Problem Set I

Set-Up :

Production function :
- Cobb-Douglas
- Labour, capital, TFP
- All firms use same capital
- Price takers in labour and product markets

Question 1

Q.1.1

(a)

$$Y = A \cdot L^{\alpha_L} \cdot K^{\alpha_K}$$

Taking logs,

$$y = \alpha_L l + \alpha_K k + \omega$$

(b)

Profit function:

$$\Pi = P.Y - W.L - R.K$$

Or equivalently,

$$\Pi = P.\left(A.L^{\alpha_L}.K^{\alpha_K}\right) - W.L - R.K$$

(c)

First order condition w.r.t. labour:

$$\frac{\partial \Pi}{\partial L} = P\alpha_L A.L^{\alpha_L - 1}.K^{\alpha_K} - W = 0$$

(d)

Re-arranging the equation,

$$\alpha_L A.L^{\alpha_L - 1}.K^{\alpha_K} = \frac{W}{P}$$

Or equivalently,

$$\alpha_L \frac{Y}{L} = \frac{W}{P}$$

(e)

Taking logs, we obtain:

$$\log(\alpha_L) + y - \ell = \log\left(\frac{W}{P}\right)$$

$$\Rightarrow \quad \ell = y + \log(\alpha_L) - r$$

where r is the log-real wage.

Q. 1. 2

(a)

The simultaneous equations are:

$$\begin{cases} y = \alpha_L \ell + \alpha_K k + \omega \\ \ell = y + \log(\alpha_L) - r \end{cases}$$

(b)

Solving the system,

$$l = \alpha_L l + \alpha_K k + \omega + \log(\alpha_L) - r$$

$$l = \frac{\alpha_K k + \omega + \log(\alpha_L) - r}{(1-\alpha_L)}$$

Substituting back into the PF,

$$y = \alpha_L \left(\frac{\alpha_K k + \omega + \log(\alpha_L) - r}{(1-\alpha_L)} \right) + \alpha_K k + \omega$$

$$y = \frac{\alpha_L \alpha_K k + \alpha_L \omega + \alpha_L \log(\alpha_L) - \alpha_L r + (1-\alpha_L)\alpha_K k + (1-\alpha_L)\omega}{(1-\alpha_L)}$$

$$y = \frac{\alpha_K k + \omega + \alpha_L \log(\alpha_L) - \alpha_L r}{(1-\alpha_L)}$$

Q.1.3

Heterogeneity in log-TFP and log-real wage:

(ω_i, r_i).

(a)

We know that

$$\text{cov}(y_i, l_i) = \underbrace{\mathbb{E}\left[y_i - \mathbb{E}[y_i]\right]}_{1} \cdot \underbrace{\mathbb{E}\left[l_i - \mathbb{E}[l_i]\right]}_{2} .$$

Looking at term 1, we have:

$$\mathbb{E}\left[\frac{\alpha_K k + \omega_i + \alpha_L \log(\alpha_L) - \alpha_L r_i}{(1-\alpha_L)} - \mathbb{E}\left[\frac{\alpha_K k + \omega_i + \alpha_L \log(\alpha_L) - \alpha_L r_i}{(1-\alpha_L)}\right]\right]$$

$$= \frac{1}{(1-\alpha_L)} \mathbb{E}\left[\alpha_K k + \omega_i + \alpha_L \log(\alpha_L) - \alpha_L r_i - \alpha_K k - \mathbb{E}[\omega_i] - \alpha_L \log(\alpha_L) + \alpha_L \mathbb{E}[r_i]\right]$$

$$= \frac{1}{(1-\alpha_L)} \mathbb{E}\left[\left(\omega_i - \mathbb{E}[\omega_i]\right) - \alpha_L\left(r_i - \mathbb{E}[r_i]\right)\right] \quad (1)$$

Similarly for term (2),

$$\mathbb{E}\left[\frac{\alpha_K k + \omega_i + \log(\alpha_L) - r_i}{(1-\alpha_L)} - \mathbb{E}\left(\frac{\alpha_K k + \omega_i + \log(\alpha_L) - r}{(1-\alpha_L)}\right)\right]$$

$$= \frac{1}{(1-\alpha_L)} \mathbb{E}\left[\alpha_K k + \omega_i + \log(\alpha_L) - r_i - \alpha_K k - \mathbb{E}[\omega_i] - \log(\alpha_L) + \mathbb{E}[r_i]\right]$$

$$= \frac{1}{(1-\alpha_L)} \mathbb{E}\left[\left(\omega_i - \mathbb{E}[\omega_i]\right) - \left(r_i - \mathbb{E}[r_i]\right)\right] \quad (2)$$

