup t_2 \cup \mathcal{I}_1 \cup \mathcal{I}_2)$ is common knowledge to both players. This is the common prior assumption. Even though it plays no role in the analysis of level-1 rational behavior, the common prior assumption is important for higher levels of rationality. We consider strategies (decision rules) for player $p$ that are threshold functions of $t_p$,

$$Y_p = \mathbb{1}\{t_p \geq \mu_p\} \quad \text{for } p = 1, 2. \tag{5}$$

It follows from the normal-form payoffs in Table 1 that this family of decision rules includes those of all expected utility-maximizing players in this simple binary choice game with incomplete information. Level-1 rational players and those who play a BNE are two special cases. In the construction of his expected utility, player $p$ forms *subjective* beliefs about $\mu_{-p}$ that can be summarized by a probability distribution for $\mu_{-p}$ given $\mathcal{I}_p$. These beliefs are derived as part of a solution concept. They may include BNE beliefs as a special case (in which case all players know those equilibrium beliefs to be correct). Here let $\widehat{G}_1(\mu_2 | \mathcal{I}_1)$ denote player 1's subjective distribution function for $\mu_2$ given $\mathcal{I}_1$, and define $\widehat{G}_2(\mu_1 | \mathcal{I}_2)$ analogously for player 2. A strategy by player $p$ is *rationalizable* if it is the best response (in the expected-utility sense) given some beliefs $\widehat{G}_p(\mu_{-p} | \mathcal{I}_p)$

that assign zero probability mass to strictly dominated strategies by player $-p$. A rationalizable strategy by player $p$ is described by

$$Y_p = \mathbb{1}\left\{t_p + \alpha_p \int_{\mathbb{S}(\widehat{G}_p)} E[\mathbb{1}\{t_{-p} \geq \mu\} | \mathcal{I}_p, \mu] \, d\widehat{G}_p(\mu | I_p) \geq 0\right\},$$
(6)

where the support $\mathbb{S}(\widehat{G}_p)$ excludes values of $\mu$ that result in dominated strategies within the class (5). Throughout, we focus on the case where $\mu_{-p}$ is continuously distributed conditional on $\mathcal{I}_p$, and ignore the distinction between strictly and weakly dominated strategies. Note that the subset of rationalizable strategies within the class (5) is of the form $\mu_p = -\alpha_p \int_{\mathbb{S}(\widehat{G}_p)} E[\mathbb{1}\{t_{-p} \geq \mu\} | \mathcal{I}_p, \mu] \, d\widehat{G}_p(\mu | I_p)$. In this setting, rationalizability requires expected utility maximization for a given set of beliefs but does not require those beliefs to be correct. It only imposes the condition that $\mathbb{S}(\widehat{G}_p)$ exclude values of $\mu_{-p}$ that are dominated. We eliminate such values by iterated deletion of dominated strategies.

Now we describe the iterative procedure that restricts $\mathbb{S}(\widehat{G}_p)$ by iterated dominance. As before, we maintain that the signs of the strategic interaction parameters $(\alpha_1, \alpha_2)$ are known. Specifically, suppose that $\alpha_p \leq 0$. Then, repeating arguments from the previous section on $k = 1$–rationalizable outcomes, looking at (6), we see that we must have eventwise comparisons

$$\mathbb{1}\{t_p + \alpha_p \geq 0\} \leq \mathbb{1}\{Y_p = 1\} \qquad \text{and}$$
$$\mathbb{1}\{t_p < 0\} \leq \mathbb{1}\{Y_p = 0\}.$$
(7)

Decision rules that do not satisfy these conditions are strictly dominated for all possible beliefs. Therefore, the subset of strategies within the class (5) that are not strictly dominated must satisfy $\Pr(t_p + \alpha_p \geq 0) \leq \Pr(t_p \geq \mu_p) \leq \Pr(t_p \geq 0)$ or, equivalently, $\mu_p \in [0, -\alpha_p]$. All other values of $\mu_p$ correspond to dominated strategies. In this setup, we refer that to the subset of strategies that satisfy $\mu_p \in [0, -\alpha_p]$ as *level-1 rationalizable strategies*. Note that, as before, these $\mu$'s do not involve the common prior distributions.

Level-2 rational players are those whose beliefs are consistent with assuming that their opponents are level-1 rational. Without any further assumptions, level-2 rational players are those whose beliefs about others satisfy (7). Consequently, a level-2 rational player must have beliefs that assign zero probability mass to values $\mu_{-p} \notin [0, -\alpha_p]$. As before, we impose no further requirements (such as having unbiased beliefs). A strategy is level-2 rationalizable if it can be justified by level-2 rationalizable beliefs, that is,

$$\mu_p = -\alpha_p \int_0^{-\alpha_{-p}} E[\mathbb{1}\{t_{-p} \geq \mu\} | \mathcal{I}_p, \mu] \, d\widehat{G}_p(\mu | \mathcal{I}_p),$$

where player $p$'s beliefs $\widehat{G}_p(\cdot | \mathcal{I}_p)$ satisfy $\widehat{G}_p(0 | \mathcal{I}_p) = 0$ and $\widehat{G}_p(-\alpha_{-p} | \mathcal{I}_p) = 1$; that is, those beliefs give zero weight to level-1–dominated strategies. Moreover, the expectation within the integral is taken with respect to the common prior conditional on $\mathcal{I}_p$, which includes player $p$'s type. Thus, exploiting this monotonicity, it is easy to see that for an outside observer, the subset of level-2 rationalizable strategies must satisfy

$$\mu_1 \in \left[-\alpha_1 E[\mathbb{1}\{t_2 \geq -\alpha_2\} | \mathcal{I}_1], -\alpha_1 E[\mathbb{1}\{t_2 \geq 0\} | \mathcal{I}_1]\right]$$

and

$$\mu_2 \in \left[-\alpha_2 E[\mathbb{1}\{t_1 \geq -\alpha_1\} | \mathcal{I}_2], -\alpha_2 E[\mathbb{1}\{t_1 \geq 0\} | \mathcal{I}_2]\right].$$

Level-$k$ rational players are those whose beliefs are consistent with assuming that their opponents are level-$(k-1)$ rational. Note that this definition is a statement about a player's higher-order beliefs up to order $k-1$; specifically, any player who believes that his opponent undertakes (at least) $k-1$ rounds of iterated deletion of dominated strategies in the construction of his expected utility will be a level-$k$ rational player. By induction, it is easy to prove the following claim.

*Claim 1.* If $\alpha_p \leq 0$, then a strategy of the type $Y_p = \mathbb{1}\{t_p \geq \mu_p\}$ is level-$k$ rationalizable if and only if $\mu_1$ and $\mu_2$ satisfy

$$\mu_p \in [0, -\alpha_{-p}] \equiv [\mu_{p,1}^L, \mu_{p,1}^U], \quad \text{for } k = 1 \text{ and } p \in \{1, 2\};$$

$$\mu_1 \in \left[-\alpha_1 E[\mathbb{1}\{t_2 \geq \mu_{2,k-1}^U\} | \mathcal{I}_1], -\alpha_1 E[\mathbb{1}\{t_2 \geq \mu_{2,k-1}^L\} | \mathcal{I}_1]\right]$$
$$\equiv [\mu_{1,k}^L, \mu_{1,k}^U], \quad \text{for } k > 1; \qquad (8)$$

$$\mu_2 \in \left[-\alpha_2 E[\mathbb{1}\{t_1 \geq \mu_{1,k-1}^U\} | \mathcal{I}_2], -\alpha_2 E[\mathbb{1}\{t_1 \geq \mu_{1,k-1}^L\} | \mathcal{I}_2]\right]$$
$$\equiv [\mu_{2,k}^L, \mu_{2,k}^U], \quad \text{for } k > 1.$$

The bounds described in (8) contain any set of beliefs that can be rationalized after $k-1$ rounds of iterated deletion of dominated strategies. We present identification results based on this entire range with no additional restrictions on how level-$k$ players actually choose their beliefs from within this space of rationalizable beliefs.

*Remark 1.* Any level-$k$ rational player also is level-$k'$ rational for any $1 \leq k' \leq k - 1$. Also, for $p \in \{1, 2\}$, with probability 1, we have that $[\mu_{p,k}^L, \mu_{p,k}^U] \subseteq [\mu_{p,k-1}^L, \mu_{p,k-1}^U]$ for any $k > 1$, with strict inclusion if $\alpha_p \neq 0$ and $t_{-p}$ has unbounded support conditional on $\mathcal{I}_p$. This monotonic feature of bounds (as $k$ increases) is a consequence of the payoff parameterization in the game. Note also that these bounds are a function of $\mathcal{I}_p$, the information on which player $p$ conditions his beliefs.

The two statements in Remark 1 follow because conditional on $\mathcal{I}_p$, the support $\mathbb{S}(\widehat{G}_p)$ of a $k$-level rational player is contained in that of a $k-1$–level rational player. In fact, if there is a unique BNE (conditional on $\mathcal{I}_p$), then $\mathbb{S}(\widehat{G}_p)$ will collapse to the singleton given by BNE beliefs as $k \to \infty$. Whenever warranted, we clarify whether a $k$-level rational player is "at most $k$-level rational" or "at least $k$-level rational." For inference based on level-2 rationality, we can use inequalities similar to (4) to map the observed choice probabilities to the predicted ones; in particular, we can use the thresholds from Claim 1 to construct a map between the model and the observable outcomes using (5). This is illustrated in Figure 5 for the case where $\mathcal{I}_p = t_p$ (i.e., players condition their beliefs exclusively on the realization of their own type). There $\mathbb{P}_{t_1}(\cdot)$ denotes the conditional distribution of $t_2 | t_1$, with $\mathbb{P}_{t_2}(\cdot)$ defined analogously. We see that, moving from level 1 to level 2, the middle square shrinks. As we show later, higher rationality levels (properly speaking, further rounds of deletion of dominated strategies) will shrink it further. The set of choice probabilities predicted by the model with level-$k$ rational players can be characterized by generalizing Result 1.
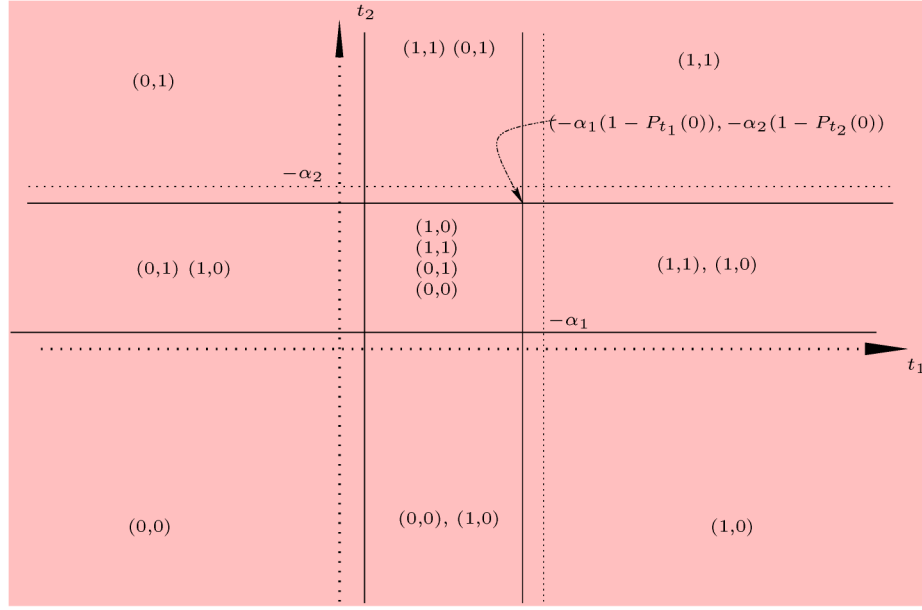
Figure 5. Observable implications of level-2 rationality.

*Result 2.* Let

$$\pi_p^L(1; \mathcal{I}_p) = 0 \qquad \text{and} \qquad \pi_p^U(1; \mathcal{I}_p) = 1$$

for $p = 1, 2$, and let for $k > 1$,

$$\pi_1^L(k; \mathcal{I}_1) = E\big[\mathbb{1}\{t_2 + \alpha_2 \pi_2^U(k-1; \mathcal{I}_2) \geq 0\}|\mathcal{I}_1\big],$$

$$\pi_1^U(k; \mathcal{I}_1) = E\big[\mathbb{1}\{t_2 + \alpha_2 \pi_2^L(k-1; \mathcal{I}_2) \geq 0\}|\mathcal{I}_1\big],$$

$$\pi_2^L(k; \mathcal{I}_2) = E\big[\mathbb{1}\{t_1 + \alpha_1 \pi_1^U(k-1; \mathcal{I}_1) \geq 0\}|\mathcal{I}_2\big],$$

$$\pi_2^U(k; \mathcal{I}_2) = E\big[\mathbb{1}\{t_1 + \alpha_1 \pi_1^L(k-1; \mathcal{I}_1) \geq 0\}|\mathcal{I}_2\big].$$

Using the notation in Section 2, the space of strategies for level-$k$ rational players is

$$\mathcal{R}^p(k) = \big\{Y_p = \mathbb{1}\{t_p + \alpha_p \pi_{-p}(\mathcal{I}_p) \geq 0\} :$$

$$\pi_{-p} \in [\pi_{-p}^L(k; \cdot), \pi_{-p}^U(k; \cdot)]\big\} \quad \text{for } p = 1, 2.$$

In the next section we parameterize the types $t_p$ to allow for observable heterogeneity and provide sufficient point identification conditions based exclusively on the restrictions implied by level-$k$ rationality.

## 4.3 Identification With Level-$k$ Rationality in a Parametric Model

From here on, we express $t_p = X_p' \beta_p - \varepsilon_p$, where $X_p$ is observable to the econometrician, $\varepsilon_p$ is not, and $\beta_p$ must be estimated (along with $\alpha_p$, the strategic interaction parameter for player $p$). Player $p$ observes the realization of his own $X_p$ and $\varepsilon_p$, where the latter is *only* privately observed. We also allow the possibility that some elements in $X_p$ are private information to player $p$ and, as before, denote the vector of signals used by player $p$ to condition his beliefs by $\mathcal{I}_p$. Throughout, we assume $(\varepsilon_1, \varepsilon_2)$ to be continuously distributed, with scale normalized to 1 and unbounded support. The results that follow only require that for each player $p$, the support of $\varepsilon_p$ be larger than that of

$X_p' \beta_p$ for all possible realizations of $\mathcal{I}_{-p}$. For simplicity, we assume that $\varepsilon_1$ is independent of $\varepsilon_2$ and denote their cumulative distribution function (cdf) as $H_p(\cdot)$ for $p = 1, 2$. Conceptually, we can extend the result that follow and obtain constructive identification results for the case where $\varepsilon_1$ and $\varepsilon_2$ are correlated, but we do not deal with that case here. For simplicity, we limit ourselves to the case where beliefs are conditioned on observables to the researcher; that is, $\mathcal{I}_p$ is observable. We define the identified set of parameters and then provide an objective function that can be used to construct the identified set. This function depends on the level $k$ of rationality that the econometrician assumes ex ante. We discuss the identification of $k$, then we provide a set of sufficient conditions to guarantee point identification under some assumptions. These point identification results provide insight into the kind of "variation" needed to shrink the identified set to a point. Our results can be extended to cases where beliefs are conditioned on unobservables to the researcher, as long as the joint distribution of all unobservables in the model is assumed known, possibly up to a finite-dimensional parameter.

