

SOCIAL INTERACTIONS IN LARGE NETWORKS: A GAME THEORETIC APPROACH*

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This article studies estimation of social interactions in a large network game, where all observations come from one single equilibrium of a network game with asymmetric information. Simple assumptions about the structure are made to establish the existence and uniqueness of the equilibrium. I show that the equilibrium strategies satisfy a network decaying dependence condition requiring that dependence between two players' decisions decay with their network distance, which serves as the basis for my statistical inference. Moreover, I establish identification and propose a computationally feasible and efficient estimation method, which is illustrated by an empirical application of college attendance.

1. INTRODUCTION

Over the last decade, network effects on social behaviors has become important in social theory (see, e.g., Granovetter, 1985). In particular, economics has been encouraged to broaden its scope to the analysis of social interactions while maintaining the rigor that is emblematic of economic analysis (Manski, 2000). Recently, game theory has played a central role in this regard, and a leading example is the study of network formation by Bala and Goyal (2000). In this article, I propose a network game of incomplete information to study large network-based social interactions. Simple assumptions about the game structure are made to ensure a unique equilibrium, and the equilibrium satisfies a decaying network effects condition. I then establish identification and estimation of the structural primitives using data from a single large network.

The structure of my network game follows the “preference interaction” approach suggested by Manski (2000). Specifically, a player's payoff from choosing an action over alternatives depends on other players' simultaneous actions as well.² Instead of interacting with all players on the social network, I assume that each agent's payoff is affected by only her choice and the choices of her direct best friends. We call it “local” interactions, a notion that was first introduced by Seim (2006) in the context of industrial organization. Such a specification is parsimonious, but rich enough to generate the interdependence of all agents' choices, which is shaped by the way the network gets connected. For example, teenagers are inclined to be affected by their friends in terms of adolescent risky behaviors (see, e.g., Nakajima, 2007), but such local effects can spread through the network. In particular, they are indirectly affected by the behaviors of

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² In particular, I assume each player's payoff relevant covariates (including friend relationship) are public information, but payoff shocks (i.e., the error terms) are private information of the player.

their friends' friends, because their friends are interacting with each other. In equilibrium, all teenagers from the same connected network will affect each other directly or indirectly.

My local interaction specification differs from the "linear-in-mean" approach widely used in the literature on social interactions (see, e.g., Manski, 1993). The latter captures the notion that an individual's behavior depends on the average behavior of all other social members. The local interaction approach is attractive in the study of large network-based social interactions: First, my model allows us to study counterfactuals and policy effects from the change of network graph under the network decaying dependence (NDD) condition. In contrast, much of the theoretical literature on network interventions has long focused attention on qualitative features like the stability, but not quantitative effects. A second advantage is that an equilibrium in my local interaction model exhibits features that reflect how the large network connects players to each other. Last but not least, peer effects between any pair of friends in my model are not diluted by the large size of the network, which is a typical feature in the linear-in-mean network model literature.

By restricting the interaction strength to be sufficiently mild, I establish the uniqueness of the equilibrium. Uniqueness of the equilibrium is crucial and of particular interest to both theoretical and empirical sides of game theory. In the presence of multiple equilibria, it is difficult to characterize the whole set of equilibria in a large network game: In particular, when the network is not regular, we cannot use a Markov type of equilibrium solution concept to simplify empirical analysis like that in dynamic models. Another more fundamental concern is the "incompleteness" of the econometric model due to the existence of multiple equilibria (see Tamer, 2003).

Although there are strategic interactions among friends in a large network, players' choices are mutually dependent on each other. Intuitively, one would expect such a dependence to decay with the players' network distance. By restricting the strength of peer effects, I show that the dependence of a player's equilibrium strategy on her friends' choices decays (exponentially) with network distance, a so-called *network decaying dependence condition* (NDD condition) that more or less amounts to the restrictions for a stationary solution in the autoregressive model. My NDD condition is related to a number of dependence decay conditions used in the time series and spatial analysis (see, e.g., Jenish and Prucha, 2009). When the data come from the equilibrium of a single large network game, all observations are dependent on each other due to strategic interactions. The NDD implies that any two players' decisions are closer to being independent if they are farther away from each other. The formulation of NDD is novel and serves as the basis for my statistical inference.

For estimation, a key challenge arises because it is costly to solve the equilibrium analytically or numerically. We propose a new approach that approximates the equilibrium solution of a large n -player Bayesian games by solving games of much smaller size, one for each player. Specifically, for player i , a Bayesian game is tailored from the original one by cutting off all players whose network distances from i are larger than h ($h \in \mathbb{N}$). The set of players left on the subnetwork, as well as their payoffs, action space, information structure and so on, defines a smaller sized Bayesian game. I solve this subnetwork game and use the equilibrium solution of player i to approximate her equilibrium strategy in the original large network game. The tuning parameter h is chosen carefully depending on the network size; that is, h increases with the network size at an exponential rate such that approximation errors are negligible with respect to the limiting distribution of the estimator. By this approximation, I then define an approximated maximum likelihood estimator (AMLE), which is asymptotically equivalent to the infeasible MLE. My Monte Carlo experiments results perform well.

It is worth pointing out that my asymptotic analysis is based on the number of players in a single game going to infinity instead of the infinite repetition of the same game with a small fixed number of players. The latter asymptotic approach is used by most of the existing empirical game literature, for example, Bjorn and Vuong (1984), Bresnahan and Reiss (1991), Brock and Durlauf (2001), and Tamer (2003). My asymptotic analysis applies to observations coming from one or a small number of large networks. In a recent paper, Menzel (2015a) characterizes

the asymptotic distribution of a large matching market. The analysis is similar in spirit to my approach in terms of using the limiting distribution as the number of players goes to infinity to approximate the distribution of the equilibrium in a large game. An important difference is that in Menzel (2015a), the strategic effects that cause the endogeneity issue become negligible as the number of players increases to infinity, which is not the case in my asymptotic analysis.

We apply our method to study college attendance decisions of high school students, using the data from the National Longitudinal Study of Adolescent Health (Add Health). The Add Health is a longitudinal survey containing a nationally representative sample of adolescents in the United States during the 1994–5 school year. A unique feature in this data set is the availability of respondents' social network information, which is reconstructed by students' best friends nominations in the survey. Applying the proposed estimation procedure, I find statistically significant, positive peer effects, which has a scale similar to other empirical findings of peer effects on youth behaviors using the same or similar data sets. See, for example, Calvó-Armengol et al. (2009) and Gaviria and Raphael (2001).

The rest of the article is organized as follows. In Section 2, I describe the data and provide descriptive statistics. In Section 3, I introduce the network game model and establish the uniqueness of the Bayesian Nash Equilibrium (BNE) and the NDD condition. In Section 4, I establish identification of the model. In Section 5, I propose an estimation procedure and establish its asymptotic properties. Moreover, I study its finite sample performance in Monte Carlo experiments and then illustrate it by an imperial application. Section 6 concludes, and all the proofs are provided in the Appendix.

2. DATA

I study peer effects on college attendance of high school students by using data from the National Longitudinal Study of Adolescent Health (Add Health), which is a longitudinal survey containing a nationally representative sample of adolescents in the United States during the 1994–5 school year. A unique feature of the Add Health data is the availability of respondents' social network information, as well as their social and economic characteristics (including college attendance): Each respondent provides his or her friendship information by nominating at most five male and female best friends, respectively. Intuitively, one can then reconstruct the whole friendship network among respondents. All the respondents in my empirical study come from three high school networks, and the total number of observations is $n = 831$. A detailed description of the data can be found on the Web site of the Carolina Population Center.³

The college attendance decisions must have been made by individual families during a short period. Following the literature (see, e.g., Christensen et al., 1975; Leslie and Brinkman, 1988), the exogenous covariates that affect college attendance include age, household income, grade point average (GPA), parents' education level, race, gender, etc. Descriptive statistics are presented in Table 1. The demographic variables, that is, Household Income, Mother's Education, and Father's Education, are recorded by some codes. These codes are natural numbers increasing with the actual value of variables. The median Household Income is between \$50,000 and \$74,999. Mother's Education and Father's Education are coded as 1 = never went to school, 2 = not graduate from high school, 3 = high school graduate, 4 = graduated from college or a university, 5 = professional training beyond a four-year college. There is a severe missing data issue in these two variables: I treat missing observations as value 0. For the observed subsample, both the median of Mother's Education and that of Father's Education are high school graduate. Over the whole sample (i.e., including observations with value 0 for Mother's and/or Father's Education), however, both medians are 0.

As a matter of fact, I only use observations from three largest schools. For schools with small numbers of respondents, the missing data issue is severe. Therefore, the descriptive statistics in

³ See <http://www.cpc.unc.edu/projects/addhealth/data>.

TABLE 1
DESCRIPTIVE STATISTICS: THREE SCHOOL NETWORKS; YEAR 1994–5

Variable	Mean	SD	Min	Max
Age	17.088	1.138	15	21
Female	0.502	0.500	0	1
Household Income	8.827	2.122	1	12
Mother's Education*	0.516	1.676	0	11
Father's Education	1.709	2.955	0	12
Overall GPA	2.376	0.772	0.11	4
American Indian**	0.039	0.193	0	1
Asian	0.140	0.347	0	1
Black	0.084	0.278	0	1
Hispanic	0.348	0.477	0	1
White	0.651	0.477	0	1
Other Race	0.153	0.360	0	1
Number of Friends	1.303	1.575	0	8
Network Centrality	1.303	1.780	0	13
College Attendance	0.535	0.499	0	1

*Missing observations have been treated as 0.
**Some observations are associated with more than one race.

Table 1 are slightly different from other studies on social interactions that use the whole Add Health data set (see, e.g., Calvó-Armengol et al., 2009).

The number of friends and the network centrality are two descriptive statistics on the network structure. Player i 's network centrality is defined by the number of players who take i as a friend, that is, $\sum_{j \neq i} 1(i \in F_j)$. In the data, the standard deviation of the number of friends is less than the standard deviation of the network centrality, which is a typical feature in many social networks.

3. THE MODEL

Following my empirical application, I consider a game theoretic model on social interactions of high school students' college attendance decisions. All these students are denoted as players indexed by $i \in N \equiv \{1, \dots, n\}$, with exogenously determined locations on the school network. Using the terminology in graph theory, a vertex of the network denotes a student and a directed edge connects vertex i to j if student j is (one of) i 's best friends. Following the network distribution theory (see, e.g., Barabási and Albert, 1999), we can view the high school network as a random graph with vertex connectivities governed by some probability distribution, and then the observed network in my data is a single realization of the large random network. I denote F_i as the group of i 's best friends; that is, the set of students is directly connected to i . Note that friendship may not be symmetric; that is, $j \in F_i$ does not necessarily imply $i \in F_j$, which is an important feature in my data. Moreover, let $Q_i = \#F_i$ be the number of i 's best friends. In my game theoretic model, I assume the school network structure is public information. Therefore, F_i is known to every player.

Given the network, I assume each player i simultaneously chooses a discrete action $Y_i \in A \equiv \{0, 1, 2, \dots, K\}$. Following the convention, let Y_{-i} denote a profile of actions of all other players except for i . Let further $X_i \in S_X \subseteq \mathbb{R}^d$ be a vector of player i 's payoff relevant state variables, which are publicly observed by all players, as well as the researcher. Before making the decision, player i observes a vector of action-specific payoff shocks labeled as $\epsilon_i \equiv (\epsilon_{i0}, \dots, \epsilon_{iK}) \in \mathbb{R}^{K+1}$. I assume that ϵ_i is i 's private information; that is, ϵ_i is not observed by $j \neq i$.⁴ In my application, Y_i is binary indicating college attendance; X_i is a vector of demographic variables including, for example, age, gender, GPA, parents' education, household income and race. Moreover, ϵ_i is an

⁴ It should be noted my specification rules out unobserved heterogeneity, that is, some variables observed by players, but not by the researcher.

idiosyncratic preference shock for college attendance. For expositional simplicity, I denote all the public state variables associated with student i by $S_i \equiv (X_i', F_i)'$.

Players interact with each other through their utilities. Specifically, I assume player i 's payoff from choosing an action $k \in A$ as follows:

$$(1) \quad U_{ik}(Y_{-i}, S_i, \epsilon_i) = \beta_k(X_i) + \sum_{j \in F_i} \alpha_k(Y_j, X_i, Q_i) + \epsilon_{ik},$$

where $\beta_k(\cdot)$ is a choice-specific function, and $\alpha_k(\cdot, \cdot, \cdot)$ measures the strategic effects on i 's payoff (of choosing k) from her best friend j 's decision. Note that the strategic effects depend on the state variable X_i as well as i 's network degree Q_i . Because only the differences of choice-specific payoffs matter to decision makers, w.l.o.g., I normalize the mean utility of action 0 by setting $\beta_0(x) = \alpha_0(\ell, x, q) = 0$ for all $x \in S_X$, $\ell \in A$ and $q \in \mathbb{N}$. Let $\theta_k = (\beta_k, \alpha_k)'$ and $\theta = (\theta_1', \dots, \theta_K)'$ be the structural parameters of the game, which are unknown functions.