Therefore, we have that

$$\operatorname{cov}(y_i, l_i) =$$

$$\frac{1}{(1-\alpha_L)^2} \mathbb{E}\left[\left(\omega_i - \mathbb{E}[\omega_i]\right) - \alpha_L\left(r_i - \mathbb{E}[r_i]\right)\right] \cdot \mathbb{E}\left[\left(\omega_i - \mathbb{E}[\omega_i]\right) - \left(r_i - \mathbb{E}[r_i]\right)\right]$$

By the linearity of the expectation operator, we obtain:

$$\mathrm{cov}(y_i, l_i) = \frac{1}{(1-\alpha_L)^2}\left(\sigma_w^2 - (1+\alpha_L)\sigma_{wr} + \alpha_L\sigma_r^2\right)$$

(b)

We know that

$$\mathrm{var}(l_i) = \mathbb{E}\left[l_i - \mathbb{E}[l_i]\right]^2$$

$$= \frac{1}{(1-\alpha_L)^2} \cdot \mathbb{E}\left[\alpha_K k + \omega_i + \log(\alpha_L) - r_i - \alpha_K k - \mathbb{E}[\omega_i] - \log(\alpha_L) + \mathbb{E}[r_i]\right]^2$$

$$= \frac{1}{(1-\alpha_L)^2} \cdot \mathbb{E}\left[(\omega_i - \mathbb{E}[\omega_i]) - (r_i - \mathbb{E}[r_i])\right]^2$$

De-composing,

$$= \frac{1}{(1-\alpha_L)^2} \cdot \mathbb{E}\left[(\omega_i - \mathbb{E}[\omega_i]) - (r_i - \mathbb{E}[r_i])\right] \cdot \left[(\omega_i - \mathbb{E}[\omega_i]) - (r_i - \mathbb{E}[r_i])\right].$$

By the linearity of the expectation operator,

$$var(l_i) = \frac{1}{(1-\alpha_2)^2} \left(\sigma_w^2 - 2\sigma_{wr} + \sigma_r^2 \right)$$

Q.1.4

(a) OLS estimator for α_2 :

$$\hat{\alpha}_L = \frac{\sum_{i=1}^{N} (y_i - \bar{y})(l_i - \bar{l})}{\sum_{i=1}^{N} (l_i - \bar{l})^2}$$

(b) Transforming the previous expression,

$$\hat{\alpha}_L = \frac{\frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})(l_i - \bar{l})}{\frac{1}{N} \sum_{i=1}^{N} (l_i - \bar{l})}$$

By the L.L.N., sample moments tend

to their population moments as N increases.

Therefore, as $N \longrightarrow \infty$, we have

$$\hat{\alpha}_L = \frac{cov(y, \ell)}{var(\ell)}$$

and plugging in our previous answers,

$$\hat{\alpha}_L = \frac{\sigma_w^2 - (1+\alpha_L)\sigma_{wr} + \alpha_L \sigma_r^2}{\sigma_w^2 - 2\sigma_{wr} + \sigma_r^2}$$

Q.1.5

We now examine the bias of $\hat{\alpha}_L$ under different scenarios.

(a) $\sigma_r^2 = 0$

Under no variance of r, note that

We also have $\sigma_{wr} = 0$

(i.e. a random variable cannot be correlated with a constant).

Therefore, we have:

$$\hat{\alpha}_L = \frac{\sigma_w^2}{\sigma_w^2} = 1$$

<u>Recall that</u>: $\text{Bias}(\hat{\alpha}_L) = \mathbb{E}[\hat{\alpha}_L] - \alpha_L$

Thus,

$$\text{Bias}(\hat{\alpha}_L) = (1 - \alpha_L).$$

Since usually we have $\alpha_L \in [0, 1)$, the bias is positive.

(b) $\sigma_w^2 = 0$

Again, this implies $\sigma_{wr} = 0$

$$\hat{\alpha}_L = \frac{\alpha_2 \sigma_r^2}{\sigma_r^2} = \alpha_2$$

Therefore, $\text{bias}(\hat{\alpha}_L) = 0$.