As in the previous section, we make a common prior assumption. This assumption is only needed to compute bounds on beliefs for levels of rationality $k$ that are strictly larger than 1. Specifically, we assume that $H_1$ and $H_2$ are common knowledge among the players, and also assume that the econometrician knows these common prior distributions. We assume that players use the true distributions as their priors for payoff covariates $X_p$ and signals $\mathcal{I}_p$, both of which are observed by the econometrician. Implicitly, we also assume that the true values of $\beta_p$ and $\alpha_p$ are common knowledge to both players. Given this setup, we can construct bounds on beliefs iteratively. For any parameter value and any "rationality level," these bounds are identified, and they constitute the foundation for our identification results.

*Iterated Dominance and Bounds for Beliefs.* For ease of exposition, we assume that both players condition on the same

vector of signals, which we denote by $\mathcal{I}$. This would include the case where the only source of private information for player $p$ is $\varepsilon_p$ and $\mathcal{I} = X_1 \cup X_2$. We will return to the more general case and allow for $\mathcal{I}_1 \neq \mathcal{I}_2$ later. As in Section 4.2, we derive bounds for the range of rationalizable beliefs iteratively by deleting those that assign positive probability to opponents' dominated strategies. For each player $p$, let

$$\pi_{-p}^L(\theta|k=1, \mathcal{I}) = 0, \quad \text{and} \quad \pi_{-p}^U(\theta|k=1, \mathcal{I}) = 1,$$

and for $k \geq 2$, let

$$
\begin{aligned}
\pi_1^L(\theta|k, \mathcal{I}) &= E\big[H_1\big(X_1'\beta_1 + \alpha_1\pi_2^U(\theta|k-1, \mathcal{I})\big)|\mathcal{I}\big]; \\
\pi_1^U(\theta|k, \mathcal{I}) &= E\big[H_1\big(X_1'\beta_1 + \alpha_1\pi_2^L(\theta|k-1, \mathcal{I})\big)|\mathcal{I}\big]; \\
\pi_2^L(\theta|k, \mathcal{I}) &= E\big[H_2\big(X_2'\beta_2 + \alpha_2\pi_1^U(\theta|k-1, \mathcal{I})\big)|\mathcal{I}\big]; \\
\pi_2^U(\theta|k, \mathcal{I}) &= E\big[H_2\big(X_2'\beta_2 + \alpha_2\pi_1^L(\theta|k-1, \mathcal{I})\big)|\mathcal{I}\big],
\end{aligned}
\tag{9}
$$

where $\pi_{-p}^L(\theta|k, \mathcal{I})$ and $\pi_{-p}^U(\theta|k, \mathcal{I})$ are the lower and upper bounds for level-$k$ rationalizable beliefs by player $p$ for $\Pr(Y_{-p}|\mathcal{I})$. Given our foregoing assumptions, these bounds are identified for any $\theta$ and $k$. In the case where we want to allow for correlation in $\varepsilon_1$ and $\varepsilon_2$, the belief function for player $p$ will depend on $\varepsilon_p$, which would be part of a player-specific information set, and $H_p$ would be the conditional cdf of $\varepsilon_p|\varepsilon_{-p}$. By induction, it is easy to show that

$$
\begin{aligned}
&[\pi_{-p}^L(\theta|k; \mathcal{I}), \pi_{-p}^U(\theta|k; \mathcal{I})] \\
&\subseteq [\pi_{-p}^L(\theta|k-1; \mathcal{I}), \pi_{-p}^U(\theta|k-1; \mathcal{I})] \\
&\quad \text{with probability 1 in } \mathbb{S}(\mathcal{I}).
\end{aligned}
\tag{10}
$$

This monotonic feature holds even if players condition on different information sets. Moreover, the inclusion in (10) is strict if the strategic interaction coefficients are nonzero and if $\varepsilon_p$ has unbounded support conditional on $X_p'\beta_p$ and $\mathcal{I}$. Figure 6 depicts this case for a fixed realization $\mathcal{I}$, a given parameter vector $\theta$, and $k \in \{2, 3, 4, 5\}$.

*Identified Set for $\theta$ Based on Level-$k$ Rationality.* Let $W_p = X_p \cup \mathcal{I}$. It follows from the discussion in Section 4.2 (see Result 2) that the identified set for $\theta$ under the assumption that players are level-$k$ rational is given by

$$
\begin{aligned}
\Theta_I(k) = \big\{ &\theta \in \Theta : \exists \pi_1(\cdot), \pi_2(\cdot) \in [\pi_1^L(\theta|k; \cdot), \pi_1^U(\theta|k; \cdot)] \\
&\times [\pi_2^L(\theta|k; \cdot), \pi_2^U(\theta|k; \cdot)] \\
&\text{such that } E[Y_p|W_p] = H_p(X_p'\beta_p + \alpha_p\pi_{-p}(\mathcal{I})) \\
&\text{with probability 1 for } p = 1, 2 \big\}.
\end{aligned}
\tag{11}
$$

We exploit the fact that under our assumptions, the bounds for level-$k$ rational beliefs are identified to characterize a set $\Theta(k)$ that includes $\Theta_I(k)$. Our characterization constructive and based on conditional moment inequalities. To proceed, note that player $p$ is level-$k$ rational if and only if

$$
\begin{aligned}
&\mathbb{1}\{X_p'\beta_p + \alpha_p\pi_{-p}^U(\theta|k; \mathcal{I}) \geq \varepsilon_p\} \\
&\leq \mathbb{1}\{Y_p = 1\} \\
&\leq \mathbb{1}\{X_p'\beta_p + \alpha_p\pi_{-p}^L(\theta|k; \mathcal{I}) \geq \varepsilon_p\} \quad \text{with probability 1.}
\end{aligned}
$$

Recall that we are studying the case where $\alpha_p \leq 0$ for $p = 1, 2$. These inequalities must hold with probability 1 for all realizations of $(X_p, \varepsilon_p, \mathcal{I})$. It follows that level-$k$ rational players must satisfy

$$
\begin{aligned}
&H_p\big(X_p'\beta_p + \alpha_p\pi_{-p}^U(\theta|k; \mathcal{I})\big) \\
&\leq E[Y_p|W_p] \\
&\leq H_p\big(X_p'\beta_p + \alpha_p\pi_{-p}^L(\theta|k; \mathcal{I})\big) \quad \text{with probability 1,}
\end{aligned}
$$

where $W_p = X_p \cup \mathcal{I}$. Define the set

$$
\begin{aligned}
\Theta(k) = \big\{ &\theta \in \Theta : H_p\big(X_p'\beta_p + \alpha_p\pi_{-p}^U(\theta|k; \mathcal{I})\big) \leq E[Y_p|W_p] \\
&\leq H_p(X_p'\beta_p + \alpha_p\pi_{-p}^L(\theta|k; \mathcal{I})) \\
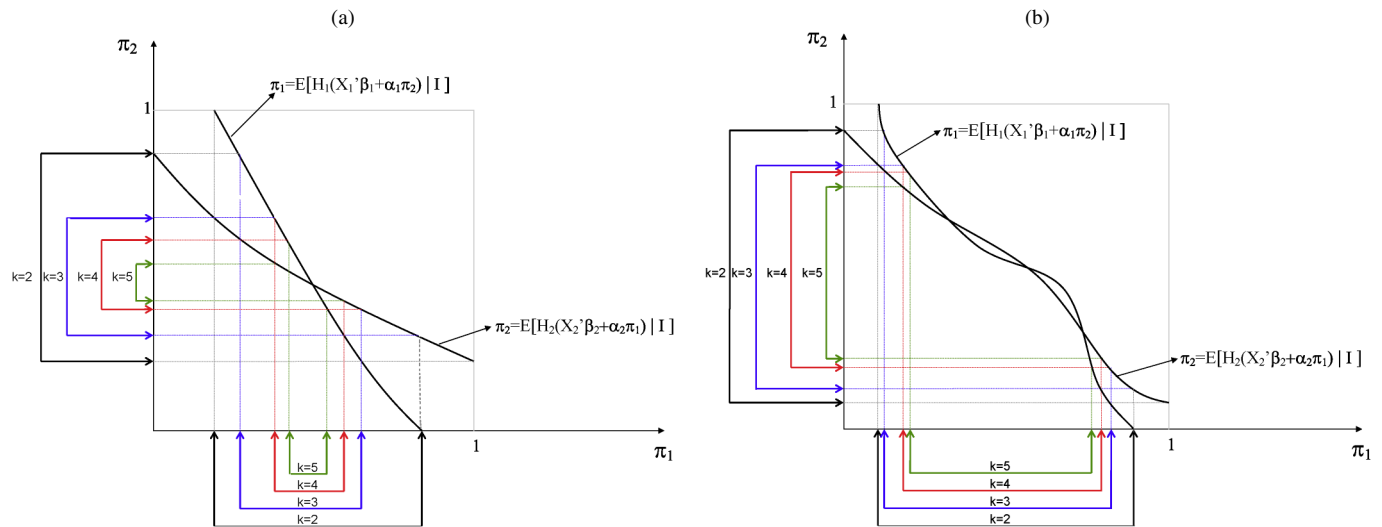&\text{with probability 1, } p = 1, 2 \big\}.
\end{aligned}
\tag{12}
$$



Figure 6. Rationalizable beliefs for $k = 2, 3, 4,$ and 5. (a) Belief iterations with a unique BNE. (b) Belief iterations with a multiple BNE. Bounds for level-$k$ rationalizable beliefs when $\mathcal{I}_1 = \mathcal{I}_2 \equiv \mathcal{I}$ (players condition on the same set of signals). The vertical axis shows level-$k$ rationalizable bounds for player 1's beliefs about $\Pr(Y_2 = 1|\mathcal{I})$. The horizontal axis shows the equivalent objects for player 2. The graphs correspond to a particular realization $\mathcal{I}$ and a given parameter value $\theta$.

Clearly, if players are level-$k$ rational, then we have $\Theta_I(k) \subseteq \Theta(k)$. If $X_p \in \mathcal{I}$ for both players, then it is easy to show that $\Theta_I(k) = \Theta(k)$. This follows because the set $[\pi_1^L(\theta|k; \cdot), \pi_1^U(\theta|k; \cdot)] \times [\pi_2^L(\theta|k; \cdot), \pi_2^U(\theta|k; \cdot)]$ is connected and the distributions $H_1$ and $H_2$ are continuous. The characterization $\Theta(k)$ is constructive and is the one that we use even though it might be a strict superset of the sharp set $\Theta_2(k)$. This is because dealing with the set $\Theta(k)$ is simple and computationally attractive. To allow for the case where $W_p$ has continuous support, we reexpress $\Theta(k)$ as the set of minimizers of an objective function (see Dominguez and Lobato 2004). For two vectors $a, b \in \mathbb{R}^{\dim(W_p)}$, let

$$
\begin{aligned}
&\Lambda_p(\theta|a, b; k) \\
&= E\Big[\big(1 - \mathbb{1}\{H_p(X_p'\beta_p + \alpha_p\pi_{-p}^U(\theta|k; \mathcal{I})) \leq \Pr(Y_p = 1|W_p) \\
&\qquad\qquad \leq H_p(X_p'\beta_p + \alpha_p\pi_{-p}^L(\theta|k; \mathcal{I}))\}\big) \\
&\qquad \times \mathbb{1}\{a \leq W_p \leq b\}\Big];
\end{aligned}
\tag{13}
$$

$$
\Gamma_p(\theta|k) = \int\int \Lambda_p(\theta|a, b; k)\, dF_{W_p}(a)\, dF_{W_p}(b);
$$

$$
\Gamma(\theta|k) = (\Gamma_1(\theta|k), \Gamma_2(\theta|k))',
$$

where the inequality $a \leq W_p \leq b$ is elementwise and $W_p \sim F_{W_p}(\cdot)$. Take any $2 \times 2$ positive definite matrix $\Omega$. The set in (12) can be expressed as

$$
\Theta(k) = \{\theta \in \Theta : \Gamma(\theta|k)'\Omega\Gamma(\theta|k) = 0\}. \tag{14}
$$

This is the definition of identified set for $\theta$ that we use under the assumption that all players in the game are level-$k$ rational. More precisely, returning to Remark 1, $\Theta(k)$ is the identified set if we assume that all players in the population are at least level-$k$ rational. By construction, $\Theta(k+1) \subseteq \Theta(k)$ for all $k$. Methods meant for set inference can be adapted to construct a sample estimator of $\Theta(k)$ based on a random sample of games where all players are level-$k$ rational for a given $k$. Note also that, compared with the Bayesian Nash solution, here we do not need to solve a fixed-point map to obtain the equilibrium; rather, rationalizability requires restrictions on player beliefs, which can be implemented iteratively. We formally show that $\Theta(k)$ contains the set of BNE for any $k > 0$. Having $\Theta(k) = \emptyset$ would reject the hypothesis that all players are at least level-$k$ rational.

*Remark 2.* Note that when $k = 1$, we do not need to specify the common prior assumption, because here beliefs play no role. Thus results will be robust to this assumption. However, depending on the magnitude of the $\alpha_p$'s, the bounds on choice probabilities predicted by such a model (where $k = 1$) can be wide.

Under certain conditions, the identified set in (14) would consist only of $\theta_0$, the true parameter value. An example of this is a case in which there exist realizations of the vector of signals $\mathcal{I}$ where the players are "forced" to take one of their actions with probability 1 regardless of their beliefs. To be concrete, suppose that the linear index $X_p'\beta_p$ has unbounded support for both players, and suppose that both of them are at least level-2 rational (i.e., they both perform at least one round of deletion of dominated strategies). Then, if the vector of signals $\mathcal{I}$ is such that

there exist regions of $\mathbb{S}(\mathcal{I})$ such that $\mathbb{S}(X_p'\beta_p|\mathcal{I})$ is concentrated around arbitrarily large positive or arbitrarily large negative values, the identified set $\Theta(k = 2)$ defined in (14) would collapse to a singleton $\theta_0$, the true parameter value. We refer to this as a case of "informative signals" and formalize this point identification result in the next section.

## 4.4 Sufficient Point Identification Conditions

In this section we study the problem of point identification of the parameter of interests in the foregoing game. In particular, we provide sufficient point identification conditions for level-1 rational play and for levels $k > 1$. These conditions can provide insight into what is required to shrink the identified set to a point (or a vector). Here we allow for the information sets to be different; that is, player $p$ conditions on $\mathcal{I}_p$ when making decisions and allow for exclusion restrictions where $\mathcal{I}_1 \neq \mathcal{I}_2$. We start with sufficient conditions for level-1 rationalizability.