It is worth pointing out that my model can be extended to allow for exogenous interaction effects; that is, player i 's payoffs U_{ik} depend on X_j for all $j \in F_i$. See, for example, Manski (1993) and Bramoullé et al. (2009). My approach could be modified to accommodate such an extension.⁵ In my empirical context, however, it seems unlikely that high school students make their college attendance decisions according to friends' demographic variables (e.g., Household Income, Parents' education level, and Overall GPA). On the other hand, my specification allows friends' payoff-relevant covariates to affect players' decisions indirectly through their beliefs/expectations on friends' equilibrium choices.

In my setting, direct interactions on payoffs only occur among friends. Although direct interactions are local, strategic effects can spread throughout the whole network if no subnetwork is isolated from the rest. For instance, a player needs to consider the decisions made by the friends of her friends, since those decisions are relevant to her friends' choices, which thereafter affect her payoffs. In the equilibrium, each player's strategy depends on all other players' public observables $\{(X_j, F_j)\}_{j \neq i}$ as well as her own state variables (X_i, F_i) .⁶

3.1. Bayesian Nash Equilibrium. Let $S_n = (S_1, \dots, S_n)$ be all the public information of the network game. For simplicity, throughout I suppress the subscript n in S_n unless the subscript is necessary. To discuss the equilibrium solution, I fix the public state variable S .

In this Bayesian game, player i 's strategy is a function that maps her private information ϵ_i to a discrete choice Y_i . Following the BNE solution concept, player i 's equilibrium strategy, denoted by $r_i^*(\cdot|S; \theta)$, maximizes her (conditional) expected payoff given all other players' equilibrium strategies $r_{-i}^*(\cdot|S; \theta)$, that is,

$$(2) \quad \begin{aligned} r_i^*(\epsilon_i|S; \theta) &= \arg \max_{k \in A} E[U_{ik}(Y_{-i}, S_i, \epsilon_i)|S, \epsilon_i] \\ &= \arg \max_{k \in A} \left[\beta_k(X_i) + \sum_{\ell=0}^K \left\{ \alpha_k(\ell, X_i, Q_i) \times \sum_{j \in F_i} \Pr(r_j^*(\epsilon_j|S; \theta) = \ell|S, \epsilon_i) \right\} + \epsilon_{ik} \right], \quad \forall i. \end{aligned}$$

Thus, Equation (2) defines a simultaneous equation system in terms of (r_1^*, \dots, r_n^*) .

To characterize the BNE solution, I first make the following assumption on ϵ_i .

ASSUMPTION 1. Let ϵ_{ik} be i.i.d. across both actions and players and conform to an extreme value distribution with a density function $f(t) = \exp(-t) \exp[-\exp(-t)]$.

⁵ I thank a referee for this point.

⁶ A recent work by Manresa (2013) develops a reduced form to assess the dependence structure from social interactions in a linear setting.

Assumption 1 has been widely assumed in the discrete choice model literature, as well as in empirical discrete games (see, e.g., Brock and Durlauf, 2002; Bajari et al., 2010). The independence of ϵ_i across players implies that players' equilibrium choices are conditionally independent given S . Therefore, the network dependences of players' decisions are all characterized by the dependence of players equilibrium strategies r_i^* on the common state variable S .

By Assumption 1, we can rewrite (2) in terms of equilibrium choice probabilities. Let $\sigma_{ik}^*(S; \theta) = \Pr(r_i^*(\epsilon_i|S; \theta) = k|S)$ and $\sigma_i^*(S; \theta) = (\sigma_{i0}^*(S; \theta), \dots, \sigma_{iK}^*(S; \theta))'$ be the equilibrium choice probabilities of action k and the equilibrium choice profile, respectively. Let further $\Sigma^*(S; \theta) = (\sigma_1^*(S; \theta), \dots, \sigma_n^*(S; \theta))$ be the sequence of equilibrium choice probability profiles of all players. By (2) and Assumption 1, we have

$$(3) \quad \sigma_{ik}^*(S; \theta) = \frac{\exp \left[\beta_k(X_i) + \sum_{l=0}^K \left\{ \alpha_k(\ell, X_i, Q_i) \times \sum_{j \in F_i} \sigma_{jl}^*(S; \theta) \right\} \right]}{1 + \sum_{q=1}^K \exp \left[\beta_q(X_i) + \sum_{l=0}^K \left\{ \alpha_q(\ell, X_i, Q_i) \times \sum_{j \in F_i} \sigma_{jl}^*(S; \theta) \right\} \right]}, \quad \forall i, k.$$

Note that solving the BNE solution $\{r_1^*(\cdot|S; \theta), \dots, r_n^*(\cdot|S; \theta)\}$ to Section 2 is equivalent to solving $\{\sigma_1^*(S; \theta), \dots, \sigma_n^*(S; \theta)\}$ from (3). See Bajari et al. (2010).

Equation (3) is the common logit functional form, except for the presence of the equilibrium choice probabilities of i 's friends. The existence of a solution follows Brouwer's fixed point theorem. Next, I establish the uniqueness of the equilibrium and show that the equilibrium satisfies a decaying dependence condition. Both uniqueness and the decaying dependence condition are crucial for my empirical analysis.

3.2. Unique Equilibrium. The insight for deriving the unique equilibrium comes from the linear spatial autoregressive model literature: Strong interactions breeds multiple equilibria in a simultaneous equation system. Hence, I first introduce an assumption to restrict the interaction strength.

ASSUMPTION 2. Denote $\lambda \equiv \frac{K}{K+1} \times \sup_{(x,q) \in S_{XQ}} \max_{k,m,\ell \in A} q|\alpha_k(\ell, x, q) - \alpha_m(\ell, x, q)|$. Let $\lambda < 1$.

Similar to the requirement that all roots lie outside of the unit circle in spatial autoregressive models, such a condition ensures weak dependence. In the estimation, I parametrize $\alpha_k(\ell, x, q)$ by $\alpha_{k\ell}/q$ for some $\alpha_{k\ell} \in \mathbb{R}$. Then, Assumption 2 becomes

$$\max_{k,m,\ell \in A} |\alpha_{k\ell} - \alpha_{m\ell}| < (K+1)/K.$$

In my empirical application, because each student takes a binary decision for college attendance, then the above condition can be further rewritten as

$$\max \{|\alpha_{10} - \alpha_{00}|, |\alpha_{11} - \alpha_{01}|\} < 2.$$

Note that $\alpha_{00} - \alpha_{10}$ and $\alpha_{11} - \alpha_{01}$ describe peer effects, that is, the amount friends benefit from choosing the same action. Following the principal in social interactions, $\alpha_{00} - \alpha_{10} \geq 0$ and $\alpha_{11} - \alpha_{01} \geq 0$. Therefore, Assumption 2 requires peer effects to be bounded above. In my context, such a condition means that the college attendance decisions are mainly determined by students' own social and economic characteristics like GPA, household income, etc., and their idiosyncratic preference shock on college attendance as well. If the average probabilities of friends' college attendance increase 1 percentage point, then the peer effects on her own college attendance probability is limited by $\lambda < 1$ percentage point.⁷

⁷ To see this, note that Assumption 2 ensures that a quasi-Lipschitz condition holds for the best response function: The best response function $\Gamma_i(s_i, \{\sigma_j : j \in F_i\})$ defined by (A.1) in the Appendix satisfies the following condition:

$$\|\Gamma_i(s_i, \{\sigma_j : j \in F_i\}) - \Gamma_i(s_i, \{\tilde{\sigma}_j : j \in F_i\})\|_1 \leq \lambda \cdot \max_{j \in F_i} \|\sigma_j - \tilde{\sigma}_j\|_1,$$

Assumption 2 generally holds in a wide range of empirical studies of youth behaviors, including, for example, substance use, church attendance, academic performance, and academic cheating. See, for example, Gaviria and Raphael (2001), Sacerdote (2001), Kawaguchi (2004), Carrell et al. (2008), and Calvó-Armengol et al. (2009). In these studies, the effects on a player's equilibrium choice probabilities from her friends' choices are significantly smaller than 1. Although the NDD is a natural condition for peer effects in my empirical context, numerous prominent exceptions exist. For example, adolescent risky behaviors like substance (marijuana, alcohol, or tobacco) use are mainly driven by influence from friends. See, for example, Gaviria and Raphael (2001) and Kawaguchi (2004). Another leading example is the butterfly effects widely applied to phenomena like, for example, fashion, financial crisis, and gold rush, which characterize the sensitive dependence of players' choices on each other.

LEMMA 1. *Suppose Assumptions 1 and 2 hold. Then, there always exists a unique BNE, regardless of the number of players n and the realization of the state variable S .*

The proof of the uniqueness relies on a contraction mapping argument. We can generalize such a result to the exponential family distribution for the private information ϵ_i .

3.3. Network Decaying Dependence. For any positive integer $h \in \mathbb{N}$, let $N_{(i,h)}$ be the subset of players defined inductively:

$$N_{(i,0)} = \{i\} \quad \text{and} \quad \forall h \geq 1, \quad N_{(i,h)} = N_{(i,h-1)} \cup \left(\bigcup_{j \in N_{(i,h-1)}} F_j \right).$$

By definition, $N_{(i,h)}$ is the set of players on the social network within h -distance of i (including i herself). Moreover, let $G_{(i,h)}$ be the network graph that uses vertices and edges to describe all the connections within the subnetwork $N_{(i,h)}$. Let $S_{(i,h)} = (\{X_j : j \in N_{(i,h)}\}; G_{(i,h)})$. By definition, $S_{(i,h)}$ describes the subnetwork centered around i within her h -distance, that is, how do these players connect to each other and what are the state variables at each node of the graph. Note that players' identities do not matter in the definition of $S_{(i,h)}$.

The idea of the NDD condition is to examine how player i 's equilibrium choice probability $\sigma_i^*(S; \theta)$ responds to counterfactual changes of another player j 's state variable S_j . Note that in equilibrium $\sigma_i^*(S; \theta)$ depends on all the public information S , including S_j no matter whether j is i 's friend or not. In a "stable" equilibrium, intuitively such a dependence should decay with i and j 's network distance. Therefore, the statistical dependence between Y_i and Y_j also diminishes with the distance.

DEFINITION 1 (NDD). In above network game, the equilibrium satisfies the NDD condition if there exists a deterministic sequence $\{\xi_h : h = 1, \dots, \infty\}$ with $\xi_h \downarrow 0$ as $h \rightarrow \infty$ such that for any network size n and integer $h \geq 1$,

$$(4) \quad \sup_{s, s' \in \mathcal{S}_S : s_{(i,h)} = s'_{(i,h)}} \|\sigma_i^*(s; \theta) - \sigma_i^*(s'; \theta)\|_1 \leq \xi_h, \quad \forall i = 1, \dots, n,$$

where $\|\cdot\|_1$ is the L_1 -norm; that is, for any $z \in \mathbb{R}^k$, $\|z\|_1 = \sum_{\ell=1}^k |z_\ell|$.

My notion of NDD is related to the weak dependence in the time-series/spatial literature. In particular, NDD implies the near-epoch dependence (NED) condition in, for example, Andrews (1988).⁸ Different from the time-series/spatial statistical literature that usually assumes weak dependence of unobserved errors across observations, the dependence of players' decisions

where $\|\cdot\|$ is the L^1 -norm. See the proof of Lemma 1.

⁸ This is because by Equation (4), $\Pr(Y_i \neq E[Y_i | (\epsilon_j, S_j) : j \in N_{(i,h)}])$ is bounded by $K\xi_h$.

results from network-based strategic interactions. Moreover, conditional on S , players' decisions are mutually independent under Assumption 1.

In Definition 1, NDD requires the causal effects from S_j on σ_i^* to be bounded above by $\xi_{\rho(i,j)}$, where $\rho(i,j)$ denotes the network distance from j to i . Note that the network size n is treated as a state variable. With NDD (and Assumption 10 below), if I increase the network size to infinity such that the probability distribution of any subgraph around player i converges to a limit, then i 's equilibrium choice probability will converge to some limit as well. The next lemma shows that the equilibrium in my network game satisfies NDD under weak conditions.

LEMMA 2. *Suppose Assumptions 1 and 2 hold. Then the BNE satisfies NDD with $\xi_h = 2\lambda^{h+1}$.*

With the NDD, the probability distribution of Y_i given S can be nonparametrically estimated by using data from a single large network as the network size n goes to infinity. See Appendix Section A.4.

4. IDENTIFICATION

In this section, I discuss the identification of the structural parameter θ . Following Hurwicz (1950) and Koopmans and Reiersol (1950), the definition of identification in a structural model requires that there is a unique value of the structural parameter θ that generates the distribution of the observable variables, denoted by $F_{Y_1, \dots, Y_n|S}$.