(c) $\sigma_{wr} = 0$, $\sigma_w^2 > 0$, $\sigma_r^2 > 0$.

$$\hat{\alpha}_L = \frac{\sigma_w^2 + \alpha_L \sigma_r^2}{\sigma_w^2 + \sigma_r^2}$$

The bias is :

$$\mathbb{E}\left[\hat{\alpha}_L\right] - \alpha_L = \frac{\sigma_w^2 + \alpha_L \sigma_r^2}{\sigma_w^2 + \sigma_r^2} - \alpha_2$$

$$= \frac{\sigma_w^2 + \alpha_L \sigma_r^2 - \alpha_L \sigma_w^2 - \alpha_2 \sigma_r^2}{\sigma_w^2 + \sigma_r^2}$$

$$= \frac{\sigma_w^2}{\sigma_w^2 + \sigma_r^2} (1 - \alpha_L)$$

For this bias to be positive, we need

$$\frac{\sigma_w^2}{\sigma_w^2 + \sigma_r^2} (1 - \alpha_L) > 0$$

Since $(\sigma_w^2, \sigma_r^2) > 0$ by definition, the equation holds as long as

$$\alpha_L < 1.$$

Since α_L is usually $\in [0, 1)$, we have that the bias is positive.

$\boxed{\text{Question 2}}$

→ Same assumptions on the Cobb - Douglas production function :

$$Y = A \cdot L^{\alpha_L} \cdot K^{\alpha_K}$$

<u>Q.2.1</u>

We can write $C(Y)$ as :

$$C(Y) = \min_{K,L} \left\{ w_K \cdot K + w_L \cdot L \right\}$$

subject to the technology

$$Y = A \cdot L^{\alpha_L} \cdot K^{\alpha_K}$$

Equivalently, we can write the Lagrangian :

$$\mathcal{L} = w_K \cdot K + w_L \cdot L - \lambda \left(A \cdot L^{\alpha_L} \cdot K^{\alpha_K} - Y \right)$$

Taking the first order conditions :

$$\frac{\partial \mathcal{L}}{\partial K} = w_K - \lambda \alpha_K A . L^{\alpha_L} . K^{\alpha_K - 1} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial L} = w_L - \lambda \alpha_L A . L^{\alpha_L - 1} . K^{\alpha_K} = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = - A . L^{\alpha_L} . K^{\alpha_K} + Y = 0 \qquad (3)$$

Re-arranging (1),

$$\lambda = \frac{w_K}{\alpha_K} \cdot \frac{K}{Y} .$$

Plugging into (2),

$$\frac{w_L}{\alpha_L} \cdot \frac{L}{Y} = \frac{w_K}{\alpha_K} \cdot \frac{K}{Y}$$

which yields

$$L = \frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L} \cdot K$$

Plugging into (3),

$$Y = A \cdot \left(\frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L} \cdot K \right)^{\alpha_L} \cdot K^{\alpha_K}$$

$$K^{(\alpha_L + \alpha_K)} = \frac{Y}{A} \cdot \left(\frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L} \right)^{-\alpha_L}$$

$$\boxed{K^* = \left(\frac{Y}{A} \right)^{\frac{1}{(\alpha_L + \alpha_K)}} \cdot \left(\frac{\alpha_K}{\alpha_L} \cdot \frac{w_L}{w_K} \right)^{\frac{\alpha_L}{\alpha_L + \alpha_K}}}$$

Plugging back into our expression for labour,

$$L = \frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L} \cdot \left(\frac{Y}{A}\right)^{\frac{1}{\alpha_L+\alpha_K}} \cdot \left(\frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L}\right)^{\frac{-\alpha_L}{\alpha_L+\alpha_K}}$$

$$\boxed{L^* = \left(\frac{Y}{A}\right)^{\frac{1}{\alpha_L+\alpha_K}} \cdot \left(\frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L}\right)^{\frac{\alpha_K}{\alpha_L+\alpha_K}}}$$