*4.4.1 Identification Results With Level-1 Rationality.* Let $\theta_p = (\beta_p, \alpha_p)$ and $\theta = (\theta_1, \theta_2)$; then we have the following identification result.

*Theorem 1.* Suppose that $X_p$ has full rank for $p = 1, 2$, and let $X \equiv (X_1, X_2)$; assume that $\alpha_p < 0$ for $p = 1, 2$, and let $\Theta$ denote the parameter space. Let there be a random sample of size $N$ from the foregoing game. Consider the following conditions:

A1.1  For each player $p$, there exists a continuously distributed $X_{\ell,p} \in X_p$ with nonzero coefficient $\beta_{\ell,p}$ and unbounded support conditional on $X \setminus X_{\ell,p}$ such that for any $c \in (0, 1)$, $b \neq 0$, and $q \in \mathbb{R}^{\dim(X_{-\ell,p})}$, there exists $C_{b,q,m} > 0$ such that

$$
\Pr(\varepsilon_p \leq bX_{\ell,p} + q'X_{-\ell,p}|X) > m
$$
$$
\forall X_{\ell,p} : \text{sign}(b) \cdot X_{\ell,p} > C_{b,q,m}. \tag{15}
$$

A1.2  For $p = 1, 2$, let $X_{d,p}$ denote the regressors that have bounded support but are not constant. Suppose that $\Theta$ is such that for any $\beta_{d,p}, \widetilde{\beta}_{d,p} \in \Theta$ with $\widetilde{\beta}_{d,p} \neq \beta_{d,p}$ and for any $\alpha_p \in \Theta$,

$$
\Pr\big(|X_{d,p}'(\beta_{d,p} - \widetilde{\beta}_{d,p})| > |\alpha_p||X \setminus X_{d,p}\big) > 0. \tag{16}
$$

If all we know is that players are level-1 rational, then the following hold:

a. If (A1.1) holds, then the coefficients $\beta_{\ell,p}$ are identified.
b. If (A1.2) holds, then the coefficients $\beta_{d,p}$ are identified.
c. We say that player $p$ is pessimistic with positive probability if for any $\Delta > 0$, there exists $\mathcal{X}_\Delta \in \mathbb{S}(X_p)$ such that $\Pr(Y_p = 1|X) < \Pr(\varepsilon_p \leq X_p'\beta_{b_0} + \alpha_{p_0}|X) + \Delta$ whenever $X_p \in \mathcal{X}_\Delta$. If (A1.1) and (A1.2) hold and player $p$ is pessimistic with positive probability, then the identified set for $\alpha_p$ is $\{\alpha_p \in \Theta : \alpha_p \leq \alpha_{p_0}\}$. (Here we refer to the identified set as the set of values of $\alpha_p$ that are observationally equivalent, conditional on observables, to the true value $\alpha_{p_0}$.)

The results in Theorem 1 imposed no restrictions on $\mathcal{I}_p$. In particular, players can condition their beliefs on unobservables to the econometrician. A special case of condition (A1.1) is when $\varepsilon_p$ is independent of $X$. The condition in (A1.2) says how rich the support of the bounded shifters must be in relation to

the parameter space. Covariates with unbounded support satisfy this condition immediately given the full-rank assumption. Finally, similar identification results to Proposition 1 hold for the cases where $\alpha_p \geq 0$ and $\alpha_1 \alpha_2 \leq 0$. The proof of Theorem 1 is given in the Appendix.

*4.4.2 Identification With Level-k Rationality.* We now move on to the case of rationalizable beliefs of higher order. Our goal is to investigate whether a higher degree of rationality will the task of point-identifying $\alpha_p$. To simplify the analysis, we assume from here on that $\varepsilon_p$ is independent of $X$ and of $\mathcal{I} \equiv (\mathcal{I}_1, \mathcal{I}_2)$. This assumption could be replaced with one along the lines of (A1) in Theorem 1. We make the assumption that $\mathcal{I}$ is observed by the econometrician; we relax it later. Again, denote the common prior assumption by $H_p(\cdot)$. The beliefs of the players for any level-$k$ rationality can be constructed as done in the previous section. Our point identification–sufficient conditions are summarized in Theorem 2.

*Theorem 2.* Suppose that there exists a subset $\mathcal{X}_1^* \subseteq \mathbb{S}(X_1)$ where $X_1$ has full-column rank such that for any $X_1 \in \mathcal{X}_1^*$, $\varepsilon > 0$, and $\theta_2 \in \Theta$, there exist $\mathfrak{I}_{1_\varepsilon}^* \subset \mathbb{S}(\mathcal{I}_1|X_1)$ and $\mathfrak{I}_{1_\varepsilon}^{**} \subset \mathbb{S}(\mathcal{I}_1|X_1)$ such that

for all $\mathcal{I}_1 \in \mathfrak{I}_{1_\varepsilon}^*$,

$$\max\{1 - E[H_2(X_2'\beta_2 + \Delta_2)|\mathcal{I}_1],$$
$$E[H_2(X_2'\beta_2 + \Delta_2)|\mathcal{I}_1] - E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1]\}$$
$$< \varepsilon,$$

for all $\mathcal{I}_1 \in \mathfrak{I}_{1_\varepsilon}^{**}$,

$$\max\{E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1],$$
$$E[H_2(X_2'\beta_2 + \Delta_2)|\mathcal{I}_1] - E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1]\}$$
$$< \varepsilon. \tag{17}$$

A special case in which (17) holds is when there exists $X_{2\ell} \in (X_2 \cap W_1)$ with nonzero coefficient in $\Theta$ such that $X_{2\ell}$ has unbounded support conditional on $(X_2 \cup W_1) \setminus X_{2\ell}$. We can

call (17) an "informative signal" condition. Note that implicit in (17) is an exclusion restriction in the parameter space that precludes $\beta_2 = 0$ for any $\theta_2 \in \Theta$. If (17) holds, then for any $\theta \in \Theta$ such that $\theta_1 \neq \theta_{1_0}$, there exists either $\mathcal{W}_1^* \subset \mathbb{S}(W_1)$ or $\mathcal{W}_1^{**} \subset \mathbb{S}(W_1)$ such that

$$H_1\left(X_1'\beta_1 + \Delta_1 + \alpha_1 \pi_2^L(\theta|k; \mathcal{I}_1)\right)$$
$$< H_1\left(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0} \pi_2^U(\theta_0|k; \mathcal{I}_1)\right)$$
$$\forall W_1 \in \mathcal{W}_1^*, \ k \geq 2;$$
$$H_1\left(X_1'\beta_1 + \Delta_1 + \alpha_1 \pi_2^U(\theta|k; \mathcal{I}_1)\right)$$
$$> H_1\left(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0} \pi_2^L(\theta_0|k; \mathcal{I}_1)\right)$$
$$\forall W_1 \in \mathcal{W}_1^{**}, \ k \geq 2. \tag{18}$$

Therefore, for any $k \geq 2$, the level-$k$ rationalizable bounds for player 1's conditional choice probability of $Y_1 = 1|W_1$ that correspond to $\theta$ will be disjoint with those of $\theta_0$ with positive probability. Consequently, if (17) holds and the population of player 1's are at least level-2 rational, $\theta_{1_0}$ is identified. By symmetry, $\theta_{2_0}$ will be point-identified if the foregoing conditions hold, with the subscripts "1" and "2" interchanged.

For the case in which $\mathcal{I}_1 = \mathcal{I}_2 = X$ and the only source of private information in payoffs is $\varepsilon_p$, Figures 7 and 8 illustrate four graphical examples of how the "informative signals" condition (17) in Theorem 2 yields disjoint level-2 bounds.

The ability to shift the upper and lower bounds for level-2 rationalizable beliefs arbitrarily close to 1 or 0 is essential for the point-identification result in Theorem 2. For simplicity, the intercept $\Delta_1$ is subsumed in $X_1'\beta_1$ in the labels of these figures.

## 4.5 On Identification of Players' Rationality Level

Without further structure, our setup is not capable of identifying each individual player's rationality level (measured by $k$). Furthermore, without strong assumptions about the support of $\varepsilon_p$ relative to that of $X_p'\beta_p$, it is not possible to reject a value of
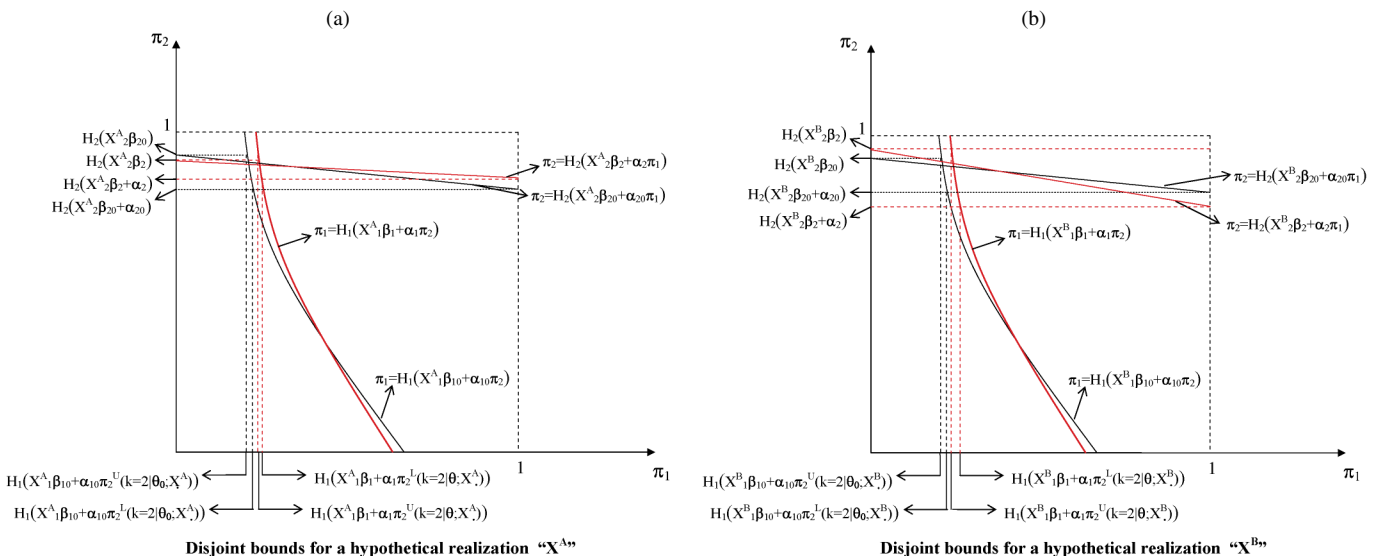


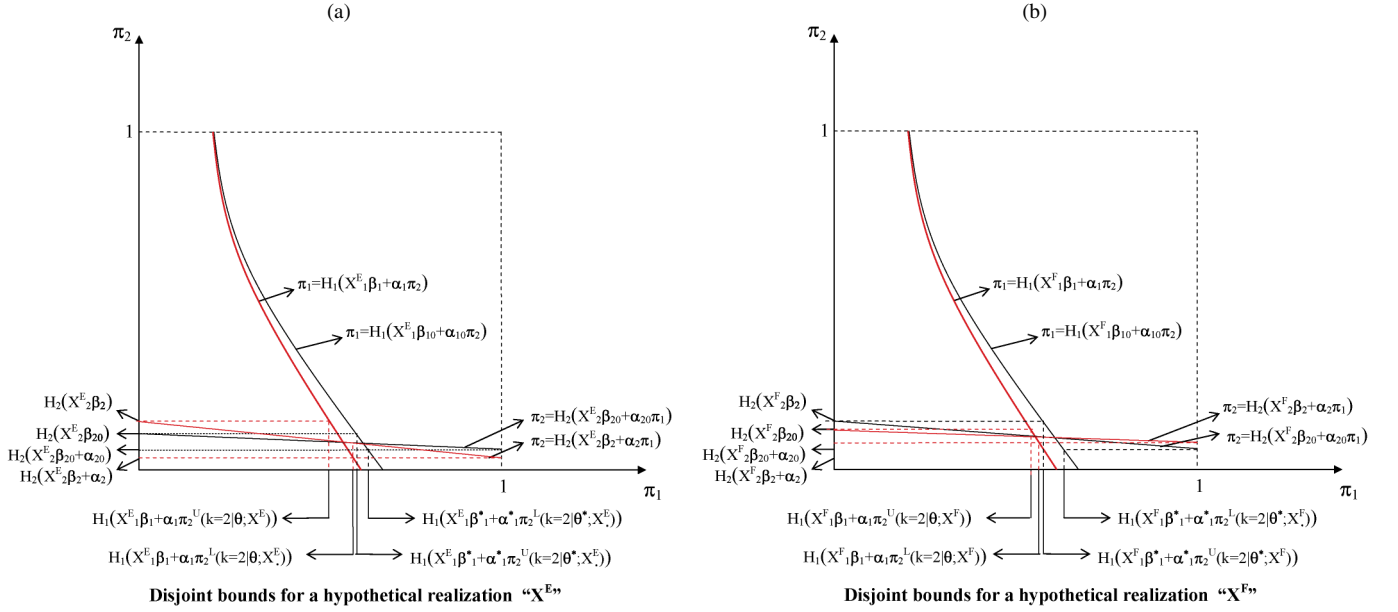Figure 7. Graphical examples of informative signals, I.

Figure 8. Graphical examples of informative signals, II.

$k$ on the basis of observed choices. But our setup is capable of producing identification results for the value $k_0$ such that players in the population are at most level-$k_0$ rational. This refers to the value such that the level-$k_0$ bounds hold with probability 1, but the level-$(k_0 + 1)$ bounds are violated with positive probability in the population. In other words, our setup has the potential to identify the rationality level $k_0$ such that a portion of players in the population have beliefs that violate the level-$(k_0 + 1)$ rationalizable bounds. Whether or not we can identify $k_0$ depends on how much we can identify about $\theta$. If all players are at least level-2 rational and the conditions for point identification of $\theta$ described in Theorem 2 hold, then $k_0$ would be point-identified because $Q(\theta_0|k) = 0$ if and only if $k \leq k_0$, where $Q(\theta|k)$ is as defined in (14). To see why this is not true when $\theta$ is set-identified, refer to parts (a) and (b) following (19). Otherwise, if the conditions for Theorem 2 do not hold, then suppose that we maintain the assumption $k_0 \geq 1$ (the only interesting case). We can start with $k = 1$ and construct $\Theta(1)$, as defined in (14). Next, for any $k \geq 2$, define

$$\underline{Q}(k) = \min_{\theta \in \Theta(1)} Q(\theta|k), \tag{19}$$

where $Q(\theta|k)$ is as defined in (14). Then the following hold:

a. $\underline{Q}(k) = 0$ for all $k \leq k_0$; however, $\underline{Q}(k) = 0$ does not imply, $k \leq k_0$.
b. $\underline{Q}(k) > 0$ implies that $k > k_0$.