Because of the uniqueness of the equilibrium by Lemma 1, $\sigma_{ik}^*(S; \theta)$ is identified by $\sigma_{ik}^*(S; \theta) = \Pr(Y_i = k|S)$. Let $\delta_{ik}(S) = \ln \Pr(Y_i = k|S) - \ln \Pr(Y_i = 0|S)$ for each $k \in \mathcal{A}$. By definition, $\delta_{ik}(S)$ is also identified. Moreover, by (3),

$$\delta_{ik}(S) = \beta_k(X_i) + \sum_{\ell \in \mathcal{A}} \left[\alpha_k(\ell, X_i, Q_i) \times \sum_{j \in F_i} \Pr(Y_j = \ell|S) \right], \quad \forall i, k.$$

Let further $\phi_{i\ell}(S) = \sum_{j \in F_i} \Pr(Y_j = \ell|S)$. By definition, $\sum_{\ell \in \mathcal{A}} \phi_{i\ell}(S) = Q_i$. It follows that

$$(5) \quad \delta_{ik}(S) = \beta_k(X_i) + \alpha_k(0, X_i, Q_i) \times Q_i + \sum_{\ell=1}^K \{[\alpha_k(\ell, X_i, Q_i) - \alpha_k(0, X_i, Q_i)] \times \phi_{i\ell}(S)\}.$$

Similar to Robinson (1988), Equation (5) is essentially a partial linear model as shown in Lemma 3.

Equation (5) suggests that $\beta_k(\cdot)$ and $\alpha_k(0, \cdot, \cdot)$ are not identified separately unless $0 \in \mathcal{S}_Q$.⁹ Hence, I introduce the following normalization on α_k .

ASSUMPTION 3. *Let $\alpha_k(0, \cdot, \cdot) = 0$ for all $k \in \mathcal{A}$.*

Next, I assume a rank condition for identification. Let $\varphi_i(S) = (1, \phi_{i1}(S), \dots, \phi_{iK}(S))'$.

ASSUMPTION 4 (RANK CONDITION). *Given the game size n , the matrix $E[\varphi_i(S) \times \varphi_i(S)'|X_i = x, Q_i = q]$ is invertible for all $(x, q) \in \mathcal{S}_{XQ}$.*

Assumption 4 is testable given that the conditional choice probabilities can be consistently estimated.

The next theorem establishes the identification of the model. For the sake of simplicity, let $\alpha_k(\cdot, \cdot) = (\alpha_k(1, \cdot, \cdot), \dots, \alpha_k(K, \cdot, \cdot))'$ be a K -dimensional vector of functions.

⁹ To see this, consider the following specification: $\alpha_k(0, X_i, Q_i) = \tilde{\alpha}_k(0, X_i)/Q_i$ for $Q_i \geq 1$.

LEMMA 3. *Fix arbitrary n . Suppose Assumptions 1–4 hold. Then the structural parameter θ is identified; that is, $F_{Y_1, \dots, Y_n|S}(\theta') \neq F_{Y_1, \dots, Y_n|S}(\theta)$ for all $\theta' \neq \theta$. Specifically, for any $(x, q) \in \mathcal{S}_{XQ}$,*

$$\begin{pmatrix} \beta_k(x) \\ \alpha_k(x, q) \end{pmatrix} = \{E[\varphi_i(S) \times \varphi'_i(S)|X_i = x, Q_i = q]\}^{-1} E[\varphi_i(S) \times \delta_{ik}(S)|X_i = x, Q_i = q].$$

Note that the identification result in Lemma 3 is established for each fixed n . For the purpose of estimation and asymptotic analysis, however, we need that n goes to infinity. Hence, I replace the rank condition 4 by the following assumption.

ASSUMPTION 5 (RANK CONDITION FOR LARGE n). *For all sufficiently large n , the matrix $E[\varphi_i(S) \times \varphi_i(S)'|X_i = x, Q_i = q]$ is invertible, that is,*

$$n \xrightarrow{\liminf} \infty \quad \det(E[\varphi_i(S) \times \varphi_i(S)'|X_i = x, Q_i = q]) > 0, \quad \forall (x, q) \in \mathcal{S}_{X_i, Q_i}.$$

By relaxing conditions in Lemma 3, the next theorem establishes identification of the model for all sufficiently large n .

THEOREM 1. *Suppose Assumptions 1–3 and 5 hold. Then the structural parameter θ is identified for all n sufficiently large.*

The semiparametric identification in Theorem 1 helps the applied researcher to get a better sense of whether a fully parametric approach relies on ad hoc specification (of the payoff function) for identification or merely for simplicity of estimation. An analogous rank condition can be formulated in the fully parametric model that is used for my estimation. Let $\beta_k(x) = x' \beta_k$ and $\alpha_k(\ell, x, q) = \alpha_{k\ell}/q$, where $\beta_k \in \mathbb{R}^d$ and $\alpha_k \equiv (\alpha_{k1}, \dots, \alpha_{kK})' \in \mathbb{R}^K$. Let $W_i = (X'_i, \phi_{i1}(S), \dots, \phi_{iK}(S))'$.

ASSUMPTION 6 (RANK CONDITION FOR LINEAR-INDEX SETUP). *The matrix $E(W_i W'_i)$ is invertible for all n sufficiently large.*

Replace Assumption 5 with 6 in Theorem 1; then we obtain identification of $\theta_k = (\beta'_k, \alpha'_k)'$ as follows: for sufficiently large n ,

$$\theta_k = [E(W_i W'_i)]^{-1} E[W_i \delta_{ik}(S)].$$

Clearly, variations in the aggregated friends' choice probabilities $\phi_{i\ell}(S)$ identify the strategic coefficients $\alpha_{k\ell}$.

5. ESTIMATION

This section discusses the parametric estimation of the structural parameter θ . In particular, I specify the payoff function by

$$(6) \quad U_{ik}(Y_{-i}, S_i, \epsilon_i) = X'_i \beta_k + \sum_{\ell=1}^K \alpha_{k\ell} \times \frac{1}{Q_i} \sum_{j \in F_i} 1(Y_j = \ell) + \epsilon_{ik},$$

where $\beta_k \in \mathbb{R}^d$ and $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kK})' \in \mathbb{R}^K$. Let $\theta_k = (\beta'_k, \alpha'_k)' \in \mathbb{R}^{K+d}$. Moreover, let $\beta = (\beta'_1, \dots, \beta'_K)' \in \Theta_\beta \subseteq \mathbb{R}^{Kd}$ and $\alpha = (\alpha'_1, \dots, \alpha'_K)' \in \Theta_\alpha \subseteq \mathbb{R}^{K \times K}$, where Θ_β and Θ_α are the parameter space for β and α , respectively. Denote $\theta = (\theta'_1, \dots, \theta'_K)'$ and $\Theta = \Theta_\beta \times \Theta_\alpha$.

Let $\{X_i, F_i, Y_i\}_{i=1}^n$ be the data generated from the equilibrium of a single large network game. For asymptotic analysis, I consider the network size n goes to infinity, since my empirical

application involves a few large networks. My data-generating process also requires the probability distributions of $G_{(i,h)}$ with fixed h converge to the same limiting distribution for all i as the network size goes to infinity, and random graph $G_{(i,h)}$ be independent of $G_{(j,h)}$, given they do not share any vertex in common.

I now proceed to motivate my estimation procedure. First, note that conditional on S , the actions profile $\{Y_1, \dots, Y_n\}$ are conditionally independent under Assumption 1. Thus, we can derive the (conditional) log-likelihood function as follows:

$$(7) \quad \hat{L}(\theta) = \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} 1(Y_i = k) \times \ln \sigma_{ik}^*(S; \theta).$$

Let $\hat{\theta}_{MLE} = \arg \max_{c \in \Theta} \hat{L}(c)$ be the MLE, which requires us to solve $\{\sigma_{ik}^*(S; \theta) : i \in N; k \in A\}$ to the equation system (3). For the log-likelihood function (7), we can verify that all the regularity conditions hold under additional weak conditions.¹⁰ In practice, however, $\hat{\theta}_{MLE}$ is not computationally feasible when the network is large. This is because the equilibrium choice probability $\sigma_{ik}^*(S; \theta)$ has no closed-form expression, and obtaining a numerical solution is costly in the large simultaneous equation system.

The key to my approach is to approximate $\sigma_{ik}^*(S; \theta)$ by some computable solution $\sigma_{ik}^h(S; \theta)$ to be defined below, where h is an integer that depends on n such that the approximation error $\|\sigma_{ik}^h(S; \theta) - \sigma_{ik}^*(S; \theta)\|_1$ is negligible relative to the sampling error. Thus, I define my approximated log-likelihood function

$$(8) \quad \hat{Q}(\theta) = \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} 1(Y_i = k) \times \ln \sigma_{ik}^h(S; \theta).$$

Further, my estimator maximizes the approximated likelihood; that is, $\hat{\theta} = \arg \max_{c \in \Theta} \hat{Q}(c)$.

To define $\sigma_{ik}^h(S; \theta)$, we first define a Bayesian game of smaller size: let $N_{(i,h)}$ be the set of players and each player $j \in N_{(i,h)}$ simultaneously makes a discrete choice $Y_j \in A$. Moreover, each player j in $N_{(i,h)}$ has the same state variables (X_j, ϵ_j) as those in the original network game, but player j 's set of friends is restricted to be $F_j \cap N_{(i,h)}$. In other words, I artificially remove all the players outside of $N_{(i,h)}$ from the large network game. Moreover, let $\{\sigma_{ik}^h(S; \theta) : j \in N_{(i,h)}, k \in A\}$ solve

$$(9) \quad \sigma_{jk} = \frac{\exp \left[\beta'_k X_j + \sum_{\ell=1}^K \alpha_{k\ell} \times \left(\frac{1}{Q_j} \sum_{j' \in F_j \cap N_{(i,h)}} \sigma_{j'\ell} \right) \right]}{1 + \sum_{q=1}^K \exp \left[\beta'_q X_j + \sum_{\ell=1}^K \alpha_{q\ell} \times \left(\frac{1}{Q_j} \sum_{j' \in F_j \cap N_{(i,h)}} \sigma_{j'\ell} \right) \right]}, \quad \forall j \in N_{(i,h)}, k \in A.$$

By Lemma 1, there is a unique solution to (9). In the derived subnetwork game, player i is at the center of the subnetwork, and her equilibrium choice probabilities profile is denoted by $\sigma_i^h(S; \theta) = (\sigma_{i0}^h(S; \theta), \dots, \sigma_{iK}^h(S; \theta))$.

By Lemma 2, the approximation error $\|\sigma_i^*(S; \theta) - \sigma_i^h(S; \theta)\|_1$ can be bounded by $2\lambda^{h+1}$.¹¹ To control for the approximation error, I choose h to increase with n at a proper rate.

5.1. Asymptotic Analysis. I now establish the consistency and limiting distribution for the proposed estimator. First, I make the following assumptions.

ASSUMPTION 7. (i) The parameter space Θ is compact and the support S_{XQ} is bounded; (ii) The true parameter θ belongs to the interior of Θ .

¹⁰ For instance, Lemma 7 in the Appendix ensures the differentiability of the objective function.

¹¹ To apply Lemma 2, let $S_{i,h}$ denote the state of the network derived from S by eliminating all the network connections outside of $N_{(i,h)}$. By definition, $S_{i,h} \in \{s' \in \mathcal{S}_S : s'_{(i,h)} = S_{(i,h)}\}$. Moreover, note that $\sigma_i^h(S; \theta) = \sigma_i^*(S_{i,h}; \theta)$, since all players outside of $N_{(i,h)}$ have no strategic effects on players in $N_{(i,h)}$.

ASSUMPTION 8. *Let*

$$\sup_{a \in \Theta_a} \max_{k, \ell, m \in A} |a_{k\ell} - a_{m\ell}| < (K + 1)/K.$$

ASSUMPTION 9. *Given any $h \in \mathbb{N}$, the probability distribution of $G_{(i,h)}$ converges to a limiting distribution $F_{G,h}$ as $n \rightarrow \infty$ for all i , and $G_{(i,h)}$ is independent of $G_{(j,h)}$ if $N_{(i,h)} \cap N_{(j,h)} = \emptyset$. Moreover, the payoff covariates X_i are i.i.d. across players given the exogenous random network.*

ASSUMPTION 10. *There exists a positive constant $c_0 \in \mathbb{N}$, which does not depend on n , such that $\max_{i \in N} \sum_{j \neq i} 1(i \in F_j) \leq c_0$ with probability one.*

ASSUMPTION 11. (i) *Let $h \rightarrow \infty$ as $n \rightarrow \infty$; (ii) let further $h = [h_0 \cdot n^a]$ for some constant $h_0 > 0$ and $a > 0$, where $[t]$ is the largest integer that is no larger than t .*

Let $P = K(d + K)$ denote the dimension of the parameter θ . Moreover, let $f_i(Y_i | S; \theta) = \sum_{k \in A} 1(Y_i = k) \times \ln \sigma_{ik}^*(S; \theta)$ and $J_n(\theta) = E[\frac{\partial}{\partial \theta} f_1(Y_1 | S; \theta) \times \frac{\partial}{\partial \theta'} f_1(Y_1 | S; \theta)]$. The latter is indexed by n because of the dependence of f_i on n through S and σ_i^* .