Now, we know that

$$C(Y) = w_L \cdot L^* + w_K \cdot K^*$$

Therefore, denoting $\alpha \equiv (\alpha_K + \alpha_L)$,

$C(Y) =$

$$w_L \cdot \left(\frac{Y}{A}\right)^{\frac{1}{\alpha}} \left(\frac{\alpha_L}{\alpha_K} \cdot \frac{w_K}{w_L}\right)^{\frac{\alpha_K}{\alpha}} + w_K \cdot \left(\frac{Y}{A}\right)^{\frac{1}{\alpha}} \left(\frac{\alpha_K}{\alpha_L} \cdot \frac{w_L}{w_K}\right)^{\frac{\alpha_L}{\alpha}}$$

$$= \left(\frac{Y}{A}\right)^{\frac{1}{\alpha}} \left[\left(\frac{\alpha_L}{\alpha_K}\right)^{\frac{\alpha_K}{\alpha}} \cdot w_L^{\alpha_L/\alpha} \cdot w_K^{\alpha_K/\alpha} + \left(\frac{\alpha_K}{\alpha_L}\right)^{\frac{\alpha_L}{\alpha}} \cdot w_K^{\frac{\alpha_K}{\alpha}} \cdot w_L^{\frac{\alpha_L}{\alpha}}\right]$$

$$=\left(\frac{Y}{A}\right)^{\frac{1}{\alpha}}\left[w_L^{\frac{\alpha_L}{\alpha}}\cdot w_K^{\frac{\alpha_K}{\alpha}}\left[\left(\frac{\alpha_L}{\alpha_K}\right)^{\frac{\alpha_K}{\alpha}}+\left(\frac{\alpha_K}{\alpha_L}\right)^{\frac{\alpha_L}{\alpha}}\right]\right]$$

<u>Note that</u> :

$$\left(\frac{\alpha_L}{\alpha_K}\right)^{\frac{\alpha_K}{\alpha}}=\left(\frac{\alpha_K}{\alpha_L}\right)^{\frac{-\alpha_K}{\alpha}}=\left(\frac{\alpha_K}{\alpha_L}\right)^{\frac{\alpha_L-\alpha}{\alpha}}$$

Therefore,

$c(Y)=$

$$\left(\frac{Y}{A}\right)^{\frac{1}{\alpha}}\left[w_L^{\frac{\alpha_L}{\alpha}}\cdot w_K^{\frac{\alpha_K}{\alpha}}\left[\left(\frac{\alpha_K}{\alpha_L}\right)^{\frac{\alpha_L-\alpha}{\alpha}}+\left(\frac{\alpha_L}{\alpha_K}\right)^{\frac{\alpha_K-\alpha}{\alpha}}\right]\right]$$

$$=\left(\frac{Y}{A}\right)^{\frac{1}{\alpha}}\left[w_L^{\frac{\alpha_L}{\alpha}}\cdot w_K^{\frac{\alpha_K}{\alpha}}\cdot\left[\left(\frac{\alpha_K}{\alpha_L}\right)^{\frac{\alpha_L}{\alpha}}\cdot\left(\frac{\alpha_L}{\alpha_K}\right)+\left(\frac{\alpha_L}{\alpha_K}\right)^{\frac{\alpha_K}{\alpha}}\cdot\left(\frac{\alpha_K}{\alpha_L}\right)\right]\right]$$

$$= \left(\frac{Y}{A}\right)^{\frac{1}{\alpha}} \cdot w_L^{\frac{\alpha_L}{\alpha}} \cdot w_K^{\frac{\alpha_K}{\alpha}} \cdot \left(\frac{1}{\alpha_L}\right)^{\frac{\alpha_L}{\alpha}} \cdot \left(\frac{1}{\alpha_K}\right)^{\frac{\alpha_K}{\alpha}} \cdot (\alpha_L + \alpha_K)$$

And therefore we obtain :

$$\boxed{C(Y) = \alpha \left(\frac{Y}{A}\right)^{\frac{1}{\alpha}} \cdot \left(\frac{w_L}{\alpha_L}\right)^{\frac{\alpha_L}{\alpha}} \cdot \left(\frac{w_K}{\alpha_K}\right)^{\frac{\alpha_K}{\alpha}}}$$

Q.2.2

We have : $\alpha_L = 0.6$; $\alpha_K = 0.3$

We first take logs of $C(Y)$ and Y, since the interpretation of the parameters in a log-log relationship is elasticities.

$$\log(Y) = \alpha_L l + \alpha_K k + \omega$$

$$\log(C(Y)) = \log(\alpha) + \frac{1}{\alpha}\log(Y) - \frac{1}{\alpha}\omega + \frac{\alpha_L}{\alpha}\log(W_L) - \frac{\alpha_L}{\alpha}\log(\alpha_L) + \frac{\alpha_K}{\alpha}\log(W_K) - \frac{\alpha_K}{\alpha}\log(\alpha_K)$$