Suppose that different observations in the data set correspond to a game with a different level of rationality; then if $\underline{Q}(k) > 0$ and $\underline{Q}(k-1) = 0$, we would reject the hypothesis (strictly speaking, this would be a joint test of the rationality hypothesis and all other maintained assumptions) that all of the population is at least level-$k$ rational. If we assumed ex ante that $k_0 \geq \underline{k} > 1$, then we could simply replace $\Theta(1)$ with $\Theta(\underline{k})$ in the definition of $\underline{Q}(k)$ in (19). Alternatively, in settings where at least a subset of the structural parameter $\theta$ is known (e.g., experiments), we could evaluate whether players are at least level-$k_0$ rational by

testing whether or not $\theta_0 \in \Theta(k_0)$ (the identified set for level-$k_0$ rationality). Otherwise, a test that would fail to reject $\Theta(k_0 + 1) = \emptyset$ would indicate that players are at most level-$k_0$ rational.

### 4.6 Bayesian Nash Equilibria and Rationalizable Beliefs

As before, let $\mathcal{I}_p$ be the signal that player $p$ uses to condition his beliefs about his opponent's expected choice, and let $\mathcal{I} \equiv (\mathcal{I}_1, \mathcal{I}_2)$. The set of BNE is defined as any pair $(\pi_1^*(\mathcal{I}_2), \pi_2^*(\mathcal{I}_1)) \equiv \pi^*(\mathcal{I})$ that satisfies

$$\begin{aligned}
\pi_1^*(\mathcal{I}_2) &= E[H_1(X_1'\beta_1 + \alpha_1\pi_2^*(\mathcal{I}_1))|\mathcal{I}_2], \\
\pi_2^*(\mathcal{I}_1) &= E[H_2(X_2'\beta_2 + \alpha_2\pi_1^*(\mathcal{I}_2))|\mathcal{I}_1].
\end{aligned} \tag{20}$$

By construction, the set of rationalizable beliefs for $\mathcal{I}$ must include the BNE set for any rational level $k$. The following result formalizes this claim.

*Proposition 1.* Let

$$\mathcal{R}(\mathcal{I}; k) = [\pi_1^L(\theta|k; \mathcal{I}_2), \pi_1^U(\theta|k; \mathcal{I}_2)]$$
$$\times [\pi_2^L(\theta|k; \mathcal{I}_1), \pi_2^U(\theta|k; \mathcal{I}_1)]$$

denote the set of level-$k$ rationalizable beliefs. Then, with probability 1, the BNE set described in (20) is contained in $\mathcal{R}(\mathcal{I}; k)$ for *any $k \geq 1$.*

We present the proof for the case where $\alpha_p \leq 0$ for $p = 1, 2$, on which we have focused. The proof can be adapted to all other cases. We proceed by induction by first proving the following claim.

*Claim 2.* Let $\pi^*(\mathcal{I}) \equiv (\pi_1^*(\mathcal{I}_2), \pi_2^*(\mathcal{I}_1))$ be any BNE. Then, for any $k \geq 1$, with probability 1, we have that $\pi^*(\mathcal{I}) \in \mathcal{R}(\mathcal{I}; k)$ implies that $\pi^*(\mathcal{I}) \in \mathcal{R}(\mathcal{I}; k+1)$ with probability 1.

*Proof.* If $\alpha_p = 0$ for $p = 1$ or $p = 2$, then the result follows trivially. Suppose that $\alpha_1 = 0$; then $\pi_1^L(\theta|k; \mathcal{I}_2) = \pi_1^U(\theta|k; \mathcal{I}_2) = \pi_1^*(\mathcal{I}_2) = E[H_1(X_1'\beta_1)|\mathcal{I}_2]$ and $\pi_2^L(\theta|k; \mathcal{I}_1) = \pi_2^U(\theta|k; \mathcal{I}_1) = \pi_2^*(\mathcal{I}_1) = E[H_2(X_2'\beta_2 + \alpha_2\pi_1^*(\mathcal{I}_2))|\mathcal{I}_1]$ for all

$k \geq 1$. We focus on the case where $\alpha_p < 0$ for $p = 1, 2$. Now suppose that $\pi^*(\mathcal{I}) \in \mathcal{R}(\mathcal{I}; k)$ but $\pi^*(\mathcal{I}) \notin \mathcal{R}(\mathcal{I}; k)$. Suppose, for example, that $\pi_1^L(\theta|k+1; \mathcal{I}_2) > \pi_1^*(\mathcal{I}_2)$. Because $\alpha_1 < 0$, this can be true if and only if

$$\underbrace{E\big[H_1\big(X_1'\beta_1 + \alpha_1\pi_2^U(\theta|k; \mathcal{I}_1)\big)\big|\mathcal{I}_2\big]}_{=\pi_1^L(\theta|k+1; \mathcal{I}_2)}$$

$$> \underbrace{E\big[H_1\big(X_1'\beta_1 + \alpha_1\pi_2^*(\mathcal{I}_1)\big)\big|\mathcal{I}_2\big]}_{=\pi_1^*(\mathcal{I}_2)}.$$

For this inequality to be satisfied, it cannot be the case that $\pi_2^*(\mathcal{I}_1) \leq \pi_2^U(\theta|k; \mathcal{I}_1)$. But this violates the assumption that $\pi^*(\mathcal{I}) \in \mathcal{R}(\mathcal{I}; k)$; therefore, we must have $\pi_1^L(\theta|k+1; \mathcal{I}_2) \leq \pi_1^*(\mathcal{I}_2)$. Suppose now that $\pi_1^U(\theta|k; \mathcal{I}_2) < \pi_1^*(\mathcal{I}_2)$. This can be true if and only if

$$\underbrace{E\big[H_1\big(X_1'\beta_1 + \alpha_1\pi_2^L(\theta|k; \mathcal{I}_1)\big)\big|\mathcal{I}_2\big]}_{=\pi_1^U(\theta|k+1; \mathcal{I}_2)}$$

$$< E\big[H_1\big(X_1'\beta_1 + \alpha_1\pi_2^*(\mathcal{I}_1)\big)\big|\mathcal{I}_2\big].$$

For this inequality to be satisfied, it cannot be the case that $\pi_2^*(\mathcal{I}_1) \geq \pi_2^L(\theta|k; \mathcal{I}_1)$. Once again, this violates the assumption $\pi^*(\mathcal{I}) \in \mathcal{R}(\mathcal{I}; k)$; therefore, we must have $\pi_1^U(\theta|k+1; \mathcal{I}_2) \geq \pi_1^*(\mathcal{I}_2)$. These results imply that we must have $\pi_1^L(\theta|k+1; \mathcal{I}_2) \leq \pi_1^*(\mathcal{I}_2) \leq \pi_1^U(\theta|k+1; \mathcal{I}_2)$. Following the same steps, we can establish that we must have $\pi_2^L(\theta|k+1; \mathcal{I}_1) \leq \pi_2^*(\mathcal{I}_1) \leq \pi_2^U(\theta|k+1; \mathcal{I}_1)$. Combined, these yield $\pi^*(\mathcal{I}) \in \mathcal{R}(\mathcal{I}; k+1)$, as claimed.

*Proof of Proposition 1.* Follows from Claim 2 and the fact that level-1 rational players satisfy $H_p(X_p'\beta_p + \alpha_p) \leq E[Y_p|X_p] \leq H_p(X_p'\beta_p)$, which yields $\mathcal{R}(\mathcal{I}; k = 1) = [0, 1] \times [0, 1]$. Consequently, $\mathcal{R}(\mathcal{I}; k = 1)$ contains all BNE. It follows from Claim 2 that $\mathcal{R}(\mathcal{I}; k = 1)$ contains all BNE for all $k \geq 1$.

### 4.7   BNE versus Rationalizability: Identification

Naturally, it is always guaranteed that one gets a weakly *smaller* identified set with BNE assumptions, because the predicted outcomes based on equilibrium use stronger assumptions on player beliefs. The size of the rationalizable outcome set depends on the distance between the smallest and largest equilibria [more precisely, the distance (in the unit square) between the smallest and the largest equilibrium beliefs]. In the case of a unique equilibrium, we can see that in the foregoing game and as $k \to \infty$, the predicted outcomes under both solution concepts converge. This convergence feature is not a general property of rationalizability, but rather is a consequence of the normal-form payoff parameterization of the game. In addition, in the foregoing simple example, predicted outcomes based on rationality of order $k$ for any $k$ are much easier to solve for, because they do not require solutions to fixed-point problems, especially in cases of multiple equilibria.

## 5.   IDENTIFICATION IN FIRST–PRICE INDEPENDENT PRIVATE VALUE AUCTIONS WITH RATIONALIZABLE BIDS

This section considers a situation in which a population of symmetric, risk-neutral potential buyers must bid simultaneously for a single good. We focus on a first-price auction with independent private values, although our results can be adapted to the case of interdependent private values and affiliated signals. As is usually the case in the econometric analysis of auctions, the object of interest is the distribution of private values. Under the assumption that observed bids conform to a BNE, nonparametric point identification for this distribution has been established by, for example, Guerre, Perrigne, and Vuong (1999). Thus equilibrium assumptions (and other conditions) deliver point identification of the valuation distribution. Here we relax the BNE requirement and assume only that buyers are strategically sophisticated in the sense of Battigalli and Siniscalchi (2003, henceforth BS). Other strategic assumptions that can be used and that deliver qualitatively different results than BS's interim rationalizability is the $\mathcal{P}$-dominance concept introduced for auction setups by Dekel and Wolinsky (2003) and used more recently by Crawford and Iriberri (2007). Here we highlight just what can be learned with the BS setup and compare that with BNE. BNE requires rational, expected utility maximizing buyers with correct beliefs. Strategically sophisticated buyers are rational and expected-utility maximizers, but their beliefs may or may not be correct. This characterization includes BNE as a special case. The degree of sophistication will be characterized using the concept of interim rationalizability. As we show, this will lead to the notion of *level-k rationalizable bids* for $k \in \mathbb{N}$. We describe these concepts next.

Let $F_0(\cdot)$ denote the distribution of $v_i$, the private valuation of bidder $i$. We assume $F_0(\cdot)$ to be common knowledge among the bidders, and focus on the case where $F_0(\cdot)$ is log-concave and absolutely continuous with respect to Lebesgue measure. We assume its support to be of the form $[0, \omega)$ (i.e., normalize its lower bound by 0) and allow, in principle, the case where $\omega = +\infty$. Assume for the moment that the seller's reservation price $p_0$ is equal to 0. We explicitly introduce a strictly positive reservation price later.

### 5.1   Assumptions About Bidders' Beliefs

Following BS, we assume that bidders expect all positive bids to win with strictly positive probability and that this is common knowledge. This condition will ensure that it is common knowledge that no bidder will bid beyond his or her valuation irrespective of his or her beliefs. It also implies that every bidder with nonzero private value will submit a strictly positive bid. Interim rationalizability will naturally produce only upper bounds for rationalizable bids. Additional, ad hoc assumptions can be made to characterize a lower bound. Therefore, with probability 1, the number of potential bidders $\mathcal{N}$ is equal to the number of actual bidders. (Only a bidder with valuation equal to 0 is indifferent between entering the bid or not.) We restrict further attention to beliefs that assign positive probability only to increasing bidding functions. Formally, let $\mathcal{B}$ denote the space of all functions of the form

$$\mathcal{B} = \big\{b : [0, \omega) \to \mathbb{R}_+ : b(v) \leq v, \text{ and } v > v' \Rightarrow b(v) > b(v')\big\}.$$

$$(21)$$

We let $\mathcal{N}$ denote the number of potential bidders in the population and write $\mathcal{B}_{-i} = \mathcal{B}^{\mathcal{N}-1}$. Beliefs for bidder $i$ are probability distributions defined over a sigma-algebra $\Delta_{\mathcal{B}_{-i}}$, where this sigma-algebra is such that singletons in $\mathcal{B}_{-i}$ are measurable. The results that we analyze here do not depend on the specific choice of the sigma-algebra, as long as this choice satisfies the singleton-measurability mentioned here (see footnote 10 in BS). A conjecture by bidder $i$ is a degenerate belief that assigns probability mass 1 to a singleton $\{b_j\}_{j \neq i} \in \mathcal{B}_{-i}$. The distribution of valuations $F_0(\cdot)$ as well as $\mathcal{N}$ are common knowledge among potential bidders. This is similar to the common prior assumption made in the previous section.

As we show, restricting attention to beliefs in $\mathcal{B}$ will yield rationalizable upper bounds for bids that also belong in $\mathcal{B}$. It also simplifies the analysis by, for example, ruling out ties in the characterization of players' expected utility. Finally, as we argue below (and formally shown in BS), restricting attention to beliefs in $\mathcal{B}$ implies that BNE-optimal bids are always rationalizable.

## 5.2 Implications of Level-$k$ Rationality in Bids

Here we follow the setup in BS, with our notation differing from that of previous sections. We have a population of $\mathcal{N}$ risk-neutral potential buyers bidding simultaneously for a single object. With a 0 reservation price of 0, we can interpret $\mathcal{N}$ as the number of observed bids that is common knowledge among the bidders. Each bidder $i$ observes his valuation $v_i$, independent of those of other bidders with an identical log-concave, continuous distribution $F_0(\cdot)$. The highest bid wins the object, ties are broken at random and only the winner pays his bid. The space of beliefs on which we focus assigns probability 0 to ties. Therefore, the decision problem for bidder $i$ can be expressed as

$$\max_{b \geq 0}(v_i - b)\widehat{\Pr}_i\left[\max_{j \neq i} b(v_j) \leq b\right], \qquad (22)$$

where $\widehat{\Pr}_i(\cdot)$ denotes bidder $i$'s subjective probability, derived from his beliefs and knowledge of $F_0(\cdot)$. (Strictly speaking, what matters is that ties have probability 0 for the most pessimistic conjecture.) For level-1 rational bidding, any bidder $i$ whose bids satisfy

$$b \leq v_i \equiv \overline{B}_1(v_i; \mathcal{N}) \quad \text{with probability 1} \qquad (23)$$

are called *level-1 rational bidders*. Any expected-utility maximizer bidder $i$ must be level-1 rational regardless of whether or not his beliefs live in $\mathcal{B}_{-i}$. Thus we have the following:

*Result* (BS). Any bid with $b_i \leq v_i$ is level-1 rational.

This was proved by BS, who also showed that the bound is sharp; that is, that for any bid in the bound, there exists a consistent and valid level-1 belief function for which that bid is a best response. This result is interesting because in this setup, the bids cannot be bound from below. This situation is in marked contrast to the BNE prediction. Note that the bound above depends on the continuity of the valuation and the assumption that any positive bid has a positive chance of winning. In another case where the valuations are assumed to take countable values, Dekel and Wolinsky (2003) showed that a form of rationalizability implies tight bounds on the bidding function in the

limit as the number of bidders increases. Here we derive strategies for identification of $F(\cdot)$ based on the results of BS, but these strategies can be easily adapted to other strategic setups, like those suggested by Dekel and Wolinsky.