ASSUMPTION 12. *There exists a nonsingular $P \times P$ matrix $J(\theta)$ such that $J_n(\theta) \rightarrow J(\theta)$.*

Assumption 7(i) ensures that choice probabilities are bounded away from zero so that the log-likelihood function is bounded. Unbounded regressors can be accommodated using high-order moments restrictions (see, e.g., Van de Geer, 1990). Assumption 7(ii) is standard in the literature. Assumption 8 strengthens Assumption 2 to hold uniformly in Θ .

Assumptions 9 and 10 impose restrictions on the distribution of the state variables as well as the network connections. For the first half of Assumption 9, note that for any given n and h , the probability distribution of $G_{(i,h)}$ is well defined, since the subgraph $G_{(i,h)}$ can be represented by an $n \times n$ matrix with 0–1 entries. Note that the subgraph $G_{(i,h)}$ here refers to all subgraphs that are homomorphic to $G_{(i,h)}$, because players' identities do not matter in the definition of $G_{(i,h)}$. Moreover, the first half of Assumption 9 also requires that $G_{(i,h)}$ should be i.i.d. across players who are at least $2h$ -step far away from each other in the network. This condition generally holds in the random graph literature, since conditional on $G_{(i,h)}$ and $G_{(j,h)}$ do not overlap, the graph structure of $G_{(i,h)}$ does not provide additional information on how $G_{(j,h)}$ is connected in a large network. For the second half of Assumption 9, the (conditional on the network connections) independence of X_i is a strong assumption. In practice, positive statistical dependence across friends' demographic variables (e.g., age, education, race, etc.) has been emphasized in the sociology literature (see, e.g., Easley and Kleinberg, 2010), which is the so-called homophily phenomena. For my asymptotic results to be established, this assumption could be relaxed to allow for some degree of dependence at the expense of longer proofs.¹²

¹² For instance, as is suggested by spatial autoregressive models, one could assume that $X \equiv (X'_1, \dots, X'_n)'$ takes a simultaneous autoregressive dependence structure: $X = \Psi(\gamma_0) \cdot X + v$, where Ψ is an $n \times n$ weight matrix parameterized by a q -dimensional vector γ_0 such that diagonal elements of Ψ are zeros and $I_n - \Psi$ is nonsingular. Moreover, $v \in \mathbb{R}^n$ is a vector of i.i.d. errors that are independent of S and $(\epsilon_1, \dots, \epsilon_n)'$. Our asymptotic results established in Theorems 2 and 3 still hold as long as for each $k \in A$,

$$\frac{1}{n} \sum_{i=1}^n \{ \sigma_{ik}^*(S; \theta) \ln \sigma_{ik}^*(S; c) - E[\sigma_{ik}^*(S; \theta) \ln \sigma_{ik}^*(S; c)] \} \xrightarrow{P} 0, \quad \text{uniformly holds in } c \in \Theta.$$

Such a high-level condition can be satisfied if the weight matrix is modeled as $\Psi_{j\ell} = \psi(\min\{\rho(j, \ell), \rho(\ell, j)\})$, where ψ is a decreasing function that decays sufficiently fast (i.e., subject to exponential decay). Moreover, it is also possible to allow the dependence between X_j and X_ℓ , that is, $\Psi_{j\ell}$, to depend not only on the network distance between j and ℓ , but also on the distance between j (or ℓ) and other players that connect (directly or indirectly) to both of them. See, for example, Pinkse et al. (2002).

Assumption 10 imposes restrictions on the number of best friends that a single individual could have. Note that the upper bound c_0 does not depend on the network size. This condition is crucial for the \sqrt{n} -asymptotics of the proposed estimator when the data come from one single large network game: By Assumption 10 and the NDD condition, we can limit the dependence among all the observations. Similar assumptions can also be found in, for example, Morris (2000) for the contagion analysis in local interaction games.

It is worth pointing out that Assumption 10 is not imposed in most of the recent empirical network formation models. Such a restriction, however, can be easily accommodated in those models, for example, Christakis et al. (2010) and Mele (2010). On the theoretic side of network formation, for example, Jackson and Wolinsky (1996) introduce a cost for players to maintain a direct friendship link, which limits the maximum number of direct links each individual could have. In my empirical application, each student was allowed to nominate at most 10 best friends. Such a restriction is reasonable in light of capacity constraints (e.g., time and/or effort) for students to maintain their friendship. Therefore, a network formation model using this data set should impose such a restriction to rationalize the data.

Assumption 11(i) is intuitive for the approximation of $\sigma_i^*(S; \theta)$. Moreover, (ii) strengthens (i) and ensures that the approximation error using h -neighborhood game is negligible.

In Assumption 12, $J_n(\theta)$ is the Fisher information matrix of the n -player game. Assumption 12 requires that the Fisher information matrix have a nondegenerate limit when the network size goes to infinity. Note that the convergence of $J_n(\theta)$ is implied by Lemma 2 and Assumption 9, since the distribution $S_{(i,h)}$ convergence to a limit for all i as $n \rightarrow \infty$. Hence, in Assumption 12, the essential restriction is the nondegeneracy of the limit.

THEOREM 2. *Suppose that Assumptions 1 and 6, 7(i), 9, 10, and 11(i) hold. Then $\hat{\theta} \xrightarrow{p} \theta$.*

Given the consistency of $\hat{\theta}$, I now establish its limiting distribution, which is shown to be identical to $\overset{\hat{\theta}}{MLE}$ under addition conditions.

THEOREM 3. *Suppose that Assumptions 1 and 6–12 hold. Then $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, J(\theta)^{-1})$.*

Note that the infeasible likelihood function (7) is indeed what ultimately gives the information equality in Theorem 3. Furthermore, the limiting Fisher information matrix $J(\theta)$ can be consistently estimated by

$$\frac{1}{n} \sum_{i=1}^n \left[\frac{\partial}{\partial \theta} f_i^h(Y_i | S; \hat{\theta}) \times \frac{\partial}{\partial \theta'} f_i^h(Y_i | S; \hat{\theta}) \right],$$

where $f_i^h(Y_i | S; \theta) = \sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^h(S; \theta)$ and $\frac{\partial}{\partial \theta} f_i^h(Y_i | S; \theta) = \sum_{k \in A} \frac{1(Y_i = k)}{\sigma_{ik}^h(S; \theta)} \times \frac{\partial}{\partial \theta} \sigma_{ik}^h(S; \theta)$.

REMARK. It is a generic aspect of my asymptotic analysis that the size of the network goes to infinity, but the maximum number of friends each player should remain fixed (i.e., Assumption 10). Therefore, the collection of state variables $\{S_{(i,h)} : i \leq n\}$ becomes an m -dependent sequence, where $m \leq c_0^{h+1}$, which is crucial for the \sqrt{n} -consistency of the estimator in the proof of Theorem 3. This aspect rules out the “Small-World phenomenon” (see, e.g., Watts and Strogatz, 1998), often referred to as six degrees of separation (see, e.g., Guare, 1990). It should be noted that whether the network is a “small-world” is an empirical question that can be verified from the data. In a small-world network, the asymptotic analysis should allow the (average) number of friends to increase with the size of the network.¹³ It seems to be an intriguing challenge to consider Small-World asymptotics.

¹³ Watts and Strogatz (1998) develop a small-world model by rewiring a regular network with $n \gg Q_i \gg \ln n \gg 1$.

It is worth pointing out that it is possible to relax Assumption 10 to accommodate some “intermediate” case of the network structure at the expense of longer proofs. For instance, consider a network where the maximum number of friends is not bounded from above, but the distribution of Q_i is asymptotically stable (as the network size n goes to infinity) with finite mean and variance. Hence, there can be a few, but significant number of nodes with a lot of connections, which, however, does not render the network a “Small-World.” In the following Monte Carlo experiments, I consider such a specification to examine the finite sample performance of my AMLE.

5.2. Monte Carlo Experiments. This section uses Monte Carlo to illustrate the finite sample performance of the proposed estimator. In particular, I consider a binary game with payoff $U_{i1}(Y_{-i}, S_i, \epsilon_i) = X_i' \beta + \alpha \times [\frac{1}{Q_i} \sum_{j \in F_i} 1(Y_j = 1)] + \epsilon_{i1}$, where $\alpha \in \mathbb{R}$ and $X_i \in \mathbb{R}^2$.

Moreover, I consider two representative networks: First, I consider the Circle network specified in Salop (1979), where n players are equally spaced in a circle and each player has two friends. In the circle network, $Q_i = 2$ for all players, and the friendship relation between each pair of players is also symmetric. The second network is a random network. For any $i \neq j$, we use a random variable $\ell_{i,j} \in \{0, 1, 2, 3\}$ to denote “no relationship,” “ i is j ’s friend, but not vice versa,” “ j is i ’s friend, but not vice versa,” and “mutual friendship,” respectively. For $i \neq j$, $\ell_{i,j}$ is drawn independently from the probability mass distribution $(1 - \frac{4}{n}, \frac{1}{n}, \frac{1}{n}, \frac{2}{n})$. Moreover, set $\ell_{ii} = 0$ for all i . By definition, $Q_i = \sum_{j=1}^n 1(\ell_{ij} \in \{2, 4\})$, which conforms to a Binomial Distribution $B(n, 3/n)$. As n goes to infinity, the mean of Q_i remains constant and conforms to the Poisson (3) distribution asymptotically.

Moreover, I take $X_{i1} \sim U(-0.5, 0.5)$, $X_{i2} \sim N(0, 1)$, and $X_{i1} \perp X_{i2}$. The results for other distributional specifications of X are qualitatively similar. Further, I set $\beta = (1, 1)'$, which are invariant across all the experiments. According to Assumption 8, I choose $\Theta_\alpha = [-1.99, 1.99]$ and set $\alpha = 0, 0.8$, and 1.6 , respectively. In particular, for $\alpha = 0$, my setting is equivalent to the classical Logit model.

I perform experiments with the number of players $n = 500, 1,000$, and $2,000$. In each design, we first compute the unique BNE given the underlying parameter value, that is, I solve the equilibrium by finding a fixed point to (3). With the (numerical) solution in hand, I am able to simulate the equilibrium decision made by each player.

Regarding estimation, it is crucial to choose the parameter $h \in \mathbb{N}$ according to the sample size n . Following Assumption 11, I set $h = \lceil \sqrt{n}/10 \rceil$, that is, $h = 2, 3$, and 4 with respect to the three choices of sample size. It is worth pointing out that the computation time increases with h in a nonlinear pattern. For fixed n , I also investigate the performance of the proposed estimator under different choices of h . The results for different sample sizes are qualitatively similar and therefore I only report results for $n = 1,000$. In addition, I perform 500 replications to approximate the finite sample distribution of my estimator.

Tables 2 and 3 report the finite sample performance of the proposed estimator under the different settings. The numbers in parentheses are the standard deviations. The estimator is consistent for all these designs, and the standard deviation diminishes at the \sqrt{n} -rate as I increase the sample size. In Table 4, I further investigate how the choice of h affects the performance of $\hat{\theta}$. For $n = 1,000$, it shows that the approximation behaves well by using $h \geq 3$, and additional gains of accuracy are minor from choosing larger h .

5.3. Empirical Results for Peer Effects on College Attendance. I now apply my method to estimate peer effects on high school students’ college attendance decisions. The specification of the payoff function is the same as the one used in my Monte Carlo experiments.