(a)

We obtain:

$$\varepsilon_{c,y} = \frac{1}{\alpha} = \frac{1}{0.9}$$

(b)

$$\varepsilon_{c,\omega} = -\frac{1}{\alpha} = -\frac{1}{0.9}$$

(c)

$$\varepsilon_{c,w_L} = \frac{\alpha_L}{\alpha} = \frac{0.6}{0.9}$$

(d)

$$\varepsilon_{c,w_K} = \frac{\alpha_K}{\alpha} = \frac{0.3}{0.9}$$

(e)

$$\varepsilon_{y,\ell} = \alpha_L = 0.6$$

(f)

$$\varepsilon_{y,w} = 1$$

<u>Q.2.3</u>

We have :

$$P = 100 - Q$$

$$\alpha_L = \alpha_K = \frac{1}{4}$$

$$w_L = w_K = \frac{1}{4}$$

$$F = 2$$

→ The cost function (including F)

of firm i is:

$$C(Y_i) = \frac{1}{2}\left(\frac{Y_i}{A_i}\right)^2 + 2$$

Therefore, the profit function of

firm i is:

$$\Pi_i = P \cdot Y_i - \frac{1}{2}\left(\frac{Y_i}{A_i}\right)^2 - 2$$

Q.2.4

Differentiating $C(Y)$ w.r.t. Y, we

obtain marginal cost:

$$MC(Y) = \frac{Y_i}{A_i^2}$$

Then, by the optimality condition,

$$P = \frac{y_i}{A_i^2}$$

$$\Rightarrow \quad y_i = P \cdot A_i^2$$

Q2.5

This condition yields:

$$\Pi_i \geq 0$$

$$\Leftrightarrow \quad P \cdot y_i - \frac{1}{2}\left(\frac{y_i}{A_i}\right)^2 - 2 \geq 0$$

Substituting in for the optimal y_i,

$$P^2 A_i^2 - \frac{1}{2}\left(\frac{PA_i^2}{A_i}\right)^2 - 2 \geq 0$$

$\iff$

$$P^2 A_i^2 - \frac{1}{2} P^2 A_i^2 - 2 \geq 0$$

$$\iff \quad \frac{1}{2} P^2 A_i^2 \geq 2$$

$$\iff \quad P A_i \geq 2$$

Q2.6

First, we can express P^* as a function of firm outputs:

$$P^* = 100 - \sum_{i=1}^{N} y_i^*$$

Substituting in for the optimal outputs,

$$P^* = 100 - \sum_{i=1}^{N} P^* A_i^2$$

By linearity, and given the condition for which firms are active,

$$P^* = 100 - P^* \sum_{i=1}^{N} \mathbb{1}\left\{A_i \geq \frac{2}{P^*}\right\} A_i^2$$

Q2.7

We have:
$A_1 = 7$; $A_2 = 5$; $A_3 = 1$
We begin with the conjecture that only 1 and 2 produce.

This entails that

$$A_3 < \frac{2}{P^*} \iff P^* < 2$$

Since 2 produces, we have

$$A_2 \geq \frac{2}{P^*} \iff P^* \geq \frac{2}{5}$$

And since 1 produces, we have

$$A_1 \geq \frac{2}{P^*} \iff P^* \geq \frac{2}{7}$$

Thus, $P^* \in \left[\frac{2}{7}, 2\right)$.

We can obtain the equilibrium price using our equation for P:

$$p^* = 100 - p^*(7^2 + 5^2)$$

$$p^* = 100 - 74p^*$$

$$p^* = \frac{100}{75} = \frac{4}{3}$$

We check, and we confirm that

$$\frac{4}{3} \in \left[\frac{2}{7}, 2\right).$$

Then, using the optimality conditions, we have:

$$y_1 = \frac{4}{3} \cdot 49 = 65.33$$

$$y_2 = \frac{4}{3} \cdot 25 = 33.33$$

and therefore

$$y_T = y_1 + y_2 = 98.66$$

Suppose we instead thought all three firms produced:

$$P^* = 100 - P^*(49 + 25 + 1)$$

$$P^* = 100 - 75P^*$$

$$P^* = \frac{100}{76}.$$

However, for firm 3 to produce, we would need

$$A_3 \geq \frac{2}{P^*}$$

$$\Rightarrow P^* \geq 2$$

This does not hold, and therefore we confirm that only 1 and 2 produce in equilibrium.