*Higher-Order Rationality.* We now characterize the identified features in an auction with higher rationality levels. Focus on bidders with beliefs in $\mathcal{B}_{-i}$. The most pessimistic assessment in $\mathcal{B}_{-i}$ is given by the conjecture $b(v_j) = \overline{B}_1(v_j; \mathcal{N}) = v_j$ for all $j \neq i$ (the upper bound for bids for level-1 rational bidders). Because bidder $i$ knows $F_0$, his optimal expected utility for this assessment is

$$\max_{b \geq 0}(v_i - b)\Pr\left[\max_{j \neq i}\overline{B}_1(v_j; \mathcal{N}) \leq b\right]$$

$$= \max_{b \geq 0}(v_i - b)\Pr\left[\max_{j \neq i} v_j \leq b\right]$$

$$= \max_{b \geq 0}(v_i - b)F_0(b)^{\mathcal{N}-1}$$

$$\equiv \underline{\pi}_2^*(v_i; \mathcal{N}), \qquad (24)$$

where $\underline{\pi}_2^*(v_i; \mathcal{N})$ is the lower bound for optimal expected utility (22) for all beliefs in $\mathcal{B}_{-i}$. The upper expected utility bound for an arbitrary bid $b$ is trivially given by $(v_i - b)$ for any possible beliefs. (No bidder would ever expect to win the good with probability higher than 1.) Any bid submitted by a rational (i.e., expected-utility maximizer) bidder with beliefs in $\mathcal{B}_{-i}$ must satisfy

$$v_i - b \geq \underline{\pi}_2^*(v_i; \mathcal{N}) \quad \Rightarrow \quad b \leq v_i - \underline{\pi}_2^*(v_i; \mathcal{N}) \equiv \overline{B}_2(v_i; \mathcal{N})$$

$$\text{with probability 1.} \quad (25)$$

We refer to bidders who satisfy (25) as *level-2 rational bidders*. Given our assumptions, $\overline{B}_2(v_i; \mathcal{N})$ is increasing and concave and satisfies $\overline{B}_2(v_i; \mathcal{N}) \leq \overline{B}_1(v_i; \mathcal{N}) = v_i$, with strict inequality for all $v_i > 0$. These and more properties were enumerated by BS, who focused on a more general case that allows for interdependent values. Therefore, $\overline{B}_2 \in \mathcal{B}$. Let $\overline{S}_2(\cdot; \mathcal{N})$ denote the inverse of $\overline{B}_2(\cdot; \mathcal{N})$. We call *level-3 rational bidders* those whose beliefs incorporate the level-2 upper bound (25). The most pessimistic assessment for level-3 rational bidders is the conjecture $b(v_j) = \overline{B}_2(v_j; \mathcal{N})$ for all $j \neq i$. The optimal expected utility for this pessimistic assessment is

$$\max_{b \geq 0}(v_i - b)\Pr\left[\max_{j \neq i}\overline{B}_2(v_j; \mathcal{N}) \leq b\right]$$

$$= \max_{b \geq 0}(v_i - b)F_0(\overline{S}_2(b; \mathcal{N}))^{\mathcal{N}-1}$$

$$\equiv \underline{\pi}_3^*(v_i; \mathcal{N}). \qquad (26)$$

Using the same logic that led to (25), the set of rationalizable bids for level-3 rational bidders must satisfy

$$v_i - b \geq \underline{\pi}_3^*(v_i; \mathcal{N}) \quad \Rightarrow \quad b \leq v_i - \underline{\pi}_3^*(v_i; \mathcal{N}) \equiv \overline{B}_3(v_i; \mathcal{N})$$

$$\text{with probability 1.} \quad (27)$$

The level-3 upper bound for rationalizable bids, $\overline{B}_3(\cdot; \mathcal{N})$ is increasing and concave and satisfies $\overline{B}_3(\cdot; \mathcal{N}) \leq \overline{B}_2(\cdot; \mathcal{N})$, with strict inequality for nonzero valuations. To see why the last result holds, recall that $\overline{B}_2(v_i; \mathcal{N}) = v_i - \underline{\pi}_2^*(v_i; \mathcal{N}) \equiv \overline{B}_1(v_i; \mathcal{N}) - \underline{\pi}_2^*(v_i; \mathcal{N})$. Therefore, for any $b$, we have that

$\Pr[\max_{j \neq i} \overline{B}_2(v_j; \mathcal{N}) \leq b] \geq \Pr[\max_{j \neq i} \overline{B}_1(v_j; \mathcal{N}) \leq b]$. Immediately, this implies that $\underline{\pi}_3^*(\cdot; \mathcal{N}) \geq \underline{\pi}_2^*(\cdot; \mathcal{N})$ and thus $\overline{B}_3(\cdot; \mathcal{N}) \leq \overline{B}_2(\cdot; \mathcal{N})$. Because $F_0(\cdot)$ is not assumed to have point masses, all of the foregoing inequalities are strict for any $v_i > 0$. Proceeding iteratively, the level-$k$ bound for rationalizable bids is given by

$$b_i \leq v_i - \underline{\pi}_k^*(v_i; \mathcal{N}) \equiv \overline{B}_k(v_i; \mathcal{N})$$

with probability 1, where

$$\underline{\pi}_k^*(v_i; \mathcal{N}) = \max_{b \geq 0}(v_i - b)F_0(\overline{S}_{k-1}(b; \mathcal{N}))^{\mathcal{N}-1} \quad (28)$$

and $\overline{S}_{k-1}(\cdot; \mathcal{N})$ is the inverse function of $\overline{B}_{k-1}(\cdot; \mathcal{N})$. The level-$k$ upper bounds for rationalizable bids, $\overline{B}_k(\cdot; \mathcal{N})$, are increasing and concave and satisfy $\overline{B}_{k+1}(v; \mathcal{N}) \leq \overline{B}_k(v; \mathcal{N})$ for all $k$, with strict inequality for all $v > 0$. Let $b^{\mathrm{BNE}}(v; \mathcal{N})$ denote the optimal BNE bidding function, produced by self-consistent, correct beliefs. BS showed that $\overline{B}_k(\cdot; \mathcal{N}) \geq b^{\mathrm{BNE}}(\cdot; \mathcal{N})$ for all $k \in \mathbb{N}$. In particular, this is true for $\lim_{k \to \infty} \overline{B}_k(\cdot; \mathcal{N})$, which is well defined by the aforementioned monotonicity property of the sequence $\{\overline{B}_k(\cdot; \mathcal{N})\}_{k \in \mathbb{N}}$. Bidding below $b^{\mathrm{BNE}}(\cdot; \mathcal{N})$ is always rationalizable for any rationality level $k$. All results presented here are consistent with this type of behavior.

*Example.* Suppose that private values are exponentially distributed, with $F_0(v) = 1 - \exp\{-\theta v\}$ and $\theta > 0$. We have that $F_0(v)/f_0(v) = \frac{1 - \exp\{-\theta v\}}{\theta \exp\{-\theta v\}} = \frac{1}{\theta} \exp\{\theta v\} - \frac{1}{\theta}$, which is an increasing function of $v$ for all $\theta > 0$, establishing log-concavity of $F_0$. Figure 9 depicts $\overline{B}_k(\cdot; \mathcal{N})$, the level-$k$ rationalizable bounds for bids for the case where $\theta = -.25$, $\mathcal{N} = 2$ (two bidders), and $k = 1, 2, 3, 4$. This graphical example illustrates the features described earlier for these bounds, namely $\overline{B}_k(\cdot; \mathcal{N})$, is continuous, increasing, concave, and invertible and satisfies $\overline{B}_{k+1}(v; \mathcal{N}) \leq \overline{B}_k(v; \mathcal{N})$ for all $k$, with strict inequality for all $v > 0$. For this particular example, the bounds corresponding to $k \geq 5$ are graphically indistinguishable from $\overline{B}_4(v; \mathcal{N})$.

## 5.3 Identification With Level-$k$ Rationality in a Parametric Model

In this section we exploit the foregoing bounds to learn about the distribution of valuation given a random sample of bids. We first focus on the hypothetical case where there is no reserve price set by the seller and for each auction we observe all bids submitted and also know $\mathcal{N}$, the number of potential entrants. In the next section we deal with the more general case where there is a nonzero reserve price and only winning bids are observed.

We assume a semiparametric setting where $F_0$ belongs to a space of log-concave, absolutely continuous distribution functions with support $[0, \omega]$ of the form

$$\mathcal{F}_v^\Theta = \big\{ F(\cdot; \theta) : \theta \in \Theta, \text{ and } F_0(\cdot) = F(\cdot; \theta_0)$$
$$\text{for some } \theta_0 \in \Theta \big\}. \quad (29)$$

Here we also can think of $\Theta$ as a set of functions, in which case the foregoing definition accommodates nonparametric analysis. Denote the level-$k$ upper bound that corresponds to $F(\cdot; \theta)$ by $\overline{B}_k(\cdot; \mathcal{N}|\theta)$.

*Level-1 Rationality.* For rationality of level 1, the game predicts that

$$0 \leq b_i^l \leq v_i^l \quad \text{for all } i = 1, \dots, \mathcal{N}, \ l = 1, \dots, L.$$

This is a problem of inference with interval data. The $b$'s are observed and the $v$'s are not, but we observe a bound on every observation. The object of interest is the distribution function $F$ of the valuations $v$. (Here we can introduce auction heterogeneity that is observed.) This implies that

$$F_0(t; \theta) \equiv P(v \leq t) \leq P(b \leq t) \equiv G_b(t).$$

Thus, with the first level of rationality, we can bound the valuation distribution above by the observed distribution of the bids. Here inference is handled below and is based on replacing the observed bids distribution with its consistent empirical analog.
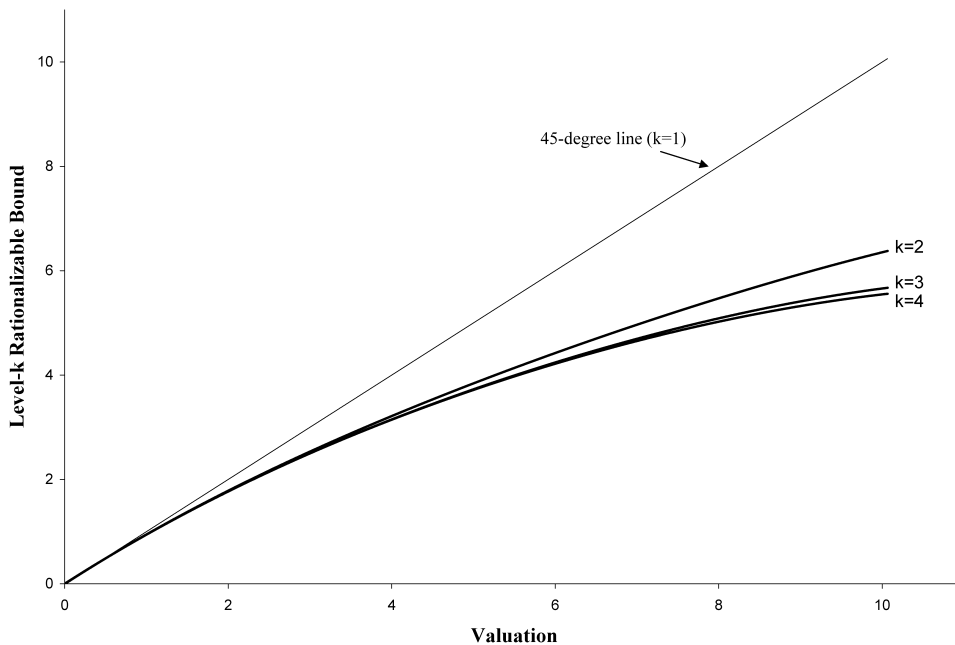


Figure 9. Level-$k$ rationalizable bounds $\overline{B}_k(\cdot; \mathcal{N})$ for $F_0(v) = 1 - e^{-.25v}$, $\mathcal{N} = 2$, and $k = 1, 2, 3, 4$.

*Level-k Rationality.* Similar to the foregoing, for level-$k$ and any $\theta \in \Theta$, we have that

$$0 \leq b_i^l \leq v_i^l - \underline{\pi}_k^*(v_i; \mathcal{N}|\theta) \equiv \overline{B}_k(v_i; \mathcal{N})$$

$$\text{for all } i = 1, \ldots, \mathcal{N}, \ l = 1, \ldots, L.$$

Thus this means that if bidders are level-$k$ rational, then

$$F(\overline{S}_k(t; \mathcal{N}|\theta_0); \theta_0) \leq P(b \leq t) \equiv G_b(t),$$

where, as before, $\overline{S}_k$ denotes the inverse function of $\overline{B}_k$. Here the bound is a bit more complicated, because the function $\overline{S}$ also depends on $F_0$. Using the notation of Section 2, the space of strategies (bidding functions) for level-$k$ rational players is

$$\mathcal{R}^i(k) = \{b \in \mathcal{B} : b(\cdot) \leq \overline{B}_k(\cdot; \mathcal{N}|\theta)\}.$$

As we did in Section 4.3, we characterize the identified set for $\theta$ based on level-$k$ rationality in terms of an objective function.

*Proposition 2.* Suppose that $F_0$ belongs to a space of distribution functions as described in (29). Moreover, suppose that we have a random sample of size $L$ of auctions, each of which has $\mathcal{N}$ bidders and where we observe all bids. Take $k \in \mathbb{N}^+$, and let

$$\Lambda(\theta|a, c; k) = \int \left(1 - \mathbb{1}\{F_b(b) \geq F(\overline{S}_k(b; \mathcal{N}|\theta); \theta)\}\right)$$

$$\times \mathbb{1}\{a \leq b \leq c\} \, dF_b(b), \quad (30)$$

$$\Gamma(\theta|k) = \int \int \Lambda(\theta|a, c; k) \, dF_b(a) \, dF_b(c).$$

Then, under the sole assumption that all bidders are level-$k$ rational, the identified set is

$$\Theta(k) = \{\theta \in \Theta : \Gamma(\theta|k)^2 = 0\}.$$

If the following condition holds for a known $k_0$, then a stronger identification result can be obtained.

*Assumption B1.* Suppose that there exists $k_0$ such that all bidders are level-$k_0$ rational and, with positive probability, bids are equal to the level-$k_0$ bounds; that is, suppose that $\Pr(b_i \leq \overline{B}_{k_0}(v_i; \mathcal{N}|\theta_0)) = 1$ and $\Pr(b_i = \overline{B}_{k_0}(v_i; \mathcal{N}|\theta_0)) > 0$.

*Proposition 3.* Suppose that Assumption B1 holds, and let $\Theta(k_0)$ be as defined in Proposition 2. For $\theta \in \Theta$, let

$$\mathcal{F}^c(\theta) = \{\theta' \in \Theta : \overline{B}_{k_0}(v_i; \mathcal{N}|\theta') < \overline{B}_{k_0}(v_i; \mathcal{N}|\theta)$$

$$\text{with probability } 1\}. \quad (31)$$

Then the identified set is

$$\Theta_0^* = \{\theta \in \Theta(k_0) : \nexists \theta' \in \Theta \text{ such that } \theta' \in \mathcal{F}^c(\theta)\}. \quad (32)$$

Consequently, if there exist $\overline{\theta} \in \Theta(k_0)$ such that

$$F(\cdot; \theta) < F(\cdot; \overline{\theta}) \quad \text{for all } \theta \in \Theta(k_0), \quad (33)$$

then $\Theta_0^* = \{\overline{\theta}\}$ and, consequently, $\theta_0 = \overline{\theta}$.