Table 5 presents my estimation results. I also provide results using the pseudo-MLE for comparison. The difference reflects the bias due to the misspecification of social interactions. Note that $\text{AMLE}(h)$ refers to the approximated MLE with the parameter value h and the pseudo-MLE is equivalent to $\text{AMLE}(0)$. From Table 5, the approximation of the equilibrium

TABLE 2
FINITE SAMPLE PERFORMANCE: $\beta = (1, 1)$ AND $(n, h) = (1,000, 3)$

True Value of α	Parameters	Circle Network	Random Network
0	β_1	1.0131 (0.2454)	1.0292 (0.2493)
	β_2	1.0036 (0.0826)	1.0058 (0.0833)
	α	0.0068 (0.1326)	0.0109 (0.1402)
0.8	β_1	1.0018 (0.2468)	1.0204 (0.2557)
	β_2	1.0091 (0.0833)	1.0060 (0.0834)
	α	0.8066 (0.1042)	0.8023 (0.1114)
1.6	β_1	1.0059 (0.2464)	1.0179 (0.2721)
	β_2	1.0008 (0.0849)	1.0064 (0.0839)
	α	1.6256 (0.0950)	1.6169 (0.0930)

TABLE 3
FINITE SAMPLE PERFORMANCE OF $\hat{\alpha}$

True Value of α	Sample Size	Circle Network	Random Network
0	500	0.0030 (0.1954)	0.0033 (0.2004)
	1,000	0.0068 (0.1326)	0.0109 (0.1402)
	2,000	0.0044 (0.0962)	0.0022 (0.0968)
0.8	500	0.8032 (0.1570)	0.8048 (0.1469)
	1,000	0.8066 (0.1042)	0.8023 (0.1114)
	2,000	0.8036 (0.0714)	0.7964 (0.0716)
1.6	500	1.6254 (0.1282)	1.6776 (0.1398)
	1,000	1.6256 (0.0950)	1.6169 (0.0930)
	2,000	1.6072 (0.0660)	1.6064 (0.0659)

NOTE: $h = 2, 3, 4$ for $n = 500, 1,000,$ and $2,000,$ respectively.

is sufficiently good for $h \geq 2$. So I can use AMLE(2) as my estimates. It is worth pointing out that the estimates of peer effects satisfy Assumption 2.

The second column of Table 5 contains the corresponding estimates of the pseudo-MLE, which has been typically adopted in the empirical analysis on college attendance. Given the pseudo-MLE estimates, the most striking difference of my estimates (i.e., AMLE(2) in the fourth column) is that the peer effects coefficient is significant at the 5% level, while the pseudo-MLE implicitly sets it to be zero. Therefore, the ignorance of peer effects in the empirical analysis on college attendance results in biased estimates, which can be corrected by increasing h from 0 to 2.

In Table 5, most of coefficients estimates are significant at the 10% significance level. Regarding race, the coefficients of American Indian, Asian, and Black are insignificant; this is

TABLE 4
FINITE SAMPLE PERFORMANCE OF $\hat{\theta}$ AT DIFFERENT h ($n = 1,000, \alpha = 0.8$)

	Parameters	$h = 0$	1	2	3	4
Circle Network	β_1	0.9790 (0.2501)	1.0121 (0.2448)	1.0155 (0.2461)	1.0157 (0.2462)	1.0157 (0.2462)
	β_2	0.9627 (0.0821)	0.9967 (0.0845)	1.0002 (0.0848)	1.0004 (0.0849)	1.0004 (0.0849)
	α	NA NA	0.8560 (0.1118)	0.8014 (0.0996)	0.7974 (0.0986)	0.7972 (0.0984)
	β_1	0.9649 (0.2568)	1.0063 (0.2575)	1.0094 (0.2584)	1.0098 (0.2585)	1.0098 (0.2585)
	β_2	0.9614 (0.0823)	0.9990 (0.0824)	1.0023 (0.0825)	1.0026 (0.0825)	1.0026 (0.0825)
Random Network	α	NA NA	0.8957 (0.1289)	0.8068 (0.1063)	0.7979 (0.1033)	0.7968 (0.1028)

TABLE 5
ESTIMATION RESULTS

Variable	Pseudo-MLE	AMLE(1)	AMLE(2)	AMLE(3)	AMLE(4)
Age	-0.140* (0.076)	-0.135* (0.076)	-0.135* (0.076)	-0.135* (0.076)	-0.135* (0.076)
Female	-0.028 (0.171)	-0.038 (0.171)	-0.035 (0.171)	-0.034 (0.171)	-0.034 (0.171)
Household Income	0.150** (0.042)	0.134** (0.043)	0.134** (0.043)	0.134** (0.043)	0.134** (0.043)
Mother's Education	0.066 (0.052)	0.064 (0.053)	0.064 (0.053)	0.064 (0.053)	0.064 (0.053)
Father's Education	0.033 (0.029)	0.035 (0.029)	0.036 (0.029)	0.036 (0.029)	0.036 (0.029)
Overall GPA	1.749** (0.147)	1.714** (0.148)	1.717** (0.148)	1.717** (0.148)	1.717** (0.148)
American Indian	-0.559 (0.418)	-0.575 (0.423)	-0.574 (0.423)	-0.574 (0.423)	-0.574 (0.423)
Asian	-0.050 (0.428)	0.035 (0.435)	0.043 (0.435)	0.043 (0.435)	0.043 (0.435)
Black	0.206 (0.455)	0.351 (0.466)	0.363 (0.467)	0.364 (0.467)	0.364 (0.467)
Hispanic	0.891** (0.223)	1.043** (0.233)	1.051** (0.234)	1.052** (0.234)	1.052** (0.234)
White	-0.703* (0.393)	-0.718* (0.401)	-0.717* (0.401)	-0.718* (0.401)	-0.718* (0.401)
Other Race	-1.024** (0.422)	-1.096** (0.430)	-1.097** (0.430)	-1.098** (0.430)	-1.098** (0.430)
Constant	-2.680* (1.441)	-2.795* (1.445)	-2.806* (1.446)	-2.806* (1.446)	-2.806* (1.446)
Peer effects	—	0.657** (0.297)	0.642** (0.286)	0.640** (0.285)	0.640** (0.285)
Log-likelihood	-437.537	-435.063	-434.988	-434.990	-434.990

* Significant at the 10% level. ** Significant at the 5% level.

simply due to the fact that all these three categories have only a few observations in the sample. Moreover, due to a missing data issue on parents' education, one would expect noisy estimates for the parents' education coefficients.

My pseudo-MLE estimates are qualitatively similar to those empirical results in Light and Strayer (2002), who estimate racial effects on college attendance with a Probit model by using the data from the 1979 National Longitudinal Survey of Youth (NLSY79), which consists of a

sample of respondents born in 1957–64. In particular, whites are less likely than minorities to attend college, given that other determinants of college attendance are held constant. For such a comparison, note that peer effects are not considered in Light and Strayer (2002). My pseudo-MLE results are also consistent with other early empirical evidence on college attendance. See, for example, Fuller et al. (1982).¹⁴

Peer effects estimates provided by AMLE(2) are related to those empirical results in Calvó-Armengol et al. (2009), who also use the Add Health data to study peer effects on school performance index. In particular, they specify a linear equation system for network-based social interactions and obtain statistically significant peer effects estimates of similar magnitude (i.e., 0.5505 with a standard error 0.1247). Moreover, Gaviria and Raphael (2001) and Kawaguchi (2004) use the National Education Longitudinal Study (NELS) data set and the National Longitudinal Survey Youth 97 (NLSY97) data set, respectively, to study peer effects on youth behaviors of high school students, for example, drug use, alcohol drinking, cigarette smoking, church attendance, and dropping out. Their empirical results also provide evidence for significant peer effects of similar magnitude to my estimates. For example, consider a typical student in my sample whose covariates take the mean values in Table 1. Suppose all her friends shift their college attendance probabilities together from 0% to 10%; then her college attendance probability would increase about 1.52% (namely, from 37.93% to 39.45%). Similarly, if all her friends' college attendance probabilities shift jointly from 0% to 50%, then it would yield an increase of 11.83%.¹⁵

6. CONCLUSION

This article provides a structural approach to study social interactions in a large network. My benchmark model assumes that individuals are affected by their friends only but all individuals are connected to each other directly or indirectly in a single network. By restricting the strength of interactions among friends, I establish the existence, uniqueness of the equilibrium, and a NDD condition. We further establish the semiparametric identification of the model and propose a computationally feasible and novel estimation procedure. The classic MLE method developed in single-agent binary response models is naturally nested in my approach.

An important extension of the benchmark model is to allow for interdependence between a pair of friends' private information. Individuals tend to bond with similar others as their friends. In sociology, such a phenomena is called "homophily"; see, for example, Easley and Kleinberg (2010). Homophily leads to friendship between people with similar characteristics (age, education, race, etc.) and with positively correlated types. The former can be directly observed from the data. To identify the latter is more challenging to the researcher. In a discrete game with a (small) fixed number of players, Liu et al. (2017) establish the nonparametric identification of homophily in a context of discrete game. Identification and estimation of homophily in a large network game is an important extension.

Allowing for possible endogeneity of the network is another important research question in the study of large network social interactions. Being popular in a high school network might be associated with a possible high draw of payoff shocks for college attendance. Part of the problem could be addressed by taking into account the network formation in the first stage; see, for example, Christakis et al. (2010), Mele (2010), Badev (2013), Leung (2014), and Menzel (2015b). In this regard, my identification and estimation results are useful for the second-stage

¹⁴ Fuller et al. (1982) use the 1972 National Longitudinal Study of the High School Class (NLSS72).

¹⁵ In a study of 10th graders' substance use, Gaviria and Raphael's (2001) estimates imply, for example, that moving a typical teenager from a school where none of his classmates use drugs to one where half use drugs would increase the probability by approximately 13%. Similar experiments would yield increases in the corresponding probabilities of 9% for alcohol use, 8% for cigarette smoking, 11% for church attendance, and 8% for dropping out of school. Moreover, Kawaguchi (2004) shows that if a teenager's perception of the percentage of his/her peers who use a substance (i.e., marijuana, alcohol, or tobacco) increases by 10%, the probability that he/she will use the substance increases from 1.4% to 2.6%.

analysis of social interactions in the subgame. In a large network game, however, difficulties arise when each player has a small opportunity set, relative to the large network size, of players to meet with, and, more importantly, such opportunity sets are not observed in the data set. For the majority of pairs of distinct individuals, it is unclear whether an unconnected link is due to the lack of opportunity, or players' unfavorable desire for such a connection.

As a matter of fact, our results go well beyond the local interaction studied here, as they can be generalized to more general social interaction games. For instance, one can consider that each player interacts directly with her friends, friends of friends, etc. In particular, the payoff function can be generalized as follows: For choosing an action $k \in A$,

$$U_{ik}(Y_{-i}, S_i, \epsilon_i) = \beta_k(X_i) + \sum_{j \neq i} \alpha_k(Y_j, d_{ij}, X_i, Q_i) + \epsilon_{ik},$$

where d_{ij} is the network distance from j to i . By such an extension, the interaction term $\alpha_k(Y_j, d_{ij}, X_i, Q_i)$ depends on player j 's choice as well as his/her network distance. In (1), direct interactions α_k have been set to zero for all $j \notin F_i$. By a similar argument, our uniqueness and NDD condition of the equilibrium can be established. A major difficulty in developing nonparametric identification and estimation, however, is to consider a model with an increasing parameter space, since the support of d_{ij} expands with the size of the network. Though significant progress has been made in the regression context (see, e.g., Belloni and Chernozhukov, 2011), the different nature of the structural analysis calls for further work.

APPENDIX

A.1. Equilibrium Uniqueness and Network Stability.

A.1.1. Proof of Lemma 1. Fix n and $S = s$. I prove by contradiction. Suppose there are two different BNEs, denoted by $\{\sigma_i^* : i = 1, \dots, n\}$ and $\{\sigma_i^\dagger : i = 1, \dots, n\}$. For notational simplicity, I suppress their dependence on S and θ .

For a given choice probability profile $(\sigma_1, \dots, \sigma_n)$ where σ_i is a $(K+1)$ -choice probability distributions, let

$$(A.1) \quad \Gamma_{ik}(s_i, \{\sigma_j : j \in F_i\}) = \frac{\exp \left\{ \beta_k(x_i) + \sum_{\ell=0}^K \left[\alpha_k(\ell, x_i, q_i) \sum_{j \in F_i} \sigma_{j\ell} \right] \right\}}{1 + \sum_{\ell'=1}^K \exp \left\{ \beta_{\ell'}(x_i) + \sum_{\ell=0}^K \left[\alpha_{\ell'}(\ell, x_i, q_i) \sum_{j \in F_i} \sigma_{j\ell} \right] \right\}}.$$

Let further $\Gamma_i(s_i, \{\sigma_j : j \in F_i\}) = (\Gamma_{i0}(s_i, \{\sigma_j : j \in F_i\}), \dots, \Gamma_{iK}(s_i, \{\sigma_j : j \in F_i\}))'$. By Equation (3), we have $\sigma_i^* = \Gamma_i(s_i, \{\sigma_j^* : j \in F_i\})$ and $\sigma_i^\dagger = \Gamma_i(s_i, \{\sigma_j^\dagger : j \in F_i\})$ for all $i \in N$.

Therefore, for any $i \in N$,

$$\begin{aligned} \sigma_i^* - \sigma_i^\dagger &= \Gamma_i(s_i, \{\sigma_j^* : j \in F_i\}) - \Gamma_i(s_i, \{\sigma_j^\dagger : j \in F_i\}) \\ &= \sum_{j \in F_i} \sum_{\ell \in A} \frac{\partial \Gamma_i(s_i, \{\tilde{\sigma}_j : j \in F_i\})}{\partial \sigma_{j\ell}} (\sigma_{j\ell}^* - \sigma_{j\ell}^\dagger), \end{aligned}$$

where $\{\tilde{\sigma}_j : j \in F_i\}$ is a vector between $\{\sigma_j^* : j \in F_i\}$ and $\{\sigma_j^\dagger : j \in F_i\}$. By definition, we have

$$\frac{\partial \ln \Gamma_{ik}}{\partial \sigma_{j\ell}} = \alpha_k(\ell, x_i, q_i) - \sum_{\ell'=1}^K \Gamma_{i\ell'} \cdot \alpha_{\ell'}(\ell, x_i, q_i) = \sum_{\ell'=0}^K \Gamma_{i\ell'} \cdot \alpha_k(\ell, x_i, q_i) - \sum_{\ell'=0}^K \Gamma_{i\ell'} \cdot \alpha_{\ell'}(\ell, x_i, q_i),$$

where the last step is because (i) $\sum_{\ell'=0}^K \Gamma_{i\ell'} = 1$ and (ii) $\alpha_0(\ell, x, q) = 0$. It follows that

$$\frac{\partial \Gamma_{ik}}{\partial \sigma_{j\ell}} = \Gamma_{ik} \sum_{k' \neq k} [\Gamma_{ik'} \cdot \{\alpha_k(\ell, x_i, q_i) - \alpha_{k'}(\ell, x_i, q_i)\}].$$

It follows that

$$\sum_{k \in A} \left| \frac{\partial \Gamma_{ik}}{\partial \sigma_{j\ell}} \right| \leq \Delta(x_i, q_i) \cdot \sum_{k \in A} [\Gamma_{ik}(1 - \Gamma_{ik})] \leq \Delta(x_i, q_i) \cdot \frac{K}{K+1},$$

where $\Delta(x, q) \equiv \max_{k, \ell, m \in A} |\alpha_k(\ell, x, q) - \alpha_m(\ell, x, q)|$ and the last step comes from the fact that (i) $0 \leq \Gamma_{ik} \leq 1$ and (ii) $\sum_{k=0}^K \Gamma_{ik} = 1$. Hence,

$$\begin{aligned} \left\| \sigma_i^* - \sigma_i^\dagger \right\|_1 &= \sum_{k \in A} \left| \sum_{j \in F_i} \sum_{\ell \in A} \frac{\partial \Gamma_{ik}(s_i, \{\tilde{\sigma}_j : j \in F_i\})}{\partial \sigma_{j\ell}} \cdot (\sigma_{j\ell}^* - \sigma_{j\ell}^\dagger) \right| \\ &\leq \sum_{j \in F_i} \sum_{\ell \in A} \left\{ \left| \sigma_{j\ell}^* - \sigma_{j\ell}^\dagger \right| \cdot \sum_{k \in A} \left| \frac{\partial \Gamma_{ik}(s_i, \{\tilde{\sigma}_j : j \in F_i\})}{\partial \sigma_{j\ell}} \right| \right\} \\ &\leq \Delta(x_i, q_i) \cdot \frac{K}{K+1} \cdot \sum_{j \in F_i} \sum_{\ell \in A} \left| \sigma_{j\ell}^* - \sigma_{j\ell}^\dagger \right| \\ &\leq \Delta(x_i, q_i) \cdot \frac{K}{K+1} \cdot q_i \cdot \max_{j \in F_i} \left\| \sigma_j^* - \sigma_j^\dagger \right\|_1 \leq \lambda \cdot \max_{j \in F_i} \left\| \sigma_j^* - \sigma_j^\dagger \right\|_1. \end{aligned}$$

Therefore,

$$\max_{i \in N} \left\| \sigma_i^* - \sigma_i^\dagger \right\|_1 \leq \lambda \cdot \max_{i \in N} \max_{j \in F_i} \left\| \sigma_j^* - \sigma_j^\dagger \right\|_1 \leq \lambda \cdot \max_{j \in N} \left\| \sigma_j^* - \sigma_j^\dagger \right\|_1,$$

which leads to contradiction by $\lambda < 1$ under Assumption 2.

A.1.2. Proof of Lemma 2. I prove by mathematical induction. Fix any $n, h \in \mathbb{N}$ and $s, s' \in \mathcal{S}$ such that $s_{(i,h)} = s'_{(i,h)}$.

First, for all $j \in N_{(i,h)}$, $s_j = s'_j$. I now derive $\sigma_j^*(s; \theta) - \sigma_j^*(s'; \theta)$ by Taylor expansion, that is,

$$\sigma_j^*(s'; \theta) - \sigma_j^*(s; \theta) = \sum_{j' \in F_j} \sum_{\ell \in A} \frac{\partial \Gamma_j(s_j, \{\tilde{\sigma}_{j'} : j' \in F_j\})}{\partial \sigma_{j'\ell}} \cdot (\sigma_{j'\ell}^*(s'; \theta) - \sigma_{j'\ell}^*(s; \theta)),$$

where $\{\tilde{\sigma}_{j'} : j' \in F_j\}$ is a vector between $\{\sigma_{j'}^*(s; \theta) : j' \in F_j\}$ and $\{\sigma_{j'}^*(s'; \theta) : j' \in F_j\}$. By a argument similar to the proof of Lemma 1, we have

$$\begin{aligned} \|\sigma_j^*(s; \theta) - \sigma_j^*(s'; \theta)\|_1 &\leq \lambda \cdot \max_{j' \in F_j} \|\sigma_{j'}^*(s; \theta) - \sigma_{j'}^*(s'; \theta)\|_1 \\ &\leq \lambda \cdot \max_{j' \in F_j} \left\{ \|\sigma_{j'}^*(s; \theta)\|_1 + \|\sigma_{j'}^*(s'; \theta)\|_1 \right\} = 2\lambda, \end{aligned}$$

where the last inequality comes from the triangular inequality. Because for all $j \in N_{(i, h-1)}$, any friend j' of j belongs to $N_{(i, h)}$, then

$$\|\sigma_j^*(s; \theta) - \sigma_j^*(s'; \theta)\|_1 \leq \lambda^2 \cdot \max_{j'' \in F_{j'}, j' \in F_j} \|\sigma_{j''}^*(s; \theta) - \sigma_{j''}^*(s'; \theta)\|_1 \leq 2\lambda^2.$$

By induction, for all $j \in N_{(i, h-q)}$ where $q \leq h$, there is

$$\|\sigma_j^*(s; \theta) - \sigma_j^*(s'; \theta)\|_1 \leq 2\lambda^{q+1}.$$

Hence, for any $q \leq h$, we have

$$\max_{j \in N_{(i, h-q)}} \|\sigma_j^*(s; \theta) - \sigma_j^*(s'; \theta)\|_1 \leq 2\lambda^{q+1}.$$

Because $i \in N_{(i, 0)}$, then $\|\sigma_i^*(s; \theta) - \sigma_i^*(s'; \theta)\|_1 \leq 2\lambda^{h+1}$. By Assumption 2, $2\lambda^{h+1} \downarrow 0$ as $h \rightarrow \infty$.

A.1.3. Proof of Lemma 3. First, by Assumption 3, (5) can be rewritten as

$$\delta_{ik}(\mathbf{S}) = \varphi'_i(\mathbf{S}) \times \begin{pmatrix} \beta_k(X_i) \\ \alpha_k(X_i, Q_i) \end{pmatrix}.$$

I further multiply by $\varphi_i(\mathbf{S})$ on both sides and obtain

$$\varphi_i(\mathbf{S}) \times \delta_{ik}(\mathbf{S}) = \varphi_i(\mathbf{S}) \times \varphi'_i(\mathbf{S}) \times \begin{pmatrix} \beta_k(X_i) \\ \alpha_k(X_i, Q_i) \end{pmatrix}.$$

Moreover, I take conditional expectation on both sides given $X_i = x$ and $Q_i = q$:

$$\mathbb{E}[\varphi_i(\mathbf{S}) \times \delta_{ik}(\mathbf{S}) | X_i = x, Q_i = q] = \mathbb{E}[\varphi_i(\mathbf{S}) \times \varphi'_i(\mathbf{S}) | X_i = x, Q_i = q] \times \begin{pmatrix} \beta_k(x) \\ \alpha_k(x, q) \end{pmatrix}$$

from which we invert the coefficients vector $(\beta_k(x), \alpha'_k(x, q))'$.

A.2. Asymptotic Properties Under Parametric Setting. For any $c \in \Theta$, let $L_n(c) = \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \mathbb{E}[\sigma_{ik}^*(\mathbf{S}; \theta) \ln \sigma_{ik}^*(\mathbf{S}; c)]$. For arbitrary $\epsilon > 0$, let $B_\epsilon(\theta)$ be an open ball centered at θ with ϵ radius in the space Θ .

A.2.1. Proof of Theorem 2. By Lemma 4, it suffices to check the conditions (i)–(iii) in the lemma. By the identification argument and Assumption 6, Condition (i) holds. Moreover, Condition (iii) also holds by Lemma 5. Hence, it suffices to verify Condition (ii), that is, $\sup_{c \in \Theta} |\hat{L}(c) - L_n(c)| \xrightarrow{P} 0$.

By Lemmas 6 and 7, $\sum_{k=0}^K 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; \cdot)$ is bounded and continuous on Θ . Since Θ is compact, then $\mathcal{F}_n = \{\sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; c) : c \in \Theta\}$ can be covered by a finite number of ϵ -brackets. To apply the classical Glivenko-Cantelli argument, it suffices to show the point-wise law of large number, that is, for any $c \in \Theta$, $\hat{L}(c) - L_n(c) \xrightarrow{P} 0$.

I pick an integer $d_n \propto 0.5 \ln n / \ln c_0$. Clearly, $d_n \rightarrow \infty$ as $n \rightarrow \infty$. Then we have

$$\begin{aligned}
 \hat{L}(c) - L_n(c) &= \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \{1(Y_i = k) - \sigma_{ik}^*(S; \theta)\} \ln \sigma_{ik}^*(S; c) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \left\{ \sigma_{ik}^*(S; \theta) \ln \sigma_{ik}^*(S; c) - \sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) \right\} \\
 &\quad + \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \left\{ \sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) - \mathbb{E} \left[\sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) \right] \right\} \\
 (A.2) \quad &\quad + \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \left\{ \mathbb{E} \left[\sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) \right] - \mathbb{E} \left[\sigma_{ik}^*(S; \theta) \ln \sigma_{ik}^*(S; c) \right] \right\}.
 \end{aligned}$$

For the first term of right-hand side in Equation (A.2), we have

$$\begin{aligned}
 &\mathbb{E} \left\{ \left[\frac{1}{n} \sum_{i=1}^n \sum_{k \in A} (1(Y_i = k) - \sigma_{ik}^*(S; \theta)) \ln \sigma_{ik}^*(S; c) \right]^2 \middle| S \right\} \\
 &= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E} \left\{ \left[\sum_{k \in A} (1(Y_i = k) - \sigma_{ik}^*(S; \theta)) \ln \sigma_{ik}^*(S; c) \right]^2 \middle| S \right\} \leq \frac{1}{n} (K+1)^2 (\ln \sigma_0)^2 \rightarrow 0,
 \end{aligned}$$

where the first step is because of the reasons that Y_i is conditionally independent given S and that $\mathbb{E}(Y_i|S) = \sigma_{ik}^*(S; \theta)$, and the last inequality is due to the fact that $\ln \sigma_0 \leq (1(Y_i = k) - \sigma_{ik}^*(S; \theta)) \ln \sigma_{ik}^*(S; c) \leq -\ln \sigma_0$ under Lemma 6.

Next, for the second term of the RHS in Equation (A.2), note that

$$\begin{aligned}
 &\mathbb{E} \left| \sigma_{ik}^*(S; \theta) \ln \sigma_{ik}^*(S; c) - \sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) \right| \\
 &\leq \mathbb{E} \left[\left| \sigma_{ik}^*(S; \theta) - \sigma_{ik}^{d_n}(S; \theta) \right| \cdot \left| \ln \sigma_{ik}^*(S; c) \right| \right] + \mathbb{E} \left[\left| \sigma_{ik}^{d_n}(S; \theta) \right| \cdot \left| \ln \sigma_{ik}^*(S; c) - \ln \sigma_{ik}^{d_n}(S; c) \right| \right] \\
 &\leq -\ln \sigma_0 \cdot \mathbb{E} \left| \sigma_{ik}^*(S; \theta) - \sigma_{ik}^{d_n}(S; \theta) \right| + \frac{1}{\sigma_0} \cdot \mathbb{E} \left| \sigma_{ik}^*(S; c) - \sigma_{ik}^{d_n}(S; c) \right| \rightarrow 0.
 \end{aligned}$$

Similarly, I can show that the last term in Equation (A.2) is also $o_p(1)$.