Under Assumption B1, $\theta \notin \Theta(k_0)$ implies that $\theta \neq \theta_0$ and $\theta \in \Theta(k_0)$ holds only if $\theta \notin \mathcal{F}^c(\theta_0)$. Suppose that we have $\theta, \theta' \in \Theta(k_0)$ such that $\theta' \in \mathcal{F}^c(\theta)$; then it cannot be the case that $\theta = \theta_0$, because then $\theta' \in \mathcal{F}^c(\theta_0)$ would imply that $\theta' \notin \Theta(k_0)$. Thus any such $\theta$ can be discarded as the true $\theta_0$. To see how this result is constructive, suppose that $\mathcal{F}_v^\Theta$ is a space of exponentially distributed valuations, Assumption B1 holds, and the largest value of $\Theta(k_0)$ is $\overline{\theta} < \infty$. This would immediately imply that $\theta_0 = \overline{\theta}$. Figure 10 illustrates this result for the exponential distribution. As shown, if Assumption B1 holds with $k_0 = 2$ and if we know that $\{.25, .50, .75, 1.00, 1.25\} \subset \Theta(k_0)$, then it will follow immediately that $\theta_0 \geq 1.25$. More generally, as in previous sections, the characterization of the identified set $\Theta(k)$ is amenable to recently developed set inference methods.
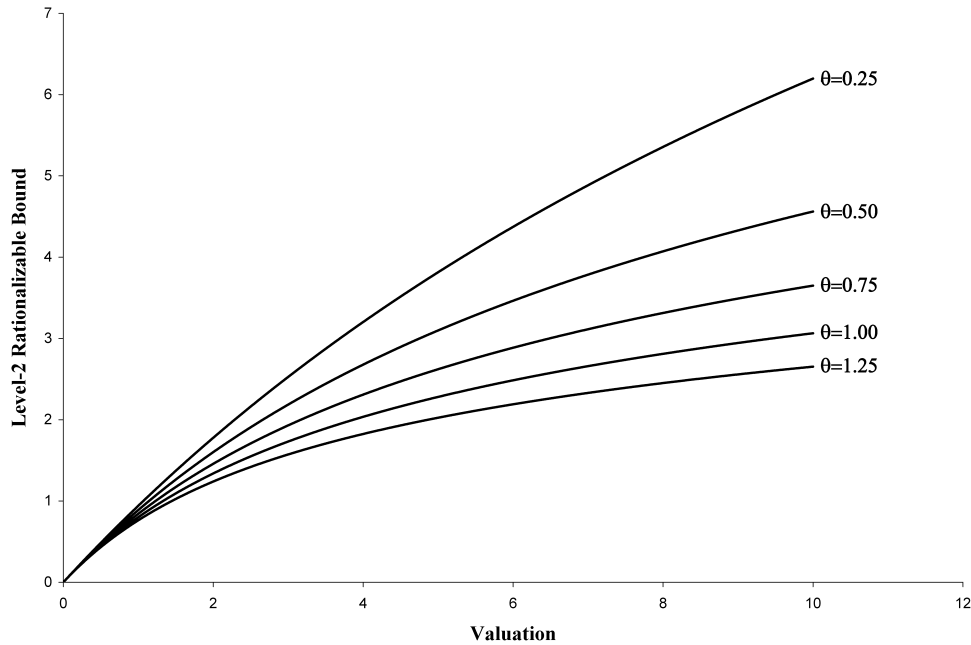


Figure 10. Level-2 upper bound for rationalizable bids for valuations with exponential distribution $F(v) = 1 - e^{-\theta v}$ and various levels of $\theta$ (number of bidders is 2). If Assumption B1 holds with $k_0 = 2$ and if we knew that $\{.25, .50, .75, 1.00, 1.25\} \subset \Theta(k_0)$, then it would follow immediately that $\theta_0 \geq 1.25$.

*Remark 3.* Let

$$\overline{B}_\infty(\cdot; \mathcal{N}|\theta) = \lim_{k \to \infty} \overline{B}_k(\cdot; \mathcal{N}|\theta). \qquad (34)$$

Given our assumptions, the results of BS can be used to show that $\overline{B}_\infty(\cdot; \mathcal{N}|\theta)$ exists and is a continuous, increasing, concave, and invertible mapping that satisfies $\overline{B}_\infty(\cdot; \mathcal{N}|\theta) \geq b^{\mathrm{BNE}}(\cdot; \mathcal{N}|\theta)$. Note that, unlike in the incomplete information game, here rationalizable behavior does not converge to BNE as $k \to \infty$; in particular, bidding below BNE is rationalizable for arbitrarily large $k$. If the rationality bound $k_0$ described in Assumption B1 does not exist, then we must have $b_i \leq \overline{B}_\infty(v_i; \mathcal{N}|\theta)$ with probability 1. The results in Proposition 3 will follow if the conditions stated there hold for the mapping $\overline{B}_\infty(\cdot; \mathcal{N}|\theta)$.

*5.3.1 Identification When Only Winning Bids Are Observed.* Suppose now that for each auction, we observe only the winning bid and the number of *actual* (as opposed to potential) entrants. We defer the introduction of nonzero reserve prices by the seller to the next section. In particular, we observe that

$$b^* = \max_{i=1,\dots,\mathcal{N}} b_i. \qquad (35)$$

Under these conditions, it follows from the monotonic nature of rationalizable upper bounds that if bidders are level-$k$ rational, with probability 1, then

$$b^* \equiv \max_{i=1,\dots,\mathcal{N}} b_i \leq \max_{i=1,\dots,\mathcal{N}} \overline{B}_k(v_i; \mathcal{N}|\theta_0)$$

$$= \overline{B}_k\left(\max_{i=1,\dots,\mathcal{N}} v_i; \mathcal{N}|\theta_0\right) \equiv \overline{B}_k(v^*; \mathcal{N}|\theta_0). \qquad (36)$$

Then we must have

$$\Pr(b^* \leq b) \geq \Pr(\overline{B}_k(v^*; \mathcal{N}|\theta_0) \leq b) \qquad \forall b \in \mathbb{R}. \qquad (37)$$

Because private values are iid, it follows that $v^* \sim F(\cdot; \theta_0)^{\mathcal{N}}$. Let $F_{b^*}(\cdot)$ denote the distribution function of $b^*$, the highest bid. Equation (37) then becomes

$$F_{b^*}(b) \geq F(\overline{S}_k(b; \mathcal{N}|\theta_0); \theta_0)^{\mathcal{N}} \qquad \forall b \in \mathbb{R}, \qquad (38)$$

where, as before, $\overline{S}_k(\cdot; \mathcal{N}|\theta)$ denotes the inverse function of the upper bound $\overline{B}_k(\cdot; \mathcal{N}|\theta)$. Clearly, by the nondecreasing properties of distribution functions, (38) holds for all $b \in \mathbb{R}$ if and only if it holds for all $b \in \mathbb{S}(b^*)$ (the support of $b^*$). We conclude that this implies that

$$F_{b^*}(b) \geq F(\overline{S}_k(b; \mathcal{N}|\theta_0); \theta_0)^{\mathcal{N}} \qquad \forall b \in \mathbb{S}(b^*). \qquad (39)$$

Equation (39) can be used as earlier to conduct inference on the set of consistent models. To do this, an objective function similar to that in Proposition 2 can be used. The results in Proposition 3 also would follow if Assumption B1 held for $b^*$. It appears that even if B1 were assumed to hold for all bids, then we would have to explicitly assume that it holds for $b^*$, because with heterogeneous beliefs, it is no longer true that the highest bid corresponds to the highest valuation among potential bidders.

## 5.4 Introducing a Binding Reserve Price

Suppose that there is a nonzero reserve price $p_0$ set by the seller and publicly observed by all potential buyers. We modify Assumption B0 accordingly as follows.

*Assumption B0′.* Assume now that all bidders expect any bid $b \geq p_0$ to win with strictly positive probability, and this is common knowledge. The implication of this for submitted bids is that $b_i \geq p_0$ if and only if $v_i \geq p_0$. We restrict attention to beliefs that assign positive probability only to bidding functions that are increasing for all $v \geq p_0$ and are equal to $p_0$ for $v = p_0$. Formally, let $\mathcal{B}(p_0)$ denote the space of all Borel-measurable functions of the form

$$\left\{ b : [0, \omega) \to \mathbb{R}_+ : b(v) < p_0 \, \forall v < p_0; \right.$$
$$b(p_0) = p_0, \text{ and for all } v > p_0 : b(v) \leq v$$
$$\left. \text{and } v > v' \Rightarrow b(v) > b(v') \right\}. \qquad (40)$$

We let $\mathcal{N}$ denote the number of potential bidders in the population and denote $\mathcal{B}_{-i}(p_0) = \mathcal{B}(p_0)^{\mathcal{N}-1}$. Beliefs for bidder $i$ are probability distributions defined over a sigma-algebra $\Delta_{\mathcal{B}_{-i}}(p_0)$, where this sigma-algebra is such that singletons in $\mathcal{B}_{-i}$ are measurable. As before, conjectures are defined as degenerate beliefs that assign probability mass 1 to a singleton $\{b_j\}_{j \neq i} \in \mathcal{B}_{-i}$. We maintain the assumption that $F_0(\cdot)$ and $\mathcal{N}$ are common knowledge among potential bidders.

A consequence of a binding reserve price is that the number of potential bidders $\mathcal{N}$ no longer may be equal to the number of bidders who participate in the auction. Potential bidders with valuation $v_i < p_0$ will not submit a bid. Beliefs for valuations $v < p_0$ will be irrelevant for participating bidders, except for the fact that it is common knowledge that $v_j < p_0$ implies that $b_j < p_0$ with probability 1 for all potential bidders. As in the case of zero reservation price, restricting attention to beliefs in $\mathcal{B}(p_0)$ will yield rationalizable upper bounds that also belong in $\mathcal{B}(p_0)$. It also rules out ties in the characterization of expected utility for bidders with valuation $v \geq p_0$ (the only ones who participate in the auction). As in the case of zero reservation price, restricting attention to beliefs in $\mathcal{B}(p_0)$ will imply that BNE-optimal bids are always rationalizable.

*5.4.1 Level-k Rationalizable Bids With a Nonzero Reserve Price.* The construction of rationalizable upper bounds will follow the same interim-rationalizability steps as in Section 5.2. Any bidder $i$ with $v_i \geq p_0$ whose bids satisfy

$$b \leq v_i \quad \text{with probability 1} \qquad (41)$$

is called *level-1 rational*. Higher-rationality levels are characterized as before. The decision problem for any bidder $i$ with $v_i \geq p_0$ now can be expressed as

$$\max_{b \geq p_0} (v_i - b) \widehat{\Pr}_i\left[\max\left\{p_0, \max_{j \neq i} b(v_j)\right\} \leq b\right], \qquad (42)$$

where $\widehat{\Pr}_i(\cdot)$ denotes bidder $i$'s subjective probability, derived from his beliefs and knowledge of $F_0(\cdot)$. The optimal bid for any assessment in $\mathcal{B}_{-i}(p_0)$ for any bidder with $v_i = p_0$ will always be $v_i = p_0$. Focusing on the case where $v_i > p_0$, the most pessimistic assessment in $\mathcal{B}_{-i}(p_0)$ is given by the conjecture

that "$b(v_j) = v_j$ for all $j \neq i$ such that $v_j \geq p_0$." The optimal expected utility for this assessment is

$$\max_{b \geq p_0}(v_i - b)F_0(b)^{\mathcal{N}-1} \equiv \pi_2^*(v_i; \mathcal{N}, p_0), \qquad (43)$$

which follows because $\widehat{\text{Pr}}_i[\max\{p_0, \max_{j \neq i} v_j\} \leq b] = F_0(b)^{\mathcal{N}-1}\mathbb{1}\{b \geq p_0\}$. (Recall that $F_0$, $\mathcal{N}$, and $p_0$ are common knowledge among bidders.) Using the same arguments that followed (24), level-2 rational bidders with $v_i \geq p_0$ must satisfy

$$p_0 \leq b \leq v_i - \pi_2^*(v_i; \mathcal{N}, p_0) \equiv \overline{B}_2(v_i; \mathcal{N}, p_0). \qquad (44)$$

$\overline{B}_2(v_i; \mathcal{N}, p_0)$ is the level-1 rationalizable upper bound for all bidders with $v_i \geq p_0$. It is continuous, increasing, and invertible for all $v_i \geq p_0$, with $\overline{B}_2(p_0; \mathcal{N}, p_0) = p_0$. In particular, the inverse function of $\overline{B}_2(\cdot; \mathcal{N}, p_0)$ is well defined for all values and bids $\geq p_0$. As before, we denote this inverse function by $\overline{S}_2(\cdot; \mathcal{N}, p_0)$. Note that in general, (43) has corner solutions; that is, there exists a range of valuations $v_i > p_0$ such that $\pi_2^*(v_i; \mathcal{N}, p_0) = (v_i - p_0)F_0(p_0)^{\mathcal{N}-1}$. This of course will not impact the continuity, monotonicity, and invertibility properties of the upper bound $\overline{B}_2(\cdot; \mathcal{N}, p_0)$ for values $v_i \geq p_0$. Nothing can be said about rationalizable upper bounds for $v_i < p_0$, except that they lie strictly beneath $p_0$. Bounds for such a range of valuations are irrelevant for the optimal decision process of bidders. Proceeding iteratively, the *level-k bound for rationalizable bids* is given by

$$b_i \leq v_i - \underline{\pi}_k^*(v_i; \mathcal{N}, p_0) \equiv \overline{B}_k(v_i; \mathcal{N}, p_0), \quad \text{where}$$

$$\underline{\pi}_k^*(v_i; \mathcal{N}) = \max_{b \geq p_0}(v_i - b)F_0(\overline{S}_{k-1}(b; \mathcal{N}, p_0))^{\mathcal{N}-1} \qquad (45)$$

and $\overline{S}_{k-1}(\cdot; \mathcal{N}, p_0)$ is the inverse function of $\overline{B}_{k-1}(\cdot; \mathcal{N}, p_0)$, well defined for all values and bids $\geq p_0$.

*5.4.2 Identification With Level-k Rationality When Only Winning Bids Are Observed.* If we replace Assumption B0 with B0′, then all of the results in Section 5.3.1 hold with a binding reserve price for all $v_i \geq p_0$ and $b_i \geq p_0$. Consider a semiparametric setting such as that described in (29), where the distribution of valuations is allowed to depend on the publicly observed reserve price $p_0$,

$$\mathcal{F}_v^{\Theta, p_0} = \{F(\cdot; \theta, p_0) : \theta \in \Theta, \text{ and } F_0(\cdot; p_0) = F(\cdot; \theta_0, p_0)$$
$$\text{for some } \theta_0 \in \Theta\}. \qquad (46)$$

Let $\overline{B}_k(\cdot; \mathcal{N}|\theta, p_0)$ denote the level-$k$ upper bound for rationalizable bids that would be induced by a given distribution $F(\cdot; \theta, p_0) \in \mathcal{F}_v^{\Theta, p_0}$, and let $\overline{B}_k(\cdot; \mathcal{N}|\theta, p_0)$ denote its inverse function. Let $b^*$ denote the winning bid, and let $F_{b^*}(\cdot; p_0)$ denote its distribution function (given $p_0$). Note that $b^*$ is $\max_{i=1,\dots,\mathcal{N}} b_i$, truncated from below at $p_0$. This automatic truncation ensures that the bounds in (45) are satisfied. As mentioned previously, bids below $p_0$ may not satisfy these bounds. If bidders are level-$k$ rational, then for any reserve price $p_0$, we must have

$$F_{b^*}(b; p_0) \geq F(\overline{S}_k(b; \mathcal{N}|\theta_0, p_0); \theta_0, p_0)^{\mathcal{N}} \qquad \forall b \in \mathbb{S}(b^*|p_0),$$
$$(47)$$

where $\mathbb{S}(b^*|p_0)$ is the support of $b^*$ given $p_0$. This result is the equivalent of (39).

*Proposition 4.* Suppose that $F_0$ belongs to a space of distribution functions as described in (46). Moreover, suppose that we have a random sample of size $L$ of auctions, each of which has $\mathcal{N}$ bidders and where we observe only the winning bid in every auction. Let the reservation price $p_0$ be known. Define

$$\Lambda(\theta|a, c; k, p_0)$$
$$= \int \left(1 - \mathbb{1}\{F_{b^*}(b; p_0) \geq F(\overline{S}_k(b; \mathcal{N}|\theta, p_0); \theta, p_0)^{\mathcal{N}}\}\right)$$
$$\times \mathbb{1}\{a \leq b \leq c\}\, dF_{b^*}(b; p_0), \qquad (48)$$

$$\Gamma(\theta|k, p_0) = \int\int \Lambda(\theta|a, c; k, p_0)\, dF_{b^*}(a; p_0)\, dF_{b^*}(c; p_0).$$

Then, under the sole assumption that all bidders are level-$k$ rational, the identified set is

$$\Theta(k, p_0) = \{\theta \in \Theta : \Gamma(\theta|k, p_0)^2 = 0\}.$$

Now suppose that we assume that winning bids satisfy Assumption B1 for some $k_0$. For any $\theta \in \Theta$, let

$$\mathcal{F}^c(\theta, p_0) = \{\theta \in \Theta : \overline{B}_{k_0}(v_i; \mathcal{N}|\theta', p_0) < \overline{B}_{k_0}(v_i; \mathcal{N}|\theta, p_0)$$
$$\text{with probability } 1\},$$
$$\Theta_0^*(p_0) = \{\theta \in \Theta(k_0, p_0) : \nexists \theta' \in \Theta(k_0, p_0)$$
$$\text{such that } \theta' \in \mathcal{F}^c(\theta, p_0)\}. \qquad (49)$$

Then the identified set is

$$\Theta_0^* = \{\theta \in \Theta : \theta \in \Theta_0^*(p_0)$$
$$\text{with probability } 1 \text{ (with respect to } p_0)\}. \qquad (50)$$

Note that the identification result in (50) requires that we explicitly assume that Assumption B1 holds for winning bids. With a nonzero reserve price, the number of actual bidders in a given auction may differ from $\mathcal{N}$. The characterization of the identified set in Proposition 4 still can be constructive in this case if we assume that $\mathcal{N}$ is the same across all auctions in the population, and if the number of actual bidders is observed. For the $\ell$th auction, denote the latter by $I_\ell$. Therefore, $I_\ell = \sum_{i=1}^{\mathcal{N}} \mathbb{1}\{v_i \geq p_{0_\ell}\}$ and

$$E[I_\ell] = \mathcal{N}E_{p_{0_\ell}}\left[F(p_{0_\ell}; \theta_0, p_{0_\ell})\right]$$

$$\Rightarrow \quad \mathcal{N} = \frac{E[I_\ell]}{E_{p_{0_\ell}}[F(p_{0_\ell}; \theta_0, p_{0_\ell})]}, \qquad (51)$$

where $E_{p_{0_\ell}}[\cdot]$ denotes the expectation taken with respect to the reserve price, which is assumed to be observed for any given auction. The foregoing result is the basis for identifying $\mathcal{N}$. It then follows that Proposition 4 is a constructive identification result.

*5.4.3 Identification Results for the Rationality Level $k_0$ in Assumption B1.* Suppose we assume that there exists a finite $k_0 \geq 2$ that satisfies the conditions of Assumption B1 (otherwise see Remark 3). The results in Proposition 3 are constructive when $k_0 \geq 2$ is assumed to be known. Naturally, we would be interested in having an identification result for both $\theta$ and $k_0$ simultaneously.

*Proposition 5.* Let $\Theta(k)$ and $\Gamma(\theta|k)$ be as defined in Proposition 2. Define

$$\underline{\Gamma}(k) = \min_{\theta \in \Theta(2)} \Gamma(\theta|k)^2. \qquad (52)$$

Then, if Assumption B1 is satisfied with $k_0 \geq 2$, the following results hold:

   a. $\underline{\Gamma}(k) = 0$ for all $k \leq k_0$; however, $\underline{\Gamma}(k) = 0$ does not imply that $k \leq k_0$.
   b. $\underline{\Gamma}(k) > 0$ implies that $k > k_0$.

It follows from Proposition 5 that any $k'$ such that $\underline{\Gamma}(k') > 0$ can be ruled out as the true $k_0$ described in Assumption B1, implying that there is a subset of bidders who are strictly less than level-$k'$ rational. At the same time, the set $\{k \in \mathbb{N} : \underline{\Gamma}(k) = 0\}$ includes all $k \leq k_0$ and also includes some values $k > k_0$.

## 6. CONCLUSION

In structural econometrics models, assumptions are implicitly grouped into behavioral assumptions and other auxiliary assumptions. Behavioral assumptions usually are unchallenged in identification analysis, and thus econometricians focus on the robustness of estimation results to those auxiliary assumptions (which are not implied by theory, such as functional forms and distributional assumptions). In this article we have explored the identifying role that some behavioral assumptions play. Mainly, we examined the identification power of equilibrium in three simple games. We replaced equilibrium with a form of rationality (i.e., interim rationalizability) that includes equilibrium as a special case, and compared the identified features of the game under rationality and under equilibrium. The games that we studied are stylized versions of empirical models considered and applied in the literature, and thus insights provided here can be carried over to those empirical frameworks. We do not advocate dropping the equilibrium assumptions from empirical work, however; rather, we have simply examined the identifying power of equilibrium in these simple setups. For example, it is not clear that we would want to drop equilibrium in a first-price auction, because the underlying interim-rationalizability based model may not provide strong restrictions on the observed bids as they relate to the underlying valuations. Ultimately, the researcher faces the usual trade-off between robustness and predictive power, requiring a balancing act guided by the economics of the particular application at hand. We also do not advocate using rationalizability per se as the basis for strategic interaction. Other frameworks are available in the literature, but, we note that the form of rationalizability used here has received much attention from game theorists (see, e.g., Morris and Shin 2003; Dekel et al. 2007; and references cited therein). Moreover, interim rationalizability allows us to incorporate the concept of higher-order beliefs into the econometric analysis through what we have defined here as rationality levels.

Some questions remain to be answered, and we leave these for ongoing and future work. As far as our results here, we are concerned with identification. A natural extension would be to study the statistical properties of estimators proposed herein and apply those estimators in empirical examples. Another important question is the issue of sharpness and whether the inequality-based inference procedures implied by the model deliver wide identified sets for parameters (as compared to the sharp identified sets). These interference procedures are attractive because of they lead to simple to compute estimators. More work needs to be done to look for other estimators that deliver sharp inferences. Another avenue of research is to extend some of the ideas to dynamic setups. It is well known that inference in dynamic games is difficult when one tries to account for the presence of multiple equilibria. Recent important contributions to this field have been made by Aguiregabiria and Mira (2007), Bajari, Benkard, and Levin (2007), Pesendorfer and Schmidt-Dengler (2004), and Pakes, Ostrovsky, and Berry (2005) (see also Berry and Tamer 1996). The identification question is complicated mainly due to the complexity of the underlying economic model and say beliefs off the equilibrium, where no data are available. From a practical perspective, estimating dynamic games while allowing for generation of different data points by a different equilibrium is a difficult problem because it involves solving for multiple fixed points in a complicated nonlinear problem; thus relaxing equilibrium in this setting might lead to enormous computational advantages because there is not need to solve for these fixed points. It also may be possible to examine the identification power of other strategic concepts that would be natural in dynamic settings, such as the self-confirming equilibria of Fudenberg and Levine (1993). In addition to examining the robustness to equilibrium assumptions, these identification framework can be used to study whether inference under these different strategic frameworks is more practically useful for applied researchers. We leave these topics for future research.

## ACKNOWLEDGMENTS

## APPENDIX: PROOFS

### Proof of Theorem 1

From our previous analysis, we know that both players are level-1 rational if and only if, with probability one in $\mathbb{S}(X)$,

$$\Pr(\varepsilon_1 > X_1'\beta_1, \varepsilon_2 > X_2'\beta_2|X)$$
$$\leq \Pr(Y_1 = 0, Y_2 = 0|X)$$
$$\leq \Pr(\varepsilon_1 > X_1'\beta_1 + \alpha_1, \varepsilon_2 > X_2'\beta_2 + \alpha_2|X),$$
$$\Pr(\varepsilon_1 \leq X_1'\beta_1 + \alpha_1, \varepsilon_2 > X_2'\beta_2|X)$$
$$\leq \Pr(Y_1 = 1, Y_2 = 0|X)$$
$$\leq \Pr(\varepsilon_1 \leq X_1'\beta_1, \varepsilon_2 > X_2'\beta_2 + \alpha_2|X),$$
$$\Pr(\varepsilon_1 > X_1'\beta_1, \varepsilon_2 \leq X_2'\beta_2 + \alpha_2|X)$$
$$\leq \Pr(Y_1 = 0, Y_2 = 1|X)$$

$$(A.1)$$

$$\leq \Pr(\varepsilon_1 > X_1'\beta_1 + \alpha_1, \varepsilon_2 \leq X_2'\beta_2|X),$$

$$\Pr(\varepsilon_1 \leq X_1'\beta_1 + \alpha_1, \varepsilon_2 \leq X_2'\beta_2 + \alpha_2|X)$$

$$\leq \Pr(Y_1 = 1, Y_2 = 1|X)$$

$$\leq \Pr(\varepsilon_1 \leq X_1'\beta_1, \varepsilon_2 \leq X_2'\beta_2|X).$$

We denote the true parameter value by $\theta_0$. To prove part a, take any $\widetilde{\beta}_1 \neq \beta_{1_0}$ such that $\widetilde{\beta}_{\ell,1} \neq \beta_{\ell,1_0}$. Given this and the support properties of $X_{\ell,1}$, for any scalar $d$, we can observe either of the following two events with positive probability: (a) $X_1'\widetilde{\beta}_1 + d > X_1'\beta_{1_0}$ or (b) $X_1'\widetilde{\beta}_1 < X_1'\beta_{1_0} + \alpha_{1_0}$. Take case (a) first. With $d = \alpha_1$ (arbitrary), if $\widetilde{\beta}_{2,\ell}\beta_{2,\ell_0} > 0$, then we can make $\widetilde{\beta}_2'X_2 \to +\infty$ and $\beta_{2_0}'X_2 \to +\infty$. By Assumption (A1), this yields $\Pr(\varepsilon_1 \leq X_1'\widetilde{\beta}_1 + \alpha_1, \varepsilon_2 \leq X_2'\widetilde{\beta}_2 + \alpha_2|X) \to \Pr(\varepsilon_1 \leq X_1'\widetilde{\beta}_1 + \alpha_1|X)$ and $\Pr(\varepsilon_1 \leq X_1'\beta_{1_0}, \varepsilon_2 \leq X_2'\beta_{2_0}|X) \to \Pr(\varepsilon_1 \leq X_1'\beta_{1_0}|X) < \Pr(\varepsilon_1 \leq X_1'\widetilde{\beta}_1 + \alpha_1|X)$. Therefore, with positive probability as $X_2$ explodes, $\Pr(\varepsilon_1 \leq X_1'\widetilde{\beta}_1 + \alpha_1, \varepsilon_2 \leq X_2'\widetilde{\beta}_2 + \alpha_2|X) > \Pr(\varepsilon_1 \leq X_1'\beta_{1_0}, \varepsilon_2 \leq X_2'\beta_{2_0}|X) > \Pr(Y_1 = 1, Y_2 = 1|X)$, which violates (A.1). If $\widetilde{\beta}_{2,\ell}\beta_{2,\ell_0} < 0$, then the result is easier to obtain by making $\widetilde{\beta}_2'X_2 \to +\infty$ and $\beta_{2_0}'X_2 \to -\infty$. For case (b), drive $\widetilde{\beta}_2'X_2 \to -\infty$ and $\beta_{2_0}'X_2 \to -\infty$ if $\widetilde{\beta}_2\beta_{2_0} > 0$, or $\widetilde{\beta}_2'X_2 \to -\infty$ and $\beta_{2_0}'X_2 \to +\infty$ if $\widetilde{\beta}_2\beta_{2_0} > 0$. In either case, we eventually obtain $\Pr(\varepsilon_1 > X_1'\widetilde{\beta}_1, \varepsilon_2 > X_2'\widetilde{\beta}_2|X) > \Pr(\varepsilon_1 > X_1'\beta_{1_0} + \alpha_{1_0}, \varepsilon_2 > X_2'\beta_{2_0} + \alpha_{2_0}|X) > \Pr(Y_1 = 0, Y_2 = 0|X)$, which violates (A.1). This establishes the identification of $\beta_{\ell,1}$; an analog proof shows that $\beta_{\ell,2}$ is identified, which proves part a.

To establish part b, focus on the worst-case scenario and take $\widetilde{\theta} \neq \theta_0$ where $\widetilde{\beta}_{d,p} \neq \beta_{d,p_0}$ but $\widetilde{\beta}_{\ell,p} = \beta_{\ell,p_0}$ for $p = 1, 2$; the parameters of the unbounded-support shifters are fixed at their true values. Here identification must rely on the properties of $X_{d,p}$, the bounded-support shifters. The condition in the statement of the proposition ensures that (a) or (b) hold even if we fix $\widetilde{\beta}_{\ell,p} = \beta_{\ell,p_0}$. To complete the proof of b we proceed as in the previous paragraph. (Note that we now have $\widetilde{\beta}_{\ell,p}\beta_{\ell,p_0} > 0$.) The case where $\widetilde{\beta}_{d,p} \neq \beta_{d,p_0}$ and $\widetilde{\beta}_{\ell,p} \neq \beta_{\ell,p_0}$ is straightforward along the same lines.

Now we proceeded to part c. Consider $\widetilde{\theta}$ to be equal to $\theta_0$ element-by-element except for $\widetilde{\alpha}_1 \neq \alpha_{1_0}$, and recall that the parameter space of interest has $\alpha_p \leq 0$. Clearly, none of the lower bounds in (A.1) evaluated at $\widetilde{\theta}$ will ever be larger than the corresponding upper bounds evaluated at $\theta_0$, and none of the upper bounds evaluated at $\widetilde{\theta}$ will ever be smaller than the corresponding lower bounds evaluated at $\theta_0$. Therefore, without further assumptions, $\widetilde{\theta}$ and $\theta_0$ are observationally equivalent and $\alpha_1$ is not identified. The only way that we can proceed is by adding more structure on $\Pr(Y_1, Y_2|X)$. We have $\Pr(\varepsilon_1 \leq X_1'\beta_{1_0} + \alpha_{1_0}) \leq \Pr(Y_1 = 1|X) \leq \Pr(\varepsilon_1 \leq X_1'\beta_{1_0})$; therefore, $\Pr(\varepsilon_1 \leq X_1'\beta_{1_0} + \widetilde{\alpha}_1) > \Pr(Y_1 = 1|X)$ only if $\widetilde{\alpha}_1 > \alpha_{1_0}$. Thus $\widetilde{\theta}$ can violate (A.1) only if $\widetilde{\alpha}_1 > \alpha_{1_0}$. For any such $\widetilde{\alpha}_1$, let $\Delta = \Pr(\varepsilon_1 \leq X_1'\beta_{1_0} + \widetilde{\alpha}_1) - \Pr(\varepsilon_1 \leq X_1'\beta_{1_0} + \alpha_{1_0}) > 0$. By the assumption in part c, there exists a subset $\mathcal{X}_1 \in \mathbb{S}(X_1)$ such that $\Pr(Y_1 = 1|X) < \Pr(\varepsilon_1 \leq X_1'\beta_{1_0} + \alpha_{1_0}) + \Delta = \Pr(\varepsilon_1 \leq X_1'\beta_{1_0} + \widetilde{\alpha}_1)$. Make $X_2'\beta_{2_0} \to +\infty$, and the lower bound on the fourth inequality in (A.1) will be violated. This establishes part c. Any $\widetilde{\theta} \neq \theta_0$ where $\widetilde{\alpha}_p \neq \alpha_{p_0}$ and either $\widetilde{\beta}_{\ell,p} \neq \beta_{\ell,p_0}$ or $\widetilde{\beta}_{d,p} \neq \beta_{d,p_0}$ can be shown to be not observationally equivalent to $\theta_0$ using the same arguments as in the previous paragraphs given the assumptions in parts a and b.

## Proof of Theorem 2

Suppose that there exists a subset of realizations in $\overline{\mathcal{X}}_1^* \subset \mathcal{X}_1^*$ such that

$$X_1'\beta_1 + \Delta_1 + \alpha_1 > X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0} \qquad \forall X_1 \in \overline{\mathcal{X}}_1^*. \quad \text{(A.2)}$$

By continuity of the linear index and of the distribution $H_1$, for any $X_1 \in \overline{\mathcal{X}}_1^*$, we can find a pair $0 \leq \overline{p}^L(X_1) < \overline{p}^U(X_1) \leq 1$ such that

$$H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\overline{p}^L(X_1)\big)$$
$$< H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\overline{p}^U(X_1)\big). \quad \text{(A.3)}$$

To see why $\overline{p}^L(X_1)$ and $\overline{p}^U(X_1)$ exist, fix $\overline{p}^U(X_1) = 1$. By continuity, there exists a small enough $\delta > 0$ such that $\overline{p}^L(X_1) \geq 1 - \delta$ satisfies (A.3). If condition (17) in Theorem 2 holds, then there exists $\mathcal{W}_1^* \subset \mathbb{S}(W_1)$ such that

$$\min\big\{E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1],$$
$$E\big[H_2\big(X_2'\beta_{2_0} + \Delta_{2_0} + \alpha_{2_0}\big)|\mathcal{I}_1\big]\big\}$$
$$\geq \overline{p}^L(X_1) \qquad \forall W_1 \in \mathcal{W}_1^*, \quad \text{(A.4)}$$
$$\max\big\{E[H_2(X_2'\beta_2 + \Delta_2)|\mathcal{I}_1], E\big[H_2\big(X_2'\beta_{2_0} + \Delta_{2_0}\big)|\mathcal{I}_1\big]\big\}$$
$$\leq \overline{p}^U(X_1) \qquad \forall W_1 \in \mathcal{W}_1^*.$$

Note trivially that because $\alpha_p \leq 0$ everywhere in $\Theta$, we have that

$$\min\big\{E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1],$$
$$E\big[H_2\big(X_2'\beta_{2_0} + \Delta_{2_0} + \alpha_{2_0}\big)|\mathcal{I}_1\big]\big\}$$
$$\leq \max\big\{E[H_2(X_2'\beta_2 + \Delta_2)|\mathcal{I}_1], E\big[H_2\big(X_2'\beta_{2_0} + \Delta_{2_0}\big)|\mathcal{I}_1\big]\big\}$$
$$\text{with probability 1.}$$

By definition, we have that

$$E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1] = \pi_2^L(\theta|k=2; \mathcal{I}_1) \qquad \text{and}$$
$$E\big[H_2\big(X_2'\beta_{2_0} + \Delta_{2_0}\big)|\mathcal{I}_1\big] = \pi_2^U(\theta_0|k=2; \mathcal{I}_1). \quad \text{(A.5)}$$

Combining (A.4) and (A.5), we have that

$$\pi_2^L(\theta|k=2; \mathcal{I}_1) \geq \overline{p}^L(X_1),$$
$$\pi_2^U(\theta_0|k=2; \mathcal{I}_1) \leq \overline{p}^U(X_1) \qquad \forall W_1 \in \mathcal{W}_1^*. \quad \text{(A.6)}$$

Combining (A.3) and (A.6), we have that

$$H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^L(\theta|k=2; \mathcal{I}_1)\big)$$
$$\leq H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\overline{p}^L(X_1)\big)$$
$$< H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\overline{p}^U(X_1)\big)$$
$$\leq H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\pi_2^U(\theta_0|k=2; \mathcal{I}_1)\big)$$
$$\forall W_1 \in \mathcal{W}_1^*. \quad \text{(A.7)}$$

This corresponds to the case described in the first line of (18). Next, suppose that (A.2) does not hold but there exists a subset of realizations $\overline{X}_1^{**} \subset \mathcal{X}_1^*$ such that

$$X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0} > X_1'\beta_1 + \Delta_1 + \alpha_1 \qquad \forall X_1 \in \overline{\mathcal{X}}_1^{**}. \quad \text{(A.8)}$$

Repeating the same arguments as before and exchanging $\theta$ and $\theta_0$, we arrive at the equivalent of (A.7), namely

$$H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\pi_2^L(\theta_0|k=2;\mathcal{I}_1)\big)$$
$$< H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^U(\theta|k=2;\mathcal{I}_1)\big)$$
$$\forall W_1 \in \mathcal{W}_1^{**}. \tag{A.9}$$

This corresponds to the case described in the second line of (18). The last remaining possibility is that neither (18) nor (A.8) holds. In this case,

$$X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0} = X_1'\beta_1 + \Delta_1 + \alpha_1 \qquad \forall X_1 \in \mathcal{X}_1^*. \tag{A.10}$$

Because $X_1$ has full column rank in $\mathcal{X}_1^*$, (A.10) implies that $\beta_{1_0} = \beta_1$ and $\Delta_{1_0} + \alpha_{1_0} = \Delta_1 + \alpha_1$. Because $\theta_1 \neq \theta_{1_0}$, we must have either

$$\Delta_1 > \Delta_{1_0} \qquad \text{or} \qquad \Delta_1 < \Delta_{1_0}. \tag{A.11}$$

Suppose that $\Delta_1 > \Delta_{1_0}$. This immediately yields $X_1'\beta_1 + \Delta_1 > X_1'\beta_{1_0} + \Delta_{1_0}$ for all $X_1 \in \mathcal{X}_1^*$. By continuity, we can find a pair $0 \leq \underline{p}^L(X_1) < \underline{p}^U(X_1) \leq 1$ such that

$$H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\underline{p}^U(X_1)\big)$$
$$> H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\underline{p}^L(X_1)\big). \tag{A.12}$$

To see why $\underline{p}^L(X_1)$ and $\underline{p}^U(X_1)$ exist, fix $\underline{p}^L(X_1) = 0$. By continuity, there exists a small enough $\delta > 0$ such that $\overline{p}^U(X_1) \leq \delta$ satisfies (A.12). If condition (17) in Theorem 2 holds, then there exists $\mathcal{W}_1^{***} \subset \mathbb{S}(W_1)$ such that

$$\min\big\{E[H_2(X_2'\beta_2 + \Delta_2 + \alpha_2)|\mathcal{I}_1],$$
$$E\big[H_2(X_2'\beta_{2_0} + \Delta_{2_0} + \alpha_{2_0})|\mathcal{I}_1\big]\big\}$$
$$\geq \underline{p}^L(X_1) \qquad \forall W_1 \in \mathcal{W}_1^{***}, \tag{A.13}$$
$$\max\big\{E[H_2(X_2'\beta_2 + \Delta_2)|\mathcal{I}_1], E\big[H_2\big(X_2'\beta_{2_0} + \Delta_{2_0}\big)|\mathcal{I}_1\big]\big\}$$
$$\leq \underline{p}^U(X_1) \qquad \forall W_1 \in \mathcal{W}_1^{***}.$$

Using the definitions of $\pi_2^L(\theta|k=2;\mathcal{I}_1)$ and $\pi_2^U(\theta_0|k=2;\mathcal{I}_1)$ [e.g., eq. (A.5)], we obtain

$$\pi_2^U(\theta|k=2;\mathcal{I}_1) \leq \underline{p}^U(X_1),$$
$$\pi_2^L(\theta_0|k=2;\mathcal{I}_1) \geq \underline{p}^L(X_1) \qquad \forall W_1 \in \mathcal{W}_1^{***}. \tag{A.14}$$

Using (A.12), this yields

$$H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^U(\theta|k=2;\mathcal{I}_1)\big)$$
$$> H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\pi_2^L(\theta_0|k=2;\mathcal{I}_1)\big)$$
$$\forall W_1 \in \mathcal{W}_1^{***}. \tag{A.15}$$

This corresponds to a case like that described in the second line of (18). If $\Delta_1 < \Delta_{1_0}$, then the same arguments as before while exchanging $\theta$ with $\theta_0$ would lead us to conclude that there exists a set $\mathcal{W}_1^{4*} \subset \mathbb{S}(W_1)$ such that

$$H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\pi_2^L(\theta_0|k=2;\mathcal{I}_1)\big)$$
$$> H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^U(\theta|k=2;\mathcal{I}_1)\big)$$
$$\forall W_1 \in \mathcal{W}_1^{4*}. \tag{A.16}$$

We have now established (18) in Theorem 2 for the where case $k = 2$. The cases where $k > 2$ follow immediately by recalling the monotonic property of rationalizable bounds, which says that, with probability 1,

$$H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^L(\theta|k+1;\mathcal{I}_1)\big)$$
$$\leq H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^L(\theta|k;\mathcal{I}_1)\big) \qquad \forall k \geq 1$$

and

$$H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^U(\theta|k+1;\mathcal{I}_1)\big)$$
$$\geq H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^U(\theta|k;\mathcal{I}_1)\big) \qquad \forall k \geq 1.$$

To see why this implies that the rationalizable bounds for player 1's conditional choice probabilities are disjoint with positive probability for all $k \geq 2$, recall that the level-2 bounds are given by

$$\begin{aligned}
&\big[H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^U(\theta|k=2;\mathcal{I}_1)\big),\\
&\quad H_1\big(X_1'\beta_1 + \Delta_1 + \alpha_1\pi_2^L(\theta|k=2;\mathcal{I}_1)\big)\big] \quad \text{(for }\theta\text{)},\\
&\big[H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\pi_2^U(\theta_0|k=2;\mathcal{I}_1)\big),\\
&\quad H_1\big(X_1'\beta_{1_0} + \Delta_{1_0} + \alpha_{1_0}\pi_2^L(\theta_0|k=2;\mathcal{I}_1)\big)\big] \quad \text{(for }\theta_0\text{)}.
\end{aligned} \tag{A.17}$$

It follows from our results that the level-2 rationalizable bounds for $\theta$ are disjoint from those of $\theta_0$ with positive probability. Because the bounds for $k > 2$ are contained in those of $k = 2$ with probability 1, it follows immediately that these bounds are also disjoint for $k > 2$. It follows that if the population of player 1 agents are at least level-2 rational, then any $\theta$ with $\theta_1 \neq \theta_{1_0}$ will produce level-2 bounds that are violated with positive probability. Thus no such $\theta$ can be observationally equivalent to one that has $\theta_1 = \theta_{1_0}$, and, consequently, $\theta_{1_0}$ is identified. Naturally, if the same conditions of Theorem 2 hold when we exchange the subscripts "1" and "2," then $\theta_{2_0}$ will be identified.

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# Comment

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In this article we study the identification of structural parameters in dynamic games when we replace the assumption of Markov perfect equilibrium (MPE) with weaker conditions, such as rational behavior and rationalizability. The identification of players' time discount factors is of especial interest. Identification results are presented for a simple two-period/two-player dynamic game of market entry-exit. Under the assumption of level-2 rationality (i.e., players are rational and know that they are rational), a exclusion restriction and a large-support condition on one of the exogenous explanatory variables are sufficient for point identification of all structural parameters.

## 1. INTRODUCTION

Structural econometric models of individual or firm behavior typically assume that agents are rational in the sense that they maximize expected payoffs given their subjective beliefs about uncertain events. Empirical applications of game-theoretic models have used stronger assumptions than rationality. Most of these studies apply the Nash equilibrium (NE) solution, or some of its refinements, to explain agents' strategic behavior. The NE concept is based on assumptions on players' knowledge and beliefs that are more restrictive than rationality. Although there is no set of necessary conditions for generating the NE outcome, the set of sufficient conditions typically includes the assumption that players' actions are common knowledge. For instance, Aumann and Brandenburger (1995) showed that mutual knowledge of payoff functions and of rationality, along with common knowledge of the conjectures (actions), imply that the conjectures form a NE. But this assumption on players' knowledge and beliefs may be unrealistic in some applications; therefore, it is relevant to study whether the *principle of revealed preference* can identify the parameters in players' payoffs under weaker conditions than NE. For instance, we would like to know whether rationality is sufficient for identification. It is also relevant to study the identification power of other assumptions that are stronger than rationality but weaker than NE, such as common knowledge rationality (e.g., everybody knows that players are rational; everybody knows that everybody knows that players are rational). Common knowledge rationality is