Therefore, it suffices to show that the third term of the RHS in Equation (A.2) is also $o_p(1)$. Note that

$$\begin{aligned}
 &\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \left[\sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) - \mathbb{E} \sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c) \right] \right\}^2 \\
 &= \frac{1}{n^2} \sum_{i,j=1}^n \text{Cov} \left(\sum_{k \in A} \sigma_{ik}^{d_n}(S; \theta) \ln \sigma_{ik}^{d_n}(S; c), \sum_{k \in A} \sigma_{jk}^{d_n}(S; \theta) \ln \sigma_{jk}^{d_n}(S; c) \right).
 \end{aligned}$$

By definition and Assumption 9, $\sigma_i^{d_n}(S; \theta)$ is independent of $\sigma_j^{d_n}(S; \theta)$ if there does not exist a player $m \in N_{(i, d_n)} \cap N_{(j, d_n)}$. By Assumption 10, there are at most $n \cdot (1 + c_0 + \dots + c_0^{d_n}) \leq n c_0^{d_n+1}$ pairs of (i, j) such that $\sigma_i^{d_n}(S; \theta)$ and $\sigma_j^{d_n}(S; \theta)$ are dependent on each other. Moreover, for any i and j ,

$$\begin{aligned}
& 2\text{Cov}\left(\sum_{k \in A} \sigma_{ik}^{d_n}(\mathbf{S}; \theta) \ln \sigma_{ik}^{d_n}(\mathbf{S}; c), \sum_{k \in A} \sigma_{jk}^{d_n}(\mathbf{S}; \theta) \ln \sigma_{jk}^{d_n}(\mathbf{S}; c)\right) \\
& \leq \mathbb{E}\left(\sum_{k \in A} \sigma_{ik}^{d_n}(\mathbf{S}; \theta) \ln \sigma_{ik}^{d_n}(\mathbf{S}; c)\right)^2 + \mathbb{E}\left(\sum_{k \in A} \sigma_{jk}^{d_n}(\mathbf{S}; \theta) \ln \sigma_{jk}^{d_n}(\mathbf{S}; c)\right)^2 \leq 2(1+K)^2(\ln \sigma_0)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathbb{E}\left\{\frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \left\{ \sigma_{ik}^{d_n}(\mathbf{S}; \theta) \ln \sigma_{ik}^{d_n}(\mathbf{S}; c) - \mathbb{E} \sigma_{ik}^{d_n}(\mathbf{S}; \theta) \ln \sigma_{ik}^{d_n}(\mathbf{S}; c) \right\}\right\}^2 \\
& \leq \frac{1}{n^2} \cdot n c_0^{d_n+1} 2(1+K)^2(\ln \sigma_0)^2 \propto \frac{1}{\sqrt{n}} 2c_0(1+K)^2(\ln \sigma_0)^2 \rightarrow 0.
\end{aligned}$$

LEMMA 4. Suppose (i) $\limsup_{n \rightarrow \infty} \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)) < 0$ holds for any $\epsilon > 0$; (ii) \hat{L}_n converges uniformly in probability to L_n , that is, $\sup_{c \in \Theta} |\hat{L}_n(c) - L_n(c)| \xrightarrow{P} 0$; and (iii) $\hat{L}_n(\hat{\theta}) \geq \sup_{c \in \Theta} \hat{L}_n(c) - o_p(1)$. Then $\hat{\theta} \xrightarrow{P} \theta$.

PROOF. To prove the lemma, I modify the proofs in Newey and McFadden (1994), Theorem 2.1. Note that the objective function $L_n(\cdot)$ in my case depends on n , and it converges to a limit as n goes to infinity. By (ii) and (iii), with probability approaching one (w.p.a.1),

$$L_n(\hat{\theta}) > \hat{L}_n(\hat{\theta}) - \eta/3 > \hat{L}_n(\theta) - 2\eta/3 > L_n(\theta) - \eta, \quad \forall \eta > 0.$$

Then, for any $\epsilon > 0$, choose $\eta = -\frac{1}{2} \limsup_{n \rightarrow \infty} \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)) > 0$. It follows that w.p.a.1,

$$L_n(\hat{\theta}) - L_n(\theta) > \frac{1}{2} \limsup_{n \rightarrow \infty} \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)).$$

Because for sufficient large n ,

$$\begin{aligned}
& \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)) - \limsup_{n \rightarrow \infty} \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)) \\
& \leq \eta = -\frac{1}{2} \limsup_{n \rightarrow \infty} \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)),
\end{aligned}$$

which implies $\frac{1}{2} \limsup_{n \rightarrow \infty} \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)) \geq \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta))$.

Therefore, w.p.a.1,

$$L_n(\hat{\theta}) - L_n(\theta) > \sup_{c \notin B_\epsilon(\theta)} (L_n(c) - L_n(\theta)),$$

which implies that $\hat{\theta} \in B_\epsilon(\theta)$ w.p.a.1. Because ϵ can be arbitrarily small, $\hat{\theta} \xrightarrow{P} \theta$. ■

LEMMA 5. Suppose that Assumptions 1, 7(i), and 8 hold. Then,

$$\hat{L}(\hat{\theta}) \geq \sup_{c \in \Theta} \hat{L}(c) - o_p(1).$$

PROOF. By the definition of $\hat{\theta}$, it suffices to show that $\sup_{c \in \Theta} |\hat{Q}(c) - \hat{L}(c)| \rightarrow 0$.
Because

$$\sup_{c \in \Theta} |\hat{Q}(c) - \hat{L}(c)| \leq \sup_{c \in \Theta} \frac{1}{n} \sum_{i=1}^n \sum_{k \in A} \left| \ln \sigma_{ik}^h(\mathbf{S}; c) - \ln \sigma_{ik}^*(\mathbf{S}; c) \right|.$$

By Taylor expansion,

$$\sum_{k \in A} \left| \ln \sigma_{ik}^h(\mathbf{S}; c) - \ln \sigma_{ik}^*(\mathbf{S}; c) \right| = \frac{1}{\sigma^\dagger} \sum_{k \in A} \left| \sigma_{ik}^h(\mathbf{S}; c) - \sigma_{ik}^*(\mathbf{S}; c) \right| \leq \frac{2\lambda^{h+1}}{\sigma_0},$$

where σ^\dagger is some real value between $\sigma_{ik}^h(\mathbf{S}; c)$ and $\sigma_{ik}^*(\mathbf{S}; c)$, and σ_0 is the lower bound of the equilibrium choice probability. The last step uses Lemmas 2 and 6. Thus,

$$\sup_{c \in \Theta} |\hat{Q}(c) - \hat{L}(c)| \leq \frac{2\lambda^{h+1}}{\sigma_0}.$$

Because of Assumption 11 and $\lambda < 1$, we have $\sup_{c \in \Theta} |\hat{Q}(c) - \hat{L}(c)| \xrightarrow{P} 0$. ■

A.2.2. *Proof of Theorem 3.* First, by the proof of Lemma 5 and Assumption 11(ii),

$$\sup_{c \in \Theta} |\hat{Q}(c) - \hat{L}(c)| \leq \frac{2(K+1)\lambda^h}{\sigma_0} = o_p(n^{-1}).$$

Hence, $\hat{L}(\hat{\theta}) \geq \sup_{c \in \Theta} \hat{L}(c) - o_p(n^{-1})$, which implies that $\partial \hat{L}(\hat{\theta}) / \partial c = o_p(n^{-1/2})$.

By the Taylor expansion, we have

$$\frac{\partial \hat{L}(\theta)}{\partial c} + \frac{\partial^2 \hat{L}(\theta^\dagger)}{\partial c \partial c'} (\hat{\theta} - \theta) = o_p(n^{-1/2})$$

for some θ^\dagger between θ and $\hat{\theta}$. Now it suffices to show

$$(A.3) \quad \sqrt{n} \times \frac{\partial \hat{L}(\theta)}{\partial c} \xrightarrow{d} N(0, J(\theta)),$$

$$(A.4) \quad \frac{\partial^2 \hat{L}(\theta^\dagger)}{\partial c \partial c'} \xrightarrow{P} -J(\theta).$$

I first show Equation (A.3). Let $\zeta_i = \frac{\partial}{\partial c} \sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; c)|_{c=\theta}$. Note that the true parameter θ always maximizes the likelihood function $E[\sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; \cdot) | \mathbf{S}]$ for any n and \mathbf{S} . Thus, $E(\zeta_i | \mathbf{S}) = 0$.

By definition, $\partial \hat{L}(\theta) / \partial c = n^{-1} \sum_{i=1}^n \zeta_i$. Then, it suffices to show that $n^{-1/2} \sum_{i=1}^n \zeta_i \xrightarrow{d} N(0, J(\theta))$. Equivalently, I need to show $n^{-1/2} \sum_{i=1}^n J(\theta)^{-\frac{1}{2}} \zeta_i \xrightarrow{d} N(0, \mathbf{1}_P)$, where $\mathbf{1}_P$ is the $P \times P$ identity matrix. For this, I show that the conditional distribution of $\sqrt{n} \sum_{i=1}^n J(\theta)^{-\frac{1}{2}} \zeta_i$ given \mathbf{S} always converges to the same limiting normal distribution $N(0, \mathbf{1}_P)$.

Because ζ_i is conditionally independent across i given \mathbf{S} . Then

$$E \left[\left(n^{-1/2} \sum_{i=1}^n \zeta_i \right) \cdot \left(n^{-1/2} \sum_{i=1}^n \zeta_i' \right) | \mathbf{S} \right] = n^{-1} \sum_{i=1}^n E(\zeta_i \cdot \zeta_i' | \mathbf{S}).$$

By an argument similar to that in the proof of Theorem 2, we have

$$n^{-1} \sum_{i=1}^n \mathbb{E}(\zeta_i \cdot \zeta'_i | \mathbf{S}) = n^{-1} \sum_{i=1}^n \mathbb{E}(\zeta_i \cdot \zeta'_i) + o_p(1) = J_n(\theta) + o_p(1) = J(\theta) + o_p(1).$$

Thus,

$$\mathbb{E} \left[\left(n^{-1/2} \sum_{i=1}^n \zeta_i \right) \cdot \left(n^{-1/2} \sum_{i=1}^n \zeta'_i \right) \middle| \mathbf{S} \right] \xrightarrow{p} J(\theta).$$

Hence, by the Lindeberg-Feller Theorem (see, e.g., Van der Vaart, 2000), conditional on \mathbf{S} ,

$$n^{-1/2} \sum_{i=1}^n J(\theta)^{-\frac{1}{2}} \zeta_i \xrightarrow{d} N(0, \mathbf{1}_P).$$

I now show Equation (A.4). Under Assumption 7, Lemmas 6 and 7 imply that $\| \frac{\partial^2}{\partial c \partial c'} \sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; c) \|$ is bounded above uniformly on n , \mathbf{S} and θ , and $\frac{\partial^2}{\partial c \partial c'} \sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; c)$ are smooth functions of $c \in \Theta$. Hence by an argument similar to the proofs in Theorem 2,

$$\sup_{c \in \Theta} \left[\frac{\partial \hat{L}(c)}{\partial c \partial c'} - \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left\{ \frac{\partial^2}{\partial c \partial c'} \sum_{k \in A} 1(Y_i = k) \ln \sigma_{ik}^*(\mathbf{S}; c) \right\} \right] \xrightarrow{p} 0.$$

Because $\theta^\dagger \xrightarrow{p} \theta$ and by Assumption 9, we have

$$\frac{\partial^2 \hat{L}(\theta^\dagger)}{\partial c \partial c'} = \mathbb{E} \left\{ \frac{\partial^2}{\partial c \partial c'} \sum_{k \in A} 1(Y_1 = k) \ln \sigma_{1k}^*(\mathbf{S}; \theta) \right\} + o_p(1).$$

Moreover, by the information matrix equality,

$$\mathbb{E} \left\{ \frac{\partial^2}{\partial c \partial c'} \sum_{k \in A} 1(Y_1 = k) \ln \sigma_{1k}^*(\mathbf{S}; \theta) \right\} = -J_n(\theta) = -J(\theta) + o(1).$$

Then Equation (A.4) is proved. ■

A.3. Auxiliary Lemmas.

LEMMA 6. *Suppose Assumption 1 and 7(i) hold. Then there exists $\sigma_0 \in (0, 1)$ such that $\sigma_{ik}^*(\mathbf{S}; c) \geq \sigma_0$ for all $n \in \mathbb{N}$, $i \in N$, $k \in A$, and $c \in \Theta$.*

PROOF. By Assumption 1, for all $(i, k) \in N \times A$,

$$\sigma_{ik}^*(\mathbf{S}; c) = \frac{\exp \left[(X'_i, Q_i) \cdot b_k + \sum_{\ell \in A} a_{k\ell} \left(\frac{1}{Q_i} \sum_{j \in F_i} \sigma_{j\ell}^*(\mathbf{S}; c) \right) \right]}{1 + \sum_{\ell'=1}^K \exp \left[(X'_i, Q_i) b_{\ell'} + \sum_{\ell \in A} a_{\ell'\ell} \left(\frac{1}{Q_i} \sum_{j \in F_i} \sigma_{j\ell}^*(\mathbf{S}; c) \right) \right]}.$$

Because $0 \leq \frac{1}{Q_i} \sum_{j \in F_i} \sigma_{j\ell}^*(\mathbf{S}; c) \leq 1$ and by Assumption 7(i), the RHS has a lower bound, denoted as $\sigma_0 > 0$. Note that the above argument does not depend on the value of n , i , k , and c . ■

LEMMA 7. Suppose that Assumptions 1 and 8 hold. Then, $\sigma_{ik}^*(S; \cdot) \in \mathcal{C}^\infty(\Theta)$ for all $n \in \mathbb{N}$, $S, i \in N, k \in A$, and $c \in \Theta$.

PROOF. I fix an arbitrary n and S in the following analysis. By Lemma 1, $\{\sigma_i^*(S; c) : i \in N\}$ is the unique solution to the equation system: for all $(i, k) \in (N, A)$,

$$\sigma_{ik}^* = \frac{\exp \left\{ b_k(X_i, Q_i) + \sum_{l=0}^K \left[a_k(\ell, X_i, Q_i) \cdot \sum_{j \in F_i} \sigma_{j\ell}^* \right] \right\}}{1 + \sum_{q=1}^K \exp \left\{ b_q(X_i, Q_i) + \sum_{l=0}^K \left[a_q(\ell, X_i, Q_i) \cdot \sum_{j \in F_i} \sigma_{j\ell}^* \right] \right\}}.$$

Let $\Sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$. Then the above equation system can be represented as

$$\Sigma^* = BR(S, \Sigma^*; c),$$

where BR is the $n(K+1)$ dimensional mapping representing the best response functions for all $(i, k) \in (N, A)$. Fix S . Clearly, BR belongs to $\mathcal{C}^\infty(\mathbb{R}^{n(K+1)} \times \Theta)$. Then by the implicit function theorem, the solution $\sigma_i^*(S; \cdot) \in \mathcal{C}^\infty(\Theta)$ for all $i \in N$. ■

A.4. Consistent Nonparametric Estimator of $F_{Y_i|S}$. The NDD condition is important for large network asymptotics. In particular, it allows us to nonparametrically estimate the probability distribution $F_{Y_i|S}$ using observations from one single large network. To illustrate, I consider the simple circle network where each player has two direct friends and the friendship is symmetric. Such a specification helps highlights key features of the consistency argument for the nonparametric estimation.

Because my asymptotic analysis considers a sequence of games with $n \rightarrow \infty$, I use S_n with subscript n to emphasize its dependence on the network size in the following analysis. The sequence of games is described as follows: Let the set of players $\{1, 2, \dots, n\}$ for $n \geq 2$ be located on a circle network as follows: First I randomly pick a location for player 1 on the circle. Next, players 2 and 3 are on 1's left and right, respectively; then players 4 and 5 are further located on 2's left and 3's right, respectively, and so on and so forth. Thus, we obtain a circle network with $n = +\infty$ in the limit. Given the network, state variables X_i are i.i.d. across all the players. Similarly to the probability theory in time series, the probability distribution of the sequence $\{S_n : n \geq 2\}$ is well defined.

For simplicity, let $A = \{0, 1\}$ and $X_i \in \mathbb{R}$. W.o.l.g., I consider the estimation of $\Pr(Y_i = 1 | S_n = s_n)$ for $i = 1$. To begin with, I first consider the case where X_i is binary, that is, $X_i \in \{0, 1\}$. It is straightforward that my arguments can be generalized to the case of multiple valued X_i s. The continuous X_i s case will be discussed later. Intuitively, a nonparametric estimator $\hat{\Pr}(Y_1 = 1 | S_n = s_n)$ can be defined as follows:

$$\frac{\sum_{j=1}^n 1(Y_j = 1) \cdot 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)}, \text{ for } \ell = -h, \dots, h]}{\sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)}, \text{ for } \ell = -h, \dots, h]},$$

where $j(\ell)$ denotes the $|\ell|$ th left vertex of j if $\ell < 0$; otherwise it refers to the $|\ell|$ th right vertex of j . Note that because of the circle network, $G_{(j,h)} = g_{(1,h)}$ a.s.. Then, the term $1[G_{(j,h)} = g_{(1,h)}]$ is redundant in the above expression. As is shown in the proof of the next lemma, the above estimator is essentially a kernel estimator with a specific choice of bandwidth and a uniform kernel.

In the above estimator, it is crucial to choose h for its consistency, which carries a bias and variance trade off: Intuitively, $h \in \mathbb{N}$ needs to increase properly with n such that $\Pr(Y_1 = 1 | S_{(1,h)})$ converges to $\Pr(Y_1 = 1 | S_n)$ (note that the approximation error is bounded by $2\xi^{h+1}$ where $|\xi| < 1$). On the other hand, I require that the number of observations $G_{(j,h)} = g_{(1,h)}$ goes to infinity with the network size, so that the variance of the estimator decreases to zero as $n \rightarrow \infty$.

W.l.o.g., suppose $\Pr(X_i = 0) \leq 1/2$. Let $p_h \equiv \Pr(S_{(1,h)} = s_{(1,h)}) = \prod_{j=1}^{2h+1} \Pr(X_j = x_j)$. By definition, $\Pr(X_i = 0)^{2h+1} \leq p_h \leq \Pr(X_i = 1)^{2h+1}$. Therefore, we have $p_h \rightarrow 0$ as $h \rightarrow \infty$.

LEMMA 8. *Suppose that Assumptions 1 and 6, 7(i), 9, and 10 hold. Suppose $h \rightarrow \infty$ and $\frac{h}{np_h} \rightarrow 0$ as $n \rightarrow \infty$. Then*

$$\hat{\Pr}(Y_1 = 1 | S_n = s_n) - \Pr(Y_1 = 1 | S_n = s_n) \xrightarrow{P} 0.$$

PROOF. First note that

$$\begin{aligned} \hat{\Pr}(Y_1 = 1 | S_n = s_n) \\ = \frac{\frac{1}{np_h} \sum_{j=1}^n 1(Y_j = 1) \cdot 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)}, \text{ for } \ell = -h, \dots, h]}{\frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)}, \text{ for } \ell = -h, \dots, h]}. \end{aligned}$$

I now show that the denominator and numerator converge to 1 and $\Pr(Y_1 = 1 | S_{(1,h)} = s_{(1,h)})$, respectively. First, I look at the denominator and show

$$(A.5) \quad \mathbb{E} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)}, \text{ for } \ell = -h, \dots, h] \right\} \rightarrow 1;$$

$$(A.6) \quad \text{Var} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)}, \text{ for } \ell = -h, \dots, h] \right\} \rightarrow 0.$$

Regarding (A.5), we have

$$\mathbb{E} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[\{X_\ell : \ell \in N_{(j,h)}\} = \{x_\ell : \ell \in N_{(1,h)}\}] \right\} = \frac{1}{p_h} \mathbb{E}\{1[S_{(1,h)} = s_{(1,h)}]\} = 1.$$

To establish (A.6), note that

$$\begin{aligned} \text{Var} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[\{X_\ell : \ell \in N_{(j,h)}\} = \{x_\ell : \ell \in N_{(1,h)}\}] \right\} \\ = \frac{1}{n^2 p_h^2} \sum_{\ell=1}^n \sum_{j \neq \ell} \text{Cov} \{1[S_{(j,h)} = s_{(1,h)}], 1[S_{(\ell,h)} = s_{(1,h)}]\} + \frac{1}{np_h^2} \text{Var} \{1[S_{(1,h)} = s_{(1,h)}]\} \\ = \frac{1}{np_h^2} \sum_{j \neq 1} \text{Cov} \{1[S_{(j,h)} = s_{(1,h)}], 1[S_{(1,h)} = s_{(1,h)}]\} + \frac{1-p_h}{np_h} \\ = \frac{1}{np_h^2} \sum_{j=2}^{2h+1} \text{Cov} \{1[S_{(j,h)} = s_{(1,h)}], 1[S_{(1,h)} = s_{(1,h)}]\} + \frac{1-p_h}{np_h}, \end{aligned}$$

where the last step comes from the assumption that $S_{(j,h)}$ is independent of $S_{(1,h)}$ if $N_{(j,h)}$ does not overlap with $N_{(1,h)}$. Thus,

$$\text{Var} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[\{X_\ell : \ell \in N_{(j,h)}\} = \{x_\ell : \ell \in N_{(1,h)}\}] \right\}$$

$$\begin{aligned}
&\leq \frac{2h}{np_h^2} \times \frac{\text{Var}\{1[S_{(j,h)} = s_{(1,h)}]\} + \text{Var}\{1[S_{(1,h)} = s_{(1,h)}]\}}{2} + \frac{1-p_h}{np_h} \\
&= \frac{(2h+1)(1-p_h)}{np_h} \propto \frac{h}{np_h} \rightarrow 0.
\end{aligned}$$

It follows that

$$\frac{1}{np_h} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)} : \ell = -h, \dots, h] \xrightarrow{P} 1.$$

By a similar argument, we have

$$\begin{aligned}
&\mathbb{E} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1(Y_j = 1) \cdot 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)} : \ell = -h, \dots, h] \right\} \\
&= \Pr(Y_1 = 1 | S_{(1,h)} = s_{(1,h)}) = \Pr(Y_1 = k | S_n = s_n) + o(|\xi|^h)
\end{aligned}$$

and

$$\text{Var} \left\{ \frac{1}{np_h} \sum_{j=1}^n 1(Y_j = 1) \cdot 1[G_{(j,h)} = g_{(1,h)}] \cdot 1[X_{j(\ell)} = x_{1(\ell)} : \ell = -h, \dots, h] \right\} \rightarrow 0.$$

Moreover, by Slutsky's theorem, I establish the consistency of the proposed estimator. \blacksquare

In Lemma 8, it is required that h should increase to infinity with n , but sufficiently slowly. In particular, the conditions imply $p_h \rightarrow 0$ and $np_h \rightarrow \infty$ as $n \rightarrow \infty$. This suggests that the term p_h plays the same role as the bandwidth in kernel estimation. In addition, because of the dependence between $S_{(j,h)}$ and $S_{(i,h)}$ for $\rho(i, j) \leq h$, I require that np_h increase to infinity faster than h . Suppose one chooses $h = [h_0 \times \ln n]$ for some constant $h_0 > 0$. Then, $p_h \propto n^{-\kappa}$, where $\kappa > 0$, which is determined by h_0 and $P(X_i = 0)$. Then the restrictions on h in Lemma 8 are satisfied if κ is sufficiently small.

Suppose X_i is continuously distributed. Let f_X be the pdf of X_i . For simplicity, I assume $0 < \inf_{x \in \mathbb{R}} f_X(x) < \sup_{x \in \mathbb{R}} f_X(x) < \infty$. As usual, additional assumptions on the structural parameters are needed to ensure $\Pr(Y_1 = 1 | S_n = s_n)$ is R th ($R \geq 2$) order continuously differentiable in each argument of S_n . Moreover, a nonparametric estimator is defined by

$$\hat{\Pr}(Y_1 = 1 | S_n = s_n) = \frac{\sum_{j=1}^n 1(Y_j = 1) \cdot 1[G_{(j,h)} = g_{(1,h)}] \cdot \prod_{\ell=1}^{2h+1} K\left(\frac{X_\ell - x_\ell}{b_\ell}\right)}{\sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot \prod_{\ell=1}^{2h+1} K\left(\frac{X_\ell - x_\ell}{b_\ell}\right)},$$

where K and b_ℓ for $\ell = 1, \dots, 2h+1$ are R th order kernel function and bandwidth, respectively.

For consistency, I need to choose $h \rightarrow \infty$ and $b_\ell \rightarrow 0$ for $\ell = 1, \dots, 2h+1$ properly as $n \rightarrow \infty$. For simplicity, let $b_\ell f_X(x_\ell) = p$ for some $p \equiv p_n > 0$. Moreover, let $h \rightarrow \infty$, $p \rightarrow 0$ and $h/(np^{2h+1}) \rightarrow 0$ as $n \rightarrow \infty$. By an argument similar to Lemma 8 and Bochner's Lemma, we can show consistency of the kernel estimator. In particular, we have

$$\mathbb{E} \left\{ \frac{1}{np^{2h+1}} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot \prod_{\ell=1}^{2h+1} K\left(\frac{X_\ell - x_\ell}{b_\ell}\right) \right\} = 1 + O(p^R)$$

and

$$\text{Var} \left\{ \frac{1}{np^{2h+1}} \sum_{j=1}^n 1[G_{(j,h)} = g_{(1,h)}] \cdot \prod_{\ell=1}^{2h+1} K \left(\frac{X_{\ell} - x_{\ell}}{b_{\ell}} \right) \right\} = o \left(\frac{h}{np^{2h+1}} \right),$$

and similar expressions hold for the numerator of the kernel estimator, which thereafter provide the consistency.